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IS ECONOPHYSICS A SOLID SCIENCE?*

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Econophysics is an approach to quantitative economy using ideas, models, conceptual and computational methods of statistical physics. In recent years many of physical theories like theory of turbulence, scaling, random matrix theory or renormalization group were successfully applied to economy giving a boost to modern computational techniques of data analysis, risk management, artificial markets, macro-economy, *etc.* Econophysics became a regular discipline covering a large spectrum of problems of modern economy. It is impossible to review the whole field in a short paper. Here we shall instead attempt to give a flavor of how econophysics approaches economical problems by discussing one particular issue as an example: the emergence and consequences of large scale regularities, which in particular occur in the presence of fat tails in probability distributions in macroeconomy and quantitative finance.

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1. Introduction

Half a decade ago, a word "econophysics" started to circulate in the community of physicists. In July 1997, "Workshop on Econophysics" was organized in Budapest by Imre Kondor and Janos Kertesz [1].

Followed by several other dedicated meetings, the field matured, reaching the state when textbooks on the subject, written by the pioneers in the field, started to appear [2–4].

The name "econophysics", a hybrid of "economy" and "physics", was coined to describe applications of methods of statistical physics to economy in general. In practice, majority of the research concerned the finances.

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In such a way, physicists entered officially and scientifically the field of financial engineering. On top of similar statistical methods used by financial mathematicians (although formulated in not so formal or "high-brow" fashion as in the textbooks on financial mathematics), physicists concentrated on the analysis of experimental data using tools borrowed from the analysis of real complex systems.

Commissioned by the Editorial Board of *Acta Physica Polonica B* to present an overview of the "econophysics" oriented towards a physicist who never really entered this interdisciplinary area, we faced the danger of an attempt to present the status of the discipline which is still *in statu nascendi*, reviewed by authors biased strongly by their personal views related to their (limited) own research in the newborn field. Therefore this mini-review is to a large extent a collection of thoughts and results from works of the three authors. As such, it is not intended to cover the whole field which has become a large discipline with many sub-branches by now but instead to present a modest sampler of scientific methods borrowed from physics to describe economical "data". We restricted to the methods which were natural extrapolation of those used in our own research in fundamental science (quantum gravity, random matrices, random geometry, complex systems). As a guiding line through this mini-review we have chosen power laws due to their omnipresence in economical data.

The review is organized as follows. We begin with a historical introduction arguing that despite the name "econophysics" entered the scientific language only half a decade ago, connections and interplay between physics and economy are more than hundred years old. The official marriage of disciplines of economy, often understood as an art, and physics being an example of a hard science, has been preceded by the continuous development of scientific methodology for a long time. One could even say that the official recognition of the close links came surprisingly late.

In the second part we concentrate on power-laws in economy. Using the system size criterion we divide the economical world into macro-, meso- and microscopic objects: the first of which are related to macro-economy, the second to stock markets and the third to individual companies. The levels are intertwined. In macro-economy one observes fat tails in the wealth and income distributions. Analysis of stock markets clearly shows the presence of large scale events, which can be described by probability distributions with fat tails. The same concerns price fluctuations of individual companies. At each of these regimes, one uses slightly different tools of the analysis. As we shall argue they all have common roots in the theory of large numbers. We shall start with the macro-economical application where we discuss the wealth and income distributions. Then we switch to the micro- and mesoscopic regimes where we shall concentrate on statistical properties of the system of fluctuating assets and on a question how the signal can be extracted in such a system. The natural language for the description of such a system is provided by the random matrix theory. We shall discuss the central limit theorem for random matrices and its consequences.

In the last, third part we very briefly mention other active areas of research which have recently attracted attention of the econophysics community. We also try to speculate on potential dangers of the approach, which may arise if methods of physics are adapted to economy to blindly. We believe that the success of scientific methods for economic applications requires broader scientific methodology, borrowing largely not only from physics, but also from other domains of science, mainly the theory of adaptive systems, studies of computer networks or the analysis of complex systems. Only successful evolution of "econophysics" into "econoscience", accompanied by rigid constraints based on careful analysis of empirical data, gives economy a chance to become a predictive theory at a high confidence level, and may acquire a status of a "hard science". We conclude that achievement of this goal, although not easy, is certainly possible.

2. Historical background

At a first glance, economy and physics do not seem to be related. Despite the fact that the literature is full of examples of famous physicists being interested in economic or financial problems, these examples are usually treated as adventures, and are sometimes anecdotical. Some well known cases are:

- unsuccessful predictions of stock prices by sir Isaac Newton, and in consequence, his terrible loss in 1720 of 20000 pounds in South Sea speculation bubble [5],
- successful management of the fund for widows of Goetingen professors, performed by Carl Friedrich Gauss,
- explanation of the Brownian random walk and the formulation of the Chapman-Kolmogorov condition for Markovian processes by Louis Bachelier in his PhD thesis on the theory of speculation done 5 years before the Smoluchowski's and Einstein's works on diffusion, on the basis of the observations of price movements on Paris stock-market [6]

and few others. These examples put forward the thesis which may sound revolutionary for a contemporary econophysicist: It was the economy which followed physics, and not *vice versa* — studies of the XVIII and XIX century classical physics made a dramatic impact on economy, and the work was done mostly by the economists, who tried to follow the scientific methodology of physical sciences (see *e.g.* [7,8]).

As a first example we mention the father of classical economy, Adam Smith. In his work *"The principles which lead and direct philosophical enquires: illustrated by the history of astronomy"*, Smith exemplifies the methodology of science by stressing the role of observing the regularities and then constructing theories (called by Smith "imaginary machines") reproducing the observations. Using the astronomy as a reference point was not accidental — it was the celestial mechanics, and the impressive amount of astronomical data, which dominated science in several cultures. It is rather amazing, that this analysis was done by a person, who is primarily identified as an economist, and not as a "physical scientist". In the end of XVIII and in XIX century, Newton's theories were transformed into more modern language of analytical mechanics in the works of Lagrange, Hamilton and others (actually, this is the formulation still used in textbooks of mechanics today). The beauty and power of the analytical mechanism did not escape the attention of the economists. In particular, the concepts of mechanics were considered as an ideal tool to be used in mathematization of economy. Again, it is perhaps surprising for a contemporary financial engineer that mathematics entered economy through physics! Economists like Walras, Jevons, Fisher, Pareto tried to map the formalism of physics onto the formalism of economy, replacing material points by economic agents, finding the analogy of the potential energy represented by "utility", and then evolving the systems by the analogs of principle of minimal action [8]. That fascination with mechanics went so far, that economists were even building mechanical models illustrating the concept of economical equilibrium. The enchantment with classical physics dated till the first half of the XX century. Again, it is surprising for a physicist, that the conceptual revolution done by Boltzmann (concepts of probability) and quantum mechanics (another meaning of probability), were missed for so long by the economists. Visionary suggestions by Majorana [9] in the 30's to use statistical physics in social science were at that time not explored neither by physicists nor by economists.

It is surprising even more, if we recall the example of the already mentioned Louis Bachelier, who formulated the theory of Brownian motion on the basis of economic data and moreover 5 years before the seminal works by Einstein and Smoluchowski. Almost half a century after the defense of his thesis *"Jeu de speculation"* (not appreciated very much by his advisor, Henri Poincaré), the ideas of Bachelier were discovered in the economy departments of major American universities. A slight modification of the Bachelier stochastic process (basically, changing the additive noise into the multiplicative) lead Osborne and Samuelson [17] to the fundamental stochastic equation governing the evolution of stock prices and serves as a cornerstone of the famous theory of Black, Scholes and Merton for calculating the correct price of an option. Technically, the Black–Scholes formula is just the solution of the heat equation, with a peculiar boundary condition. The incredible practical success of option-pricing formulae perhaps lured economists and financial engineers a bit, and maybe, to some extent, was responsible for the spectacular crash on Wall Street in August and September 1998 which ricocheted over the other markets.

Taking into account several discoveries done in physics, one could say that perhaps in the 80' the economists missed a lesson from physics. Concepts of a random walk were formulated using the assumption of the Gaussian character of a stochastic process. As such, the movement of prices was considered as memoryless, with almost negligible effects of large deviations, exponentially screened in the Gaussian world. Actually already in the 60' Mandelbrot pointed certain selfsimilarity of the behavior of commodities (cotton prices) over different time scales, interpreted as the appearance of power law. Today, for a physicist, familiar with critical phenomena, the concept of a power law and large fluctuations is rather obvious, although she or he may not be familiar with the fact that the main concepts of fractal behavior, spelled by Mandelbrot in 70', were predecessed by his study of cotton prices, done a decade earlier. Actually, stock markets exhibited large fluctuations (power behavior is usually named as "fat" or "heavy" tail behavior), but rather a limited interest in this behavior in the 90' was caused to large extent by the reservation of financial mathematics, lacking powerful mathematical methods (like Ito calculus) suited for processes with divergent moments.

The second major factor, changing the Gaussian world was a computer. In the last 40 years the performance of the computers had increased by six orders of magnitude. This fact had to have a crucial impact on economy. First, the speed and the range of transactions had changed drastically. In such a way computer started involuntarily to serve as an amplifier of fluctuations. Second, the economies and markets started to watch each other more closely, since computer possibilities allowed for collecting exponentially more data.

In this way, several nontrivial couplings started to appear in economical systems, leading to nonlinearities. Nonlinear behavior and overestimation of the Gaussian principle for fluctuations were responsible for the Black Monday Crash in 1987, and the crisis in August and September 1998.

That shock had however also a positive impact visualizing the importance of the non-linear effects. Already Poincaré has pointed the possibility of unpredictability in a *nonlinear* dynamical system, establishing the foundations of the chaotic behavior. The study of chaos turned out to be a major branch of theoretical physics. It was only a question of time, how fast these ideas will start to appear in economy. Ironically, Poincaré, who did not appreciate Bachelier's results, made himself a large impact on real complex systems as one of the discoverers of chaotic behavior in dynamical systems. Nowadays studies of chaos, self-organized criticality, cellular automata and neural networks are seriously taken into account as economical and financial tools.

One of the benefits of the computers was that economic systems started to *save* more and more data. Today markets collect incredible amount of data (practically they remember every transaction). This triggers the need for new methodologies, able to manage the data. In particular, the data started to be analyzed using methods, borrowed widely from physics, where seeking for regularities and for unconventional correlations is mandatory.

It was perhaps the reason, why several institutions (however, more financial than devoted to study the problems of macroeconomy) started hiring physicists as their "quants" or "rocket scientists". In the last ten years, another tendency appeared — physicists started to study economy scientifically. Several educational or research institutions devoted to study complexity launched the research programs in economy and financial engineering. These studies were devoted mostly to quantitative finance. To a large extent, it was triggered by vast amount of data accessible in this field. In such a way, physics started to play the role of financial mathematics — sometimes rephrasing the mathematical constructions in the language of physics, sometimes applying methods developed solely in physics, usually at the level of various effective theories of complex systems. Name "econophysics", often attributed to the activity of physicists in this field, is in our opinion rather misleading — perhaps "the physics of finances" is more adequate or even "statistical phynance" as J.P. Bouchaud jokes. Moreover, as we speculate in the conclusions of this work, name "physics" may be to restrictive to include majority of the tools of financial analysis.

Probably the most challenging questions in economy are those related to macro-economy. Extrapolating the historical perspective, briefly sketched above, to the future, one can expect methods of physics, especially those used in studies of complex and nonlinear systems, to make an impact on this field in the nearest future. In this case the meaning of econophysics would be similar to "physical economy", and econophysics could be viewed as a physicists' realization of XIX century economists' dream.

3. Macro-economy

Let us now turn to an example of econophysical reasoning in macro-economy. The term macro-economy has in general a double meaning: of a science which deals with large scale phenomena in economical systems and of a system which is the subject of the macro-economical studies. Such a macro-economical system is a complex system which consists of many individuals interacting with each other. The individuals function in the background provided by the legal and institutional frames. Individuals differ in abilities, education, mentality, historical and cultural background *etc.* They enter the system with different financial and cultural initial conditions. Each of them has his own vision of what is important and of what she or he is willing and able to achieve. It is clear that one cannot formulate a general theory of needs and financial possibilities of a single individual or to create an economical profile of a typical member of such a complicated system. There are too many random factors to be taken into account. They change in time: sometimes slowly, sometimes faster, sometimes abruptly and in an unpredictable way. Every day some individuals leave the system, some new enter it. It is impossible to follow individual changes. One can however control their statistics. Actually, it is the statistics which shapes the system on large macro-economical scale and drives the large scale phenomena observed in the whole macro-system.

The aim of macro-economical studies is to extract important factors, understand their mutual relations and describe the development of past events. The ultimate goal is to reach a level of understanding which would also permit to predict the reaction of the system to the change of macro-economical parameters in the future. Having such a knowledge at hand, macro-economists would be able to stimulate the optimal evolution by appropriately adjusting the macro-economical parameters. This level of understanding goes far beyond a formal description and requires modeling and understanding of fundamental principles which are difficult because of the complexity of the problem. Clearly, a model whose main ambition would be to realistically take into account all parameters and factors characterizing the whole network of dependencies in such a complex system would fail to be comprehensive and solvable. One would not be able to learn anything from such a model. It would be even to complicated to properly reflect what it actually intends to describe.

Obviously, one has to find a way of simplifying the underlying complexity to the level which enables a formulation of a treatable model. A danger of a simplification of a complex and non-linear problem is that by a tiny modification one can loose an important part of the information or introduce some artificial effects. There are two possible approaches to the problem of modeling complexity. One way is to follow a phenomenological reduction scheme. The first step is to introduce effective phenomenological quantities which encode the most important part of the reduced information. Of course, it is very difficult to quantify many important factors like cultural potential, historical background or influence of a change in particular law which for example regulates relation between employers and employees *etc.* Such factors play crucial role in the outcoming shape of the macro-system. The next step is to determine mutual dependencies of these quantities. This procedure usually leads to a set of non-linear differential equations describing evolution of the phenomenological quantities as a function of other parameters. At this level a new complication occurs. It is well known that nonlinear equations generally possess a very complicated spectrum of solutions whose stability depends on precise values of the parameters. Sometimes tiny changes of parameters which are irrelevant, from the point of view of the macro-economy, may be significant for the underlying mathematics, and opposite. In other words, a formal mathematical solution does not always carry a realistic economical information. One has to distinguish between the real and artificial effects. It is not always easy and one should be aware of limitations steaming from the complexity and non-linearity.

An alternative approach is the search for universal laws which govern the behavior of the complex system. Such laws may uncover global regularities which are insensitive to tiny changes of parameters within a given class of parameters. Such laws also provide a classification of possible universal large scale behaviors which can occur in the system and which can be used as a first order approximation in the course of gaining insight into the mechanisms driving the system.

This approach has been successfully used in theoretical physics for a long time where for a given model one is able with the aid of the renormalization group ideas to determine so called fixed points, each of which being related to one universality class of the model [10]. The space of all possible classes of different large scale behaviors of the model is divided into subspaces called domains (or basins) of attraction of those fixed points. The universal properties of any theory within a domain of attraction of a given fixed point are entirely determined by the properties of the renormalization group map in the nearest neighborhood of the fixed point. The number of domains of attraction is usually small. Thus typically one has only a few distinct universal large scale behaviors despite the original theory has infinitely many degrees of freedom and infinitely many coupling constants controlling the mutual interactions of those degrees of freedom. Macro-economical systems are in this respect very similar to field theoretical ones.

Another well known example of the emergence of universal laws is the central limit theorem. Saying not rigorously, the central limit theorem tells us that the sum of many independent identically distributed random numbers polled from a distribution with a finite average and a finite variance obeys a Gaussian law with the mean and the variance which scale with the number of terms in the sum independently of the particular shape of the distribution. One could say that all distributions with finite variance belong to the Gaussian basin of attraction. The Gaussian distribution is stable. Stable distributions play here the role of fixed points. We see that a regularity emerges for large sums telling us that all details of the original distribution except the mean and the variance get forgotten in the course of enlarging the number of terms in the sum. Distributions with infinite variance belong to the Lévy universality classes (or saying equivalently to the basin of attraction of the Lévy distributions) [11, 12].

One expects the large scale phenomena in economy to display a universal character because they result from a large number of events which are driven by laws of the same system and which contribute to the same statistics.

In this paper we shall take the latter approach. We shall be looking for general laws which describe large scale behavior of economical systems. We shall try to deduce them from assumptions as simple as possible, which define certain universality classes. Small refinements and perturbations are believed not to change the universality class of the large scale behavior. As an example, in the next section we shall concentrate on the issue of the wealth and income distribution. This issue, addressed already by Adam Smith, still stands in the central place in the macro-economical research.

4. Wealth and income distributions

As mentioned above, we argue that the laws governing distributions can be deduced from the mathematics of large numbers. A simple assumption about the nature of wealth fluctuations seems to capture properly the microscopical mechanism which in the large scale leads to the emergence of laws known for a long time from empirical studies in macro-economy. The first law, discovered by Pareto more than one hundred years ago [13], tells us that the wealth distribution of the richest part of the society is controlled by the power-law tail

$$dw\ p(w) \sim \frac{\alpha A^{\alpha} dw}{w^{1+\alpha}} \quad \text{for} \quad w \gg w_0 \,. \tag{1}$$

Here $p(w)dw$ stands for a probability that a randomly chosen member of the macro-economical system possesses the wealth between w and $w + dw$; w_0 has the meaning of a typical value of the individual's wealth in this system. The exponent α is called the Pareto index. Pareto himself suspected that there may exist an underlying mechanism which singles out a particular fixed value of this index. Today we know that it is not true. The value of the Pareto index α changes from macro-economy to macro-economy [14]. It also varies in time. The empirical estimates show that a value of the Pareto index in real macro-economical systems fluctuates around two.

It is worth discussing the consequences of the presence of the power-law tail in the probability distribution. An immediate consequence is that the probability that a random person from the richer part of the society is λ times richer than another person with wealth w

$$\frac{p(\lambda w)}{p(w)} \sim \lambda^{1+\alpha} \tag{2}$$

is independent of w. This distribution is scale-free, reflecting a certain self-similarity of the structure of the richest class. Actually the scale appears in the problem through the parameter w_0 which provides the lower cut-off above which $w \gg w_0$ the power-law part of the distribution sets in. The scale is provided by prices of elementary goods which one needs to function in the system, like for instance prices of houses, cars, *etc.* Being rich means to be far above this scale, to the degree that it does not matter how much the basic things cost.

Let us take a closer look at some values to gain the intuition about the consequences of the Pareto. For $\lambda = 10$ and $\alpha = 2$, the factor on the right hand side of (2) is 10^{-3}. Thus for $\alpha = 2$ the Pareto law predicts that the number of people ten times richer is roughly one thousand times smaller. The suppression factor is very sensitive to α. If the value of α moves towards unity, the suppression factor decreases, and for $\lambda = 10$ it is only 10^{-2}. In other words, in the macro-economy with a smaller value of α the tail of the distribution is fatter. This leaves more space for rich individuals. Thus one intuitively expects that for smaller α the macro-economy is more liberal. In a more restrictive macro-economical system the Pareto exponent α is larger and hence the richer population is suppressed.

The presence of heavy tails in empirical data is relatively easy to detect. One just observes cases lying far beyond the range suggested by standard estimators of the mean and width of the distribution. What is however difficult is to quantitatively estimate the values of the Pareto index. The reason for this is actually very simple. As follows from the discussion above, cases with a very large deviation from the mean are relatively rare — much more rare than those in the bulk of the distribution. Thus the statistics in the tail is very poor. The effect of small statistics is additionally amplified by the fact that for a given macro-economical system one can carry only one measurement of the wealth distribution. One thus has only one statistically independent sample. Secondly, the crossover between the bulk of the distribution coming from the lower and middle classes and the tail coming from the richest is smeared and therefore it is not entirely clear where the Pareto law sets in: the position of the termination point of the Pareto tail is not unique. This uncertainty introduces a bias to the estimators.

Moreover, gathering data about personal wealth and income is a delicate matter. It is technically very difficult, close to impossible, to collect the unbiased data, which would be free of personal, social or political factors.

Here we shall discuss only the difficulty related to poor statistics. Having the wealth distribution $p(w)dw$ one can easily estimate the probability that the wealth of a random member of the macro-economy exceeds a certain value W

$$P(W) = \int_W^\infty dw\ p(w)\,. \tag{3}$$

For the particular form of the power law (1) this probability can be calculated to be

$$P(W) \sim \left(\frac{A}{W}\right)^\alpha \quad \text{for} \quad W \gg w_0\,. \tag{4}$$

In the population of N people the number of individuals whose wealth exceeds W is roughly of the order $P(W)N$. Thus denoting the wealth of the richest by $W_{\max}$, one can estimate $P(W_{\max})N \approx 1$ and hence

$$W_{\max} \sim AN^{1/\alpha}\,. \tag{5}$$

A more involved analysis allows one to determine the distribution of wealth of the richest in the macro-economy with the power-law tail to be given by the Fréchet distribution [15]

$$d\omega\ p_F(\omega) = d\omega\,\frac{\alpha}{\omega^{1+\alpha}}\mathrm{e}^{-\omega^{-\alpha}} = d\mathrm{e}^{-\omega^{-\alpha}}\,, \tag{6}$$

where ω is a rescaled variable $\omega = W_{\max}/AN^{1/\alpha}$. The distribution of the maximal wealth inherits thus the power-law tail from the original wealth distribution $p(w)dw$. This means that in some realizations of the same macro-system the richest may be much richer that the richest in other realizations. As a consequence, the maximal wealth may undergo strong fluctuations and so may the whole empirical data points in the Pareto tail. This is an additional factor which makes the quantitative analysis of the Pareto tail in the macro-economical data difficult.

It is much easier to study empirically the distribution in the range of smaller wealths. The statistics is much better in this case since the poor and middle class sectors are more numerous. Also the income declarations are statistically more reliable. In effect, the flow of wealth is much easier to control. The statistics is thus less biased. Surprisingly the empirical law which governs this part of the income and wealth distributions was discovered only four decades after the Pareto law. It was discovered by Gibrat and named after him [16]. According to this law the wealth and income distributions for the lower and middle classes obey the log-normal law

$$dw\,p(w) = \frac{dw}{w}\frac{1}{\sqrt{2\pi\sigma^2}}\exp -\frac{\log^2 w/w_0}{2\sigma^2}\,. \tag{7}$$

The cumulative probability $P(W)$ that the wealth of a random member of the Gibrat macro-economy exceeds W is given by

$$P(W) = \int_W^\infty dw\, p(w) = \frac{1}{2}\mathrm{erfc}\left(\frac{\log w/w_0}{\sqrt{2}\sigma}\right) . \tag{8}$$

All moments of the Gibrat distribution are finite $\langle w^k \rangle = w_0^n \exp \sigma^2 n^2/2$. The parameter σ^2 gives a typical width of fluctuations of the order of magnitude of w around w_0. The values w which deviate from w_0 by few σ are strongly suppressed for the Gibrat distribution. Sometimes to distinguish between the Gibrat and Pareto distributions for large W one draws the cumulative distributions in the log-log plot [14]. The plot $\log P(W)$ versus $\log W$ has a parabolic shape for the Gibrat distribution when W goes to infinity, while the corresponding plot for the Pareto distribution is a straight line (see Fig. 1), This makes an enormous difference between the Pareto and Gibrat laws in the range of large wealths.

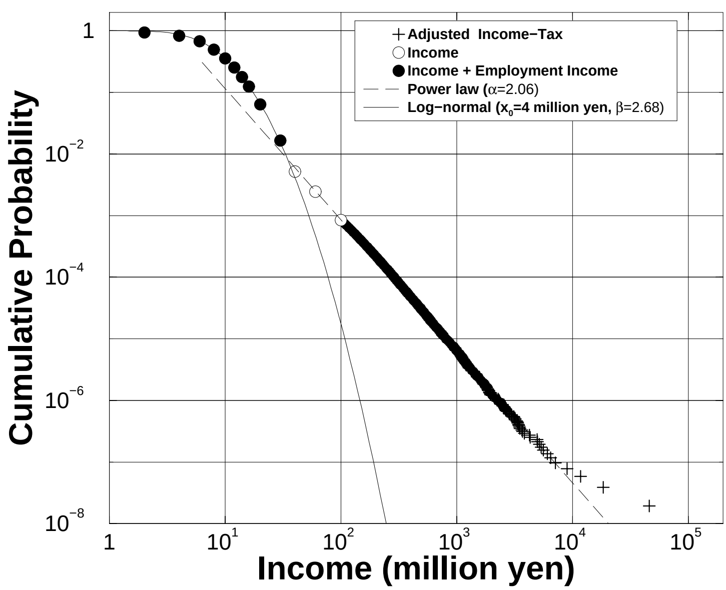

Fig. 1. The power law and the lognormal fits to the 1998 Japanese income data. The solid line represents the lognormal fit with $x_0 = 4$ million yen and $\beta = 2.68$. The straight dashed line represents the power law fit with $\alpha = 2.06$. Reprinted from the paper [14] with the kind permission of the author. The data sets presented in the figure come from three different sources. The corresponding data points are denoted by different symbols in the figure. See [14] for the detailed description.

Let us discuss mathematical mechanisms which may underlie the Gibrat and Pareto laws. Imagine a random individual in the system. Denote her or his wealth at a time t by w_t, and by w_{t+1} at a later time, separated by one unit ε of time. The wealth could increase or decrease by some factor λ_t [17]

$$w_{t+1} = \lambda_{t+1} w_t \,. \tag{9}$$

In general this factor may itself depend on many factors like which particular individual we picked up to look at, with whom she or he interacts in the system, what is his or her current financial situation *etc.* In the simplest approximation, which would be called in physics a mean-field approximation, we assume this factor to be a random number from the representative distribution which statistically characterizes the whole system. Further, the distribution is assumed to depend neither on time nor on the current wealth. The first assumption means that the process is stationary, and the second that it is linear in wealth. Although all this seems to be a crude approximation, the essential point is that it may be enough to capture the general properties of the related universality class. What seems to be significant in the assumption is that the variation of the wealth is described by a multiplicative rather than an additive process. Hopefully the large scale behavior which we want to deduce from this assumption is representative for a larger class including also more complex processes.

The assumed multiplicative nature of changes seems to well reflect the economical reality in which the primary objects which fluctuate are the rates of exchange understood in a broad sense: rates between goods, currencies, money, real estate *etc.* The prices of stocks also belong to this category. The change of wealth is proportional to the change of the exchange rate which implies the multiplicative nature of changes. In a diversified portfolios the situation is a little more complicated as we shall discuss later.

It is convenient to parameterize the changes of the factor scale λ_t by the quantity r_t which is related to λ_t as follows: $\lambda_t = \exp r_t$ or equivalently as

$$r_t = \log \lambda_t = \log w_{t+1} - \log w_t \,. \tag{10}$$

When the time unit ε between t and $t+1$ is small, the factor λ_t is close to unity. In this case it can be substituted by $\lambda_t = 1 + r_t + \ldots$ which gives the meaning of an instantaneous return to the quantity r_t. The parameterization $\lambda_t = \exp r_t$ automatically takes care of the positive definiteness of the scale factor λ_t: for r_t fluctuating in the range $(-\infty, +\infty)$, λ_t fluctuates in the range $(0, +\infty)$. In the simplest model the statistical information about the returns r_t is encoded in a probability distribution $\rho_\varepsilon(r)dr$ which characterizes the system. Successive returns r_t are assumed to be random numbers polled from the same distribution $\rho_\varepsilon(r)$. The wealth w_T and the return R_T after the time $\tau = T\varepsilon$ which elapsed from the moment $t = 0$, is given by the equation

$$R_T = \log \frac{w_T}{w_0} = \sum_{t=1}^{T} r_t \tag{11}$$

as can be directly deduced from the equation (9). If the mean and the variance

$$\begin{aligned} \bar{r} &= \langle r \rangle_\varepsilon \, , \\ \sigma^2 &= \langle (r - \bar{r})^2 \rangle_\varepsilon \end{aligned} \tag{12}$$

of the distribution $\rho_\varepsilon(r)$ are finite, the distribution of the return R_T approaches the normal law with the density

$$dR_T \; P_T(R_T) = dR_T \; \frac{1}{\sqrt{2\pi T\sigma^2}} \exp - \frac{(R_T - T\bar{r})^2}{2T\sigma^2} \tag{13}$$

as follows from the central limit theorem. We use the relation between the return R_T and the wealth w_T (11) to obtain the distribution of wealth

$$dw_T \; p_T(w_T) = \frac{dw_T}{w_T} \frac{1}{\sqrt{2\pi T\sigma^2}} \exp - \frac{\log^2 w_T / w_0 \mathrm{e}^{\bar{r}T}}{2T\sigma^2} \, . \tag{14}$$

This is the Gibrat law [16]. The typical wealth of individuals in the system changes in time as $w_0 \mathrm{e}^{\bar{r}T}$ and the range of the order of magnitude of fluctuations as $\sqrt{T}\sigma$. A few comments are in order. A typical wealth of the system increases in time if the return $\bar{r}$ is positive and decreases if the return is negative. It is constant for $\bar{r} = 0$. If one assumes it changes slowly (adiabatically) in time one can think of R as a sort of an averaged return. Thus in some periods the total wealth may grow and in some diminish.

The width of the wealth fluctuations which is given in the formula (14) by $2T\sigma^2$, grows in the model even if one assumes adiabatic changes: $\int^T dt \, \sigma^2(t)$. Thus the distribution gets flatter in time, suggesting that the differences of wealth may only grow with time: the spread between lower and upper end of middle class increases. This is what one very often observes if one surveys a macro-system over years, but not always. There are two reasons for this. Firstly, the simple model (9) seems to be inappropriate to describe the wealth evolution in turbulent periods like wars or crises. Secondly, the mean-field approximation (9) fails to reflect the conservation law for the total wealth in the macro-system. If one assumes that the total wealth W changes much slower in time than the wealths of individuals then in a short period one can treat the total wealth as constant in comparison with the wealths of individual w_i's. This means however that w_i's cannot fluctuate independently of each other as is assumed in the equation (9) because it would violate the conservation law

$$W = w_1 + w_2 + \ldots + w_N \tag{15}$$

which tells us that, unless the economy as a whole produces a new wealth, fluctuations of w_i are not independent [18]. This effect does not allow fluctuations of a typical order to grow as fast as the equation (14) would suggest. Later we shall discuss other consequences of the presence of the conservation law.

There is another economical factor which one should take into account when considering the process of wealth fluctuations (9). In each macroeconomy there is some threshold wealth which one has to posses to function in the system to fulfill minimal needs. In welfare economies it is provided by the social security system. Generally for each macro-economical system one can assume the existence of a positive cut-off $w_* > 0$ for the minimal wealth of each individual. It is easy to work out consequences of imposing the cut-off [19]

$$w > w_* \tag{16}$$

on the multiplicative process (9). The right-hand side of the equation for the return is also given by the sum of independent increments as in (11). What changes is the boundary condition: in the presence of a cut-off, R_T cannot be smaller than a certain value R_*. One can think of the equation (11) as of a random walk, which in the case of a cut-off has the lower barrier R_*. Microscopically the model with the barrier and without the barrier are identical. Thus one can check that both cases are described by an identical differential equation but with a different boundary condition. The equation reads

$$\frac{\partial P_T(R_T)}{\partial T} = -\bar{r}\frac{\partial P_T(R_T)}{\partial R_T} + \sigma^2 \frac{\partial^2 P_T(R_T)}{\partial R_T^2}\,. \tag{17}$$

By inspection one can check that indeed the probability distribution $P_T(R_T)$ (13) is a solution of the equation. In physics, the corresponding equation is called the Fokker–Planck equation. It describes a random walk with a drift. The two constants $\bar{r}$ and σ^2 in the equation correspond to the drift velocity and the diffusion constant and are related to the mean $\bar{r}$ and the variance σ^2 of the underlying distribution (12). In the presence of the cut-off in the boundary condition: $R_T > R_*$. the Fokker–Planck equation (17) possesses a stationary solution $P_T(R) = P(R)$

$$\frac{\partial P(R)}{\partial T} = 0 \tag{18}$$

if $\bar{r} < 0$. The equation obtained by comparing the right-hand side of (17) to zero can be solved with the normalization condition

$$\int_{R_*}^{\infty} P(R) dR = 1 \,. \tag{19}$$

The solution reads

$$P(R) = \alpha \exp -\alpha (R - R_*) \,, \tag{20}$$

where $\alpha = -\bar{r}/\sigma^2 > 0$. Substituting the return $\bar{r}$ by $w = w_0 \mathrm{e}^R$ (11) one eventually obtains the stationary distribution for wealth [19]

$$p(w)dw = \frac{\alpha w_*^\alpha}{w^\alpha} \frac{dw}{w} \,. \tag{21}$$

Notice that it is independent of w_0 which disappears from the solution. This is the Pareto law [13]. When the drift $\bar{r}$ is positive the exponent α is negative, the normalization condition (19) cannot be fulfilled. There is no stationary solution. For positive α the distribution flows with time and approaches the log-normal law (14) of the Gibrat universality class [16]. In this case the traces of the lower limit gradually disappear due to the positive drift which makes the bulk of the distribution depart from the lower cut-off. Now imagine that the drift changes slowly in time taking sometimes positive and sometimes negative values. In this case the system oscillates between the Gibrat and Pareto universality classes. For a finite time of the system evolution it may effectively lead to a mixed Pareto–Gibrat properties of the distribution, being in accordance with empirical observations [14].

What is counter-intuitive in this picture at the first glance is that the distribution of average returns $\rho_\varepsilon(r)$ generates the Pareto tail in the outcoming distribution of wealth when the drift $\bar{r}$ is negative. We see then that power-law tails occur in the wealth distribution when the system on the average generates negative returns. Negative returns mean that people loose wealth. Thus, paradoxically, when most of the people get poorer some get extremely rich, populating the Pareto tail. We shall see this effect more transparently below when discussing a constraint macro-economy.

To summarize this part of the discussion, the theory of large numbers explains very well the observed empirical data. Fluctuations in the empirical data may be large due to the fact that the empirical histograms are based on single measurements. Fluctuations may be particularly large in the tail of the distribution where there are only few counts in the empirical histograms and where the wealth fluctuations may be large due to the fat tails (6).

5. Wealth condensation

One of the implications of the mean-field approximation (9) is that the total wealth of the system might fluctuate with the amplitude proportional to the amplitude of individual changes and the square root of the number of individuals, or with a higher power if the fat tail properties become important. In reality the total wealth of the macro-system alternates slower in time and does not undergo such fluctuations. Therefore it is natural to introduce another time scale for changes of the total wealth than for changes of individual wealths. This leads to the constraint of the type (15) in which the value W on the left hand side changes much slower than w_i's on the right-hand side. This means that the flow of the wealth between individuals within the system is much faster than the process of change of the total wealth. Thus, if one considers changes of w_i's in a short time the constraint (15) means that w_i's cannot be treated as completely independent stochastic variables. In particular if an individual becomes very rich, amassing a substantial part of the total wealth W accumulated in the macro-economical system, this happens at a price of making others poorer. It is instructive to analyze consequences resulting from the constraint. We shall do this in the following way. In statistical mechanics of quasi-stationary systems one approximates averages over time by averages over a statistical ensemble. We shall use this approach here to represent fluctuations of the partition of wealth as a sum over all states in the ensemble of wealth partitions with the micro-canonical partition function

$$Z(W,N) \;=\; \sum_{\{w_i\geq 0\}} \prod_i p(w_i)\,\delta\left(W-\sum_{i=1}^{N} w_i\right) . \tag{22}$$

The total wealth W (15) is distributed among N individuals. This model is very close in spirit to the mean-field approximation discussed above since it assumes almost entire factorization of the probability into independent probabilities $p(w_i)$ of individuals. One could, of course, introduce interactions between different values w_i and w_j but as discussed above the mean field arguments are good enough to explain empirical data within the accuracy provided by single observations. We use here the strategy of not introducing refinements which are not necessary. The full factorization is weakly violated by the wealth conservation. The individual wealths are bounded from below $w_i > w_*$. For technical reasons it is convenient to consider integer valued w_i's. From the economical point of view this means that there exists a minimal indivisable unit in which one expresses wealth as for example the monetary unit used in the country. The only thing we shall assume about the probabilities $p(w)$, following the previous section, is that they possess a

Pareto tail (1). As will become clear, the details concerning the exact shape of the probability distribution are irrelevant for the universal large scale effects of wealth condensation. The only important parameters of the model are the value of the Pareto exponent α and the mean of the distribution

$$w_{cr} = \sum wp(w)\,. \tag{23}$$

The mean is finite for $\alpha > 1$ and infinite otherwise. In a thermalized economy where $p(w)$ is constant for a long time this average $w_{\rm cr}$ adjusts itself to the average *per capita*

$$\bar{w} = \frac{W}{N}\,, \tag{24}$$

and one has

$$w_{\rm cr} = \bar{w}\,. \tag{25}$$

The mean of the distribution $w_{\rm cr}$ may however depart from w as a result of some changes which the system may undergo. For example it may happen that for some reasons a thermalized stable economy will start to develop, increasing the total wealth W. Alternatively the economy may quickly go down decreasing the total wealth W. The question arises how the system adjusts to the new situation in which $\bar{w} \neq w_{\rm cr}$: how it redistributes the surplus if $\bar{w} > w_{\rm cr}$ or covers the deficit if $\bar{w} < w_{\rm cr}$. A potential discrepancy between $w_{\rm cr}$ and $\bar{w}$ may also occur as a result of some structural changes of the macro-economical framework, like taxation laws, employee rights *etc.*, which may lead to a change of the distribution $p(w)$ yet before the total wealth of the economy changes.

We shall try to answer this question by investigating the response of the system defined by (22). This model can be solved analytically [18, 20]. The response of the system can be determined from the shape of the effective probability distribution defined as an average over all partitions weighted by the partition function (22)

$$\widehat{p}(w) = \frac{1}{N}\left\langle \sum_i^N \delta(w_i - w) \right\rangle . \tag{26}$$

One can show that when $w_{\rm cr} = w_*$, there is a perfect matching and the effective probability

$$\widehat{p}(w) = p(w)\,. \tag{27}$$

However, when the wealth *per capita* exceeds the critical value $\bar{w} > w_{\rm cr}$ or is smaller than the critical value: $\bar{w} < w_{\rm cr}$ the system enters one of two different phases which we call the surplus phase or the deficit phase respectively.

In the surplus phase the effective probability distribution $\widehat{p}(w)$ nonuniformly approaches $p(w)$ creating a peak at the large values. For large systems $N \to \infty$ the effective probability density may be approximated by

$$\widehat{p}(w) = p(w) + \frac{1}{N}\delta(w - N\Delta w)\,, \tag{28}$$

where the second term is the Dirac delta localized at the value proportional to the system size N. The proportionality coefficient $\Delta w = \bar{w} - w_{\rm cr}$ is a deviation of the average wealth from the critical value. The coefficient $1/N$ in front of the delta function means that the probability related to the peak is $1/N$, or equivalently that the contribution comes from one out of N individuals. The wealth of this individual $w_{\rm max} = N\Delta w$ grows with the system size. He or she takes a finite fraction of the whole wealth. This effect is similar to the Bose–Einstein condensation for which a finite fraction of all particles is in the ground state. The difference between the two condensations is that in the Bose–Einstein condensation the ground state is favored by the energy, while here all individuals are identical and therefore they have *a priori* the same chance that the wealth will condense in their pocket. The condensation results from a spontaneous symmetry breaking mechanism which breaks the permutation symmetry of N individuals of the original model. In reality, of course, the position of individuals in the macro-system is not identical. This may further enhance the effect of condensation observed already in the model where those differences are neglected.

In the deficit phase ($\bar{w} < w_{\rm cr}$) the effective probability distribution $\widehat{p}(w)$ is given by

$$\widehat{p}(w) = c\mathrm{e}^{-\mu w} p(w)\,, \tag{29}$$

where μ is some positive function which depends on $\Delta w = \bar{w} - w_{\rm cr}$. The factor c is a normalization constant. The exponent μ vanishes in the limit $\Delta w \to 0^-$. We see that when the system enters the deficit phase a suppression of the fat tails occurs: these are the richest who first pay for the deficit.

The order of the transition between the deficit and surplus phases depends on α. The transition is of the third or higher order [20]. The transition becomes weaker when α approaches one or infinity. The critical value $w_{\rm cr}$ being the average of the distribution depends on the whole distribution but it is very sensitive to the tails: the fatter the tail the larger the critical value $w_{\rm cr}$. On the other hand, when the critical value $w_{\rm cr}$ is larger it is more difficult to enter the surplus phase $\bar{w} > w_{\rm cr}$ because the wealth *per capita* must exceed this critical value. This may happen in a very rich society. In the limiting case $\alpha = 1$, the critical value $w_{\rm cr}$ is infinite and the system never enters the surplus phase.

When the critical value $w_{\rm cr}$ becomes smaller it is easier for the wealth *per capita* $\bar{w}$ to exceed $w_{\rm cr}$ and to enter the surplus phase where the system has problems to redistribute the wealth of the richest. If it happens in a rich society this means that one individual creates a large fortune and the system is not able to redistribute it quickly or at least that such a redistribution is not favored statistically. The wealth condensation becomes however natural then. It is not a shame to be rich in a rich society as says Confucius.

Paradoxically, the condensation may also take place in a restrictive macro-economy. Assume that the total wealth of a poor society is fixed. Additionally imagine that the system becomes more restrictive, which results in the increase of the Pareto index and the decrease of the critical value $w_{\rm cr}$. If this value becomes smaller than the wealth *per capita* $\bar{w}$, which is fixed, the system enters the surplus phase. The wealth condensates in one pocket as a result of the surplus anomaly. Some of the richest become richer and other poorer. This clearly reveals the danger of corruption of restrictive poor macro-economies.

The main conclusion of this section is that large number theory also on the elementary level explains potential danger of statistical instability, which in the case of restrictive macro-economy may be related to the phenomenon of corruption. One can avoid this danger by making the macro-economical rules more liberal [18, 21]. For completeness let us mention that one can consider a macro-economy in contact with the external world [21]. In the language of statistical physics this corresponds to the model defined by the canonical version of the partition function (22). In addition to what we discussed here, in the canonical version of the model one can observe statistical effects of the attraction of the external wealth to the macro-economy, or the withdrawal of the internal one, depending on whether the macro-economical rules inside or outside are more liberal.

6. Modeling a financial market

Let us now turn to the mesoscopic scale and discuss financial markets. Financial market is a part of the econosystem which is easiest to quantify. We shall use a simplified picture of this market in which the only objects are the prices of assets, asset being the name commonly used to describe a financial instrument, which can be bought or sold, like currencies, bonds, shares *etc.* In the following we shall understand assets solely as shares. Asset (or stock) prices $S_i(t)$ are functions of time. A typical time step ε, when the price is changed is as short as few seconds. It will be the dynamics of price changes, which we shall discuss in this chapter.

In the analogous way as the quantity r_t (10) of the chapter about macro-economy we define the instantaneous returns, which we shall alternatively call relative price changes of the asset in the period from τ to $\tau + \varepsilon$

$$x_i(\tau;\varepsilon) = \log S_i(\tau + \varepsilon) - \log S_i(\tau). \tag{30}$$

Again the crucial ingredient of this analysis is the assumption about the multiplicative nature of price changes. The definition of return is independent of the unit in which the price is given and seems the best to capture the essential properties of the price system. Return $x_i(\tau;\varepsilon)$ can be any positive real number. Obviously the return over a larger time interval is a sum of all changes over its subintervals

$$x_i(\tau;\varepsilon_1 + \varepsilon_2) = x_i(t;\varepsilon_1) + x_i(t + \varepsilon_1;\varepsilon_2)\,. \tag{31}$$

Financial databases contain huge number of time series of asset prices, sampled at various frequencies. Phenomenologically one can observe that prices behave in a random way: relative price changes $x_i(t,\varepsilon)$ fluctuate. The empirically measured time correlations show that these fluctuations have a rather short autocorrelation time, typically of the order of several minutes. Longer autocorrelation times were observed for the absolute values of fluctuations.

If the frequency of sampling ε is chosen larger than the autocorrelation time ε_0, corresponding price changes can be viewed as independent random variables. The simplest assumption one can make is the assumption of stationarity: $x_{it} = x_i(\tau = t * \varepsilon_0; \varepsilon_0)$, where t is an integer, can be interpreted as random numbers generated with the same random number generator, independent of time. One can derive surprisingly strong predictions based on this simple assumption, using very general properties of this random number generator. Let us assume that the generator is characterized by the normalized probability distribution function (pdf) $P(x)$, with a characteristic function $\hat{P}(z)$ defined by the Fourier transform

$$\hat{P}(z) = \int_{-\infty}^{\infty} dx\, P(x) \mathrm{e}^{ixz}\,. \tag{32}$$

Define a function $\hat{R}(z) = \log \hat{P}(z)$. It is straightforward to see that the sum

$$X_n = \sum_{i=1}^{n} x_i \tag{33}$$

of independent random numbers distributed with P is again a random number with a distribution P_n being an n-fold convolution of $P(x)$. In consequence, $\hat{P}_n(z) = \hat{P}^n(z)$ and $\hat{R}_n(z) = n\hat{R}(z)$ where $\hat{R}_n(z) = \log P_n(z)$.

A special role is played by stable distributions, which have the property that the probability distribution of the sum P_n can be mapped into the original distribution by a linear change of the argument

$$dx\, P_n(x) = d(a_n x + b_n)\ P(a_n x + b_n)\,, \tag{34}$$

where a_n and b_n are suitable parameters. Saying differently, the stable distributions are self-similar under the convolution which means that the shape of pdf is preserved up to a scale factor and shift. The condition (34) can be rewritten as a condition for $\hat{R}(z)$ in the form

$$\hat{R}(z) = n\hat{R}(a_n z) + ib_n z\,. \tag{35}$$

A class of stable distributions is limited. The best known is the Gaussian distribution, for which

$$\hat{R}(z) = -\gamma z^2 + i\delta z\,, \tag{36}$$

where $\delta = \langle x \rangle$ and $\gamma = \frac{1}{2}\langle (x-\delta)^2 \rangle$. One can think of the straightforward generalizations of the last formula

$$\hat{R}(z) = -\gamma |z|^\alpha + i\delta z\,. \tag{37}$$

One can check that they indeed fulfill the stability condition (35). However only for $0 < \alpha \leq 2$ the corresponding characteristic function $\hat{P}(z) = \exp \hat{R}(z)$ leads after inverting the Fourier transform (32) to a positive definite and normalizable function $P(x)$, which only in this case can be interpreted as a probability distribution.

It is a special case of Lévy distributions characterized by the index $0 < \alpha \leq 2$ which can be further generalized to asymmetric functions. The most general form of $\hat{R}(z)$ can be shown ([12]) to be

$$\begin{aligned}
\hat{R}(z) &= -\gamma |z|^\alpha (1 + i\beta \tan(\frac{\pi\alpha}{2}) \mathrm{sign(z)}) + \mathrm{i}\delta\mathrm{z}\,, \quad \alpha \neq 1, \\
R(z) &= -\gamma |z| (1 + i\beta \frac{2}{\pi} \mathrm{sign(z)} \ln(\gamma|\mathrm{z}|) + \mathrm{i}\delta\mathrm{z}\,, \qquad \alpha = 1\,.
\end{aligned} \tag{38}$$

The asymmetry parameter β takes values in the range $[-1, 1]$. For $\alpha = 2$ we have the Gaussian distribution, the asymmetry plays no role in this case as one can see from the formula since the β-dependent term drops. Indeed the Gaussian distribution has only a symmetric realization.

One can easily check that for stable distributions the self-similarity parameter scales as $a_n = n^{-1/\alpha}$. Although $\hat{R}(z)$ is given explicitly, only in very few cases the corresponding pdf $P(x)$ is expressible in terms of simple analytical expressions. For $x \to \pm\infty$ and $\alpha < 2$

$$dx\ P(x) \propto dx\ \frac{A_{\pm}^{\alpha}}{|x|^{1+\alpha}} \tag{39}$$

and the asymmetry parameter

$$\beta = \frac{A_{+}^{\alpha} - A_{-}^{\alpha}}{A_{+}^{\alpha} + A_{-}^{\alpha}}\,. \tag{40}$$

This behavior means that Levy distributions are very different from the Gaussian distribution. For $1 < \alpha < 2$ only the first moment $\langle x \rangle$ is defined, all higher moments diverge. For $0 < \alpha \leq 1$ even the first moment diverges.

The importance of the stable distributions is demonstrated by the central limit theorem. Suppose we start with an arbitrary distribution $P(x)$, not necessarily stable. Performing the n-fold convolution of this distribution, in the limit $n \to \infty$ we necessarily end up with one of the stable distributions described above. Typically if $P(x)$ has the asymptotic behavior like (39) for arbitrary $\alpha > 0$ we shall obtain the Lévy distribution if $\alpha < 2$ or Gaussian distribution if $\alpha \geq 2$. As a consequence, if our sampling frequency in the price list is large, say one day, we may expect to a good approximation the relative price changes measured with this frequency to be random numbers obtained from one of the stable distributions.

If the idealized assumption of stationarity holds, we can represent the history of the financial market as a matrix x_{it}, with the times t measured in intervals of the sampling unit ε, corresponding to one day. In this way we lose information about the short time scale fluctuations, but we may expect that for each i the entries x_{it} will represent a sequence of random numbers drawn from the same stable distribution. It is, of course, a crucial question, which stable distribution is realized in practice. We may deduce the properties of this distribution studying a finite sample of x_{it} on a time window T, consisting of many days (say one month).

7. Gaussian world

Simplest models assume the distribution to be Gaussian. If this is the case, it can be characterized by two parameters: the shift $\delta_i = \langle x_i \rangle$ and the variance $\sigma_i^2 = 2\gamma_i^2 = \langle (x_i - \delta_i)^2 \rangle$. Both parameters can be easily determined empirically from the data on a time window T by the following estimators

$$\tilde{\delta}_i = \frac{1}{T}\sum_t^T x_{it}\,,$$
$$\tilde{\sigma}_i^2 = \frac{1}{T}\sum_t^T \left(x_{it} - \tilde{\delta}_i\right)^2\,. \tag{41}$$

Obviously these numbers would be subject to a statistical error due to the finiteness of the time window. The values of the estimators converge to the exact values $\tilde{\delta}_i \to \delta_i$, $\tilde{\sigma}^2 \to \sigma^2$ only in the limit $T \to \infty$. In the Gaussian world the evolution of the price (or in our case the logarithm of the price) is just a diffusion process with a drift. Knowledge of the parameters of the Gaussian distribution describing price changes in one day can be used to predict the distribution of the relative price changes on a longer time scales. These will again be given by the Gaussian distribution (due to its stability), but with rescaled variance and shift.

The market consists of many assets (say $i = 1, \ldots, N$). The number of assets in the market is typically a large number (the well-known Standard and Poor index SP500 quotes prices of 500 companies). The market reality is more complex than suggested by the model of independent stationary Gaussian returns discussed above.

The first problem is that the market reality is not stationary. One cannot expect that the prices will fluctuate according to the same law over twenty years. In this period many things may happen which may affect performances of individual companies. One has to weaken the stationarity assumption and to substitute it by a sort of quasi-stationarity. In practice this means that the time window T used in the estimators (41) should be limited and so should be the future time in which one uses the value of the estimators. Practitioners [22] introduce further improvements to the estimators by weighting past events with weight, which gradually decreases with time. Here we shall not discuss this issue further, assuming in what follows a quasi-stationarity.

The second correction which one has to introduce to the model discussed above is that in reality the prices of individual stocks are mutually correlated as a result of the existence of the network of inter-company dependencies. Indeed even by a purely statistical analysis of the correlation matrix [23] one can observe and determine the statistical correlations of price fluctuations of stock prices of companies from the same industrial sectors. Of course, inter-sector correlations also exist. Further, the stock market is not a closed system. The total capital invested in the market may shift between the stock market and other investments like for instance the real estate. This leads to the observed periods of flows of the capital into the stock market or out of the stock market. As a result the prices may go up or down, depending on whether the market attracts are repulses the capital. This is closely related to the effect known in sociology as herding. The effect of herding is also clearly seen in the statistical analysis of the matrix which shows the occurrence of an eigenvalue in the spectrum of the correlation matrix which is significantly larger than all other. The corresponding eigenvector is interpreted as a vector of correlations of changes of individual prices to the main market tendencies which are often referred to as β-parameters after the Capital Asset Pricing Model [24]. We shall come back to this issue later. This discussion shows that a realistic approach should allow to model the inter-company correlations.

A logical generalization of the Gaussian model described above is the model of correlated asset fluctuations generated from some multidimensional Gaussian distribution. The probability of generating a vector of returns x_{it}, $i = 1, \ldots, N$ at some time t is

$$\prod_i dx_i \; P(x_1, x_2, \ldots, x_N) \sim \prod_i dx_i \; \exp -\frac{1}{2} \sum_{ij} (x_i - \delta_i) C^{-1}_{ij} (x_j - \delta_j) \, . \quad (42)$$

The properties of this generator can be assumed, as discussed before, to be constant in the period of time for which the shifts δ_i and the correlation matrix C_{ij} are estimated (quasi-stationarity)

$$\tilde{C}_{ij} = \frac{1}{T} \sum_t^T \left(x_{it} - \tilde{\delta}_i \right) \left(x_{jt} - \tilde{\delta}_j \right) \, . \quad (43)$$

The correlations may be both positive or negative. Knowledge of the correlation matrix C_{ij} is crucial in financial engineering, and in the construction of "optimal portfolios" following the Markowitz recipe [25]. The main idea in the construction of "optimal portfolios" is to reduce the risk by diversification. The portfolio is constructed by dividing the total invested capital into fractions p_i which are held in different assets: $\sum_i^N p_i = 1$. The evolution of the return of the portfolio is now given by the stochastic linearized variable $X(\vec{p}) = \sum_i^N p_i x_i$, which produces an instantaneous return $X(\vec{p})_t = \sum_i^N p_i x_{it}$ at time t. The quintessence of the Markowitz idea is to minimize the fluctuations of the random variable $X(\vec{p})$ at a given expected return by optimally choosing the p_i's. The risk is measured by the variance of the stochastic variable $X(\vec{p})$

$$\Sigma^2 = \sum_{ij} p_i C_{ij} p_j \, . \quad (44)$$

Clearly, the information encoded in C_{ij} is crucial for the appropriate choice of p_i's. Intuitively, a diversification makes only sense when one diversifies between independent components and one does not gain too much if one redistributes capital between strongly correlated assets which make collective moves on the market.

The covariance matrix contains this precious information about the independent components. The spectrum of eigenvalues tells us about the strength of fluctuations of individual components, and the corresponding eigenvectors about the participation of different assets in this independent components.

The fundamental question which arises is how good is the estimate $\tilde{C}_{ij}$ given by the equation (42) of the underlying covariance matrix (43), in particular how good is the risk estimate

$$\tilde{\Sigma}^2 = \sum_{ij} \tilde{p}_i \tilde{C}_{ij} \tilde{p}_j \tag{45}$$

of risk (44). Although the question looks simple, the answer is not immediate. One can quantify the answer with the help of the random matrix theory. We shall sketch some ideas which one uses in this theory in the next sections. Here we shall only quote the results.

To start with, consider the simplest case of completely uncorrelated assets which are equally risky. Further, we assume that they all fluctuate symmetrically around zero $\delta_i = 0$ with the same variance $\sigma_i = 1$. The correlation matrix reads in this case $C_{ij} = \delta_{ij}$. The spectrum of eigenvalues of this matrix is $\rho(\lambda) = \delta(\lambda - 1)$ which means that it is entirely localized at unity. For the ideal diversification $p_i = 1/N$ the risk measured by Σ (44) is $\Sigma = 1/\sqrt{N}$. What shall we obtain if we use in this case the estimate $\tilde{C}_{ij}$ instead?

The random matrix theory as we shall see later gives a definite answer. The first observation is that the quality of the estimator (43) depends on the time T for which we could measure the correlation matrix. The longer time T, the better quality of the information which can be read of from $\tilde{C}_{ij}$: all diagonal elements should approach unity, and off-diagonal ones zero. In reality, as we mentioned, one never has an infinite time T at ones disposal. Geometry of the data matrix $x_{it}, i = 1, \ldots, N, t = 1, \ldots, T$ is finite. It is just a rectangular matrix with the asymmetry parameter $a = N/T < 1$. Such matrices form an ensemble called the Wishart ensemble [26]. The case $a > 1$ requires a special treatment and is not relevant in this case. For a larger than zero we expect that the spectrum of the matrix $\tilde{C}$ will be smeared in comparison with the delta spectrum of C. Indeed, as we shall see in the next sections using the methods of random matrix theory one finds

$$\tilde{\rho}(\lambda) = \frac{1}{2\pi a} \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\lambda} \tag{46}$$

with $\lambda_\pm = (1 \pm \sqrt{a})^2$. Only in the limit $a \to 0$ we get the spectrum peaked at unity. This spectrum is calculated from the random matrix theory for Wishart matrices as we discuss later.

Although the empirical matrix x_{it} is obtained from a single realization of a random matrix from the Wishart ensemble, its spectral properties are in general very similar to those described above. This is due to the self-averaging property of large matrices.

We can also explicitly find the estimate of risk (45). In doing this one should take into account that the optimal choice of probabilities $\tilde{p}_i$ which minimizes the risk $\tilde{\Sigma}$ depends on $\tilde{C}_{ij}$

$$\tilde{p}_i = \frac{\sum_j^N \tilde{C}_{ij}^{-1}}{\sum_{jk}^N \tilde{C}_{jk}^{-1}} \,. \tag{47}$$

Inserting this solution into the formula (44) we can calculate the minimal value of the estimated risk

$$\Sigma^2 = \frac{1}{N} \frac{\int d\lambda \; \rho(\lambda) \lambda^{-2}}{\left(\int d\lambda \; \rho(\lambda) \lambda^{-1}\right)^2} \tag{48}$$

which eventually gives

$$\Sigma = \frac{1}{\sqrt{N}} \frac{1}{\sqrt{1-a}} \,. \tag{49}$$

The exact relation between the spectrum of C_{ij} and $\tilde{C}_{ij}$ can be obtained in the limit $N, T \to \infty$, $a = N/T$ fixed. Again we skip here the derivation and quote only the result. A simple formula can be obtained for the Green's function

$$\tilde{\mathcal{G}}(z) = \frac{1}{N} \left\langle \mathrm{Tr} \frac{1}{z - \tilde{C}^{-1}} \right\rangle_{\mathrm{W}} \tag{50}$$

which relates it to its counterpart, in the $T \to \infty$ limit:

$$\mathcal{G}(t) = \frac{1}{N} \mathrm{Tr} \frac{1}{t - C^{-1}} \,. \tag{51}$$

The subscript W means the average over the Wishart ensemble (42). One finds [27]

$$z \tilde{\mathcal{G}}(z) = t \mathcal{G}(t) \,, \tag{52}$$

here z and t are related to each other as:

$$z \;=\; t(1 - a + at\mathcal{G}(t)) \,. \tag{53}$$

These two relations are in fact a concise way to write infinitely many relations between the moments of matrices C_{ij} and $\tilde{C}_{ij}$. Let

$$\begin{aligned} c_k &= \frac{1}{N}\mathrm{Tr}C^{-k}, \\ \tilde{c}_k &= \frac{1}{N}\langle \mathrm{Tr}\tilde{C}^{-k}\rangle_{\mathrm{W}}. \end{aligned} \tag{54}$$

On finds

$$\begin{aligned} \tilde{c}_1 &= c_1\,, \\ \tilde{c}_2 &= c_2 + ac_1^2\,, \\ \tilde{c}_3 &= c_3 + 3ac_1c_2 + a^2c_1^3 \\ \ldots \end{aligned} \tag{55}$$

At the end of this section let us come to the problem of the large eigenvalues observed in the spectra of eigenvalues of the financial covariance matrices $\tilde{C}_{ij}$. The spectra consist typically of the random part (46) which is universal as discussed above and few large eigenvalues. Among them one is particularly large. Its value is roughly speaking proportional to the number N of the assets in the market. The corresponding eigenvector contains the contribution from almost all N companies on the market. This eigenvector is called the "market". One can relatively easily understand the source of the appearance of the market in the spectrum in terms of the herding phenomena which we shortly signaled before. Imagine that there is a collective behavior of investors on the market which can be driven by some sociological factors. Mathematically such a collective movement may be in the simplest version modeled by the coupling of the individual prices to some common background, for example by substituting the generator of the vector of prices (42) by a new generator of the form

$$\prod_i dx_i\; P(\vec{x}) \sim \prod_i dx_i\; \exp -\frac{1}{2}\sum_{ij}(x_i - \beta_i m_t)C^{-1}_{ij}(x_j - \beta_j m_t)\,, \tag{56}$$

where β_i's are some constants, and m_t is a common random variable describing the market movements. This is the basic idea underlying the CAPM model [24] mentioned above. One can check that the largest eigenvalue disappears from the spectrum leaving the remaining part intact if at each t one subtracts from each return the market background represented as the instantaneous average over all companies.

The other large eigenvalues can be attributed to the real strong correlations between companies. The analysis of the eigenvectors allows to divide the market into highly correlated clusters, usually corresponding to companies from the same industrial sector. For example, one can see that the gold companies form a cluster which is anticorrelated to the market.

An example of the eigenvalue spectrum of the empirical covariance matrix $\tilde{C}$ (43), is shown in figure 2. It is calculated for the SP500 for the period.

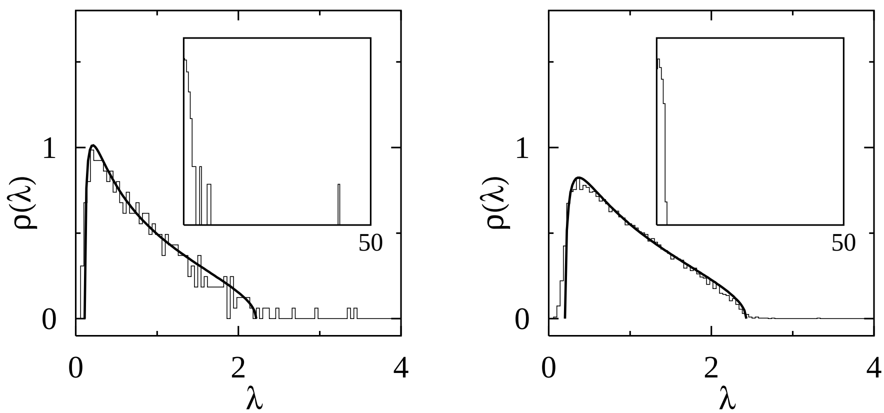

Fig. 2. The spectrum of the financial covariance matrix for the daily SP500 for $N = 406$ stocks and for $T = 1309$ days from 01.01.1991 to 06.03.1996. The left plot represents the spectrum of the covariance matrix for the normalized returns in the natural time ordering; the right one for the normalized return in the reshuffled ordering. The reshuffling destroys correlations between entries of the matrix $\tilde{C}_{ij}$. The random matrix prediction is plotted in solid line. The large eigenvalues lying outside the random matrix spectrum in the left figure disappear from the spectrum for reshuffled data shown in the right.

The data matrix x_{it} has the size $N = 406$ and $T = 1308$ which corresponds to the asymmetry parameter $a = 0.31$. In the spectral analysis of the empirical matrix one usually unifies the scale of return fluctuations of different assets by normalizing them by individual variances σ_i (41): $x_{it} \to x_{it}/\sigma_i$ which for each asset produces fluctuations of unit width. For such normalized fluctuations the formula (46) tells us that that the random part of the spectrum of the covariance matrix should be concentrated between 0.20 and 2.43. We clearly see the presence of larger eigenvalues in the spectrum presented in the left plot in figure 3, which as mentioned, can be attributed to the inter-asset correlations. However, the large eigenvalues disappear when one removes the inter-asset correlation. One can do this by random reshuffling of the time ordering of returns for each individual asset. A random reshuffling does not change the content of information stored in each separate row of data but it destroys the statistical information about the correlations between different rows. Indeed as is shown on the right plot in the figure 2, the larger eigenvalues disappear from the spectrum. The resulting spectrum of the covariance matrix of such reshuffled data is perfectly described by the random matrix formula (41).

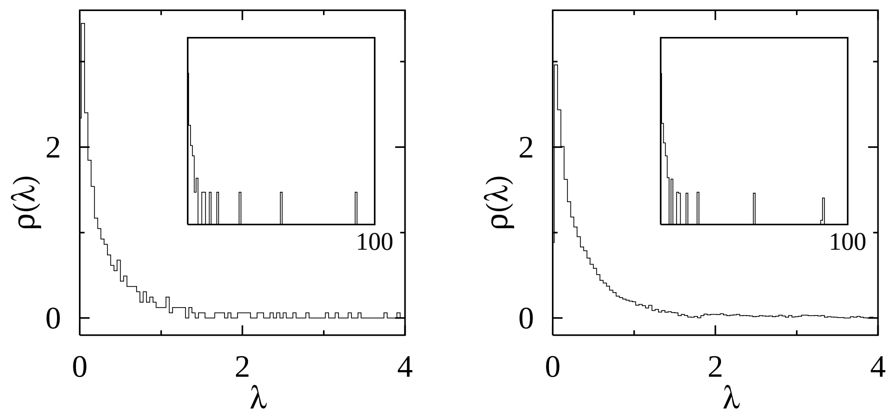

Fig. 3. The same as in Fig. 2 but for for the nonnormalized returns: the left figure for the data in the natural time ordering and the right for the reshuffled ordering. In this case reshuffling does not remove the large eigenvalues from the spectrum signaling the presence of non-Gaussian effects in the return statistics.

The above mentioned normalization of return fluctuation $x_{it} \to x_{it}/\sigma_i$ is natural if fluctuations belong to the Gaussian universality class. If the underlying distributions governing the return fluctuations have fat tails, this normalization is not appropriate since the variance of the distribution does not exist. In this case the use of the normalization $x_{it} \to x_{it}/\sigma_i$ artificially forces the resulting rescaled quantities to behave as if they belonged to the Gaussian universality class of distributions with the unit variance. This introduces a bias to the analysis in case of non-Gaussian statistics. Indeed, if one skips this normalization one observes that covariance matrices for the original SP500 data as well as for the reshuffled SP500 data both possess large eigenvalues in the spectra (see Fig. 3). What is the reason that the reshuffling does not remove them? Is the random matrix prediction (46) wrong? The random matrix prediction is not wrong of course but is valid only for matrices from the Gaussian ensemble. The removal of the normalization condition revealed the nature of the randomness of return fluctuations which contain fat tails. As we shall discuss later, the spectra of Lévy random matrices contain fat tails which means that even a completely random matrix may contain large eigenvalues. The main conclusion of this discussion is that the large eigenvalues in the spectrum of financial covariances stem both from inter-asset correlations and from the Lévy statistics of return fluctuations and therefore a proper statistical analysis of financial data, in principle of the eigenvalue content, would require the new Lévy methodology.

8. Lévy world

Indeed on closer inspection one finds that individual price fluctuations have rather heavy tails. Empirically one can fit their distribution, at least in the asymptotic limit, as a power low of the form (39) with the power $\alpha \approx 1.5 \ldots 1.8$. Following our earlier discussion this means that one should rather consider stable Lévy distributions when discussing the distribution of relative price fluctuations, for the sampling frequency of the order of one day or more.

Models of this type were proposed in the literature. For a single asset i one should in principle determine four parameters (index α_i, asymmetry β_i, range γ_i and mean δ_i), which characterize it's distribution $P^{\alpha_i}_{\beta_i \gamma_i \delta_i}(x_i)$. In practice such a determination is numerically very difficult, one can assume a value of α to be some fixed number in the range given above. Similarly one can assume the asymmetry $\beta_i = 0$ (numerically it is very difficult to distinguish the effect of asymmetry from that of a non-zero δ_i). Even with these assumptions the determination of the remaining two parameters is more difficult, because for Lévy distributions the second moment diverges.

A typical time evolution of the logarithm of price will in the Lévy world be very different than in the Gaussian world. One observes from time to time very large jumps, called Lévy flights. The practical consequence is a relatively large probability of extreme events. Since these events are responsible for possible large losses on financial market, the correct determination of the risk cannot be made if their probability is underestimated. Each investment on a financial market is risky and investors must know rather accurately the probabilities of possible gains and losses.

A Lévy market means that we should describe a multidimensional, possibly correlated, Lévy random number generator. A natural assumption, as explained above is a common value of the index α for all market components. Correlations mean that for a given moment t_j, fluctuations x_{it} can be decomposed as linear combinations of *independent* Lévy components $\Lambda_k, \quad k = 1, N$, with a factorizable probability distribution

$$P(\{\Lambda_i\}) = \prod_i P^{\alpha}_{B_i \Delta_i}(\Lambda_i). \tag{57}$$

and a unit range $\Gamma_i = 1$. Such a decomposition means that

$$x_{it} = \sum_k^N A_{ik} \Lambda_k \tag{58}$$

and that a probability distribution of this asset is (because Lévy distributions are stable) parametrized by

$$\begin{aligned}
\delta_i &= \sum_k A_{ik} \Delta_k \,, \\
\gamma_i &= \left(\sum_k |A_{ik}|^\alpha \right)^{1/\alpha} , \quad \text{and} \\
\beta_i &= \frac{\sum_k |A_{ik}|^\alpha B_k}{\sum_k |A_{ik}|^\alpha} \,.
\end{aligned} \tag{59}$$

In the simplest version described above we may take all $B_k = 0$ and in consequence have all $\beta_i = 0$. A matrix X with elements $x_{it}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T$ can be viewed as a single realization of the generalized Wishart random matrix generated with the Lévy probability distribution. Determination of the matrix A_{ij} in this case requires new methods, different than in the Gaussian case and will be discussed elsewhere [28].

One can construct the analogue of the correlation matrix $\tilde{C}_{ij}$ as

$$\tilde{C}_{ij} = \frac{1}{T^{2/\alpha}} \sum_t^T x_{it} x_{jt} = \frac{1}{T^{2/\alpha}} (XX^T)_{ij} \tag{60}$$

and discuss its spectral properties when averaged over the ensemble of Lévy matrices. The dependence on the size of the window T is different than in the Gaussian case (which corresponds to the limit $\alpha \to 2$). To understand the reason for that let us consider the uncorrelated Lévy matrix with $A_{ij} = \delta_{ij}$. The diagonal elements $d_i = \tilde{C}_{ii}$ are the sums of squares of the random Lévy variables with the index α. It is trivial to realize that such squares are themselves random variables and that their distribution has a fat tail with the index $\alpha/2$. Following the arguments of the central limit theorem given in the preceding sections we expect that if T is large enough a sum of such variables will be distributed according to the corresponding Lévy distribution. We may even argue that this distribution should by completely asymmetric ($\beta = 1$), since the squares are all positive. The factor $T^{-2/\alpha}$ is the correct scaling factor in this case. Similar arguments can be used to show that the off-diagonal elements $\tilde{C}_{ij}, \ i \neq j$ retain the original index α

and therefore in the limit $T \to \infty$ the eigenvalue spectrum of the matrix $\tilde{C}_{ij}$ is dominated by its diagonal elements. The shape of this spectrum is given by the Lévy pdf with the index $\alpha/2$ and $\beta = 1$. This pdf has a power-like behavior with a relatively low power ($\alpha/2 < 1$) and can easily be responsible for large eigenvalues, which in this version have no dynamical origin.

To assess the importance of the off-diagonal entries on the spectrum for finite T, we use the standard perturbation theory. For that, we write

$$\tilde{C}_{ij} = \left(d_i \delta_{ij} + T^{-1/\alpha} a_{ij} \right) . \tag{61}$$

In the zeroth order, the eigenvalues of $\tilde{C}_{ij}$ are just d_i. The first order corrections are zero because the matrix a_{ij} is off-diagonal. Generically, for a random matrix, d_i's are not degenerate, so up to the second order, the eigenvalues of $\tilde{C}_{ij}$ are

$$\lambda_i = d_i + \varepsilon^2 \sum_{j(\neq i)} \frac{a_{ij}^2}{d_j - d_i} = d_i + T^{-2/\alpha} \sum_{j(\neq i)} \frac{a_{ij}^2}{d_j - d_i} . \tag{62}$$

There are $N - 1$ terms in the sum, each of order unity. Thus the sum contributes a factor proportional to N, say $\approx s_i N$, and we have:

$$\lambda_i = d_i + s_i N T^{-2/\alpha} . \tag{63}$$

The off-diagonal terms compete with the diagonal ones for $N \approx T^{2/\alpha}$.

In the general case, where the matrix A_{ij} is non-trivial, the usefulness of the correlation matrix $\tilde{C}_{ij}$ to determine the real correlations in the system is limited. Looking for methods of determination of the A_{ij} is crucial to distinguish between the noise and signal.

In both approaches presented above the elements of the matrix x_{it} were treated as random numbers obtained for each time step t from the same multidimensional random number generator. This can be understood as a particular case of a situation where this generator depends also on t and where we have some non-trivial matrix probability measure $P(x)Dx$. Examples of such measures are known in the literature.

One can speculate that in reality the distribution of x_{it} comes from many different sources s and that

$$x_{it} = \sum_s x_{it}^{(s)} , \tag{64}$$

where all $x_{it}^{(s)}$ have the same matrix measure. This approach leads to the concept of non-commutative probability distributions, discussed in the next chapter.

9. Matrix economy

In the previous chapters we mentioned several consequences of the central limit theorem, one of the cornerstones of the theory of probability. We may ask a question, which at the first glance looks academic: Can one formulate an analog of the central limit theorem, if random variables $\hat{X}_1, \hat{X}_2, \ldots \hat{X}_N$ forming the sums

$$\hat{S}_N = \hat{X}_1 + \hat{X}_2 + \ldots \hat{X}_N \tag{65}$$

do not commute? In other words, we are seeking for a theory of probability, which is non-commutative, *i.e.* $\hat{X}_i$ can be viewed as operators, but which should exhibit close similarities to the "classical" theory of probability. Such theories are certainly interesting from the point of view of quantum mechanics or noncommutative field theory, but are they relevant for economic analysis? The answer is positive. Abstract operators may have matricial representations. If such construction exists, we would have a natural tool of formulating the probabilistic analysis directly in the space of matrices. Contemporary financial markets are characterized by collecting and processing enormous amount of data. Statistically, they may come from a processes of the type (64) and may obey the matrix central limit theorems. Matrix-valued probability theory is then ideally suited for analyzing the properties of arrays of data (like the ones encountered in the previous chapter), analyzing signal to noise ratio and time evolution of large portfolios. It allows also to recast standard multivariate statistical analysis of covariances [29] into novel and powerful language. Spectral properties of large arrays of data may also provide a rather unique tool for studying chaotic properties, unraveling correlations and identifying unexpected patterns in very large sets of data.

The origins of non-commutative probability is linked with abstract studies of von Neumann algebras done in the 80'. A new twist was given to the theory, when it was realized, that noncommuting abstract operators, called free random variables, can be represented as infinite matrices [30]. Only very recently the concept of FRV started to appear explicitly in physics [31–33].

In this paper, we abandon a formal way and we shall follow the intuitive approach, using frequently a physical intuition.

Our main goal is to study the *spectral* properties of large arrays of data. Such analysis turned out to be relevant for the source detection and bearing estimations in many problems related to signal processing [34]. Since large stochastic matrices obey central limit theorems with respect to their *measure*, spectral analysis is a powerful tool for establishing a stochastic feature of the whole set of matrix-ordered data, simply by comparing their spectra to the analytically known results of random matrix theory. Simultaneously, the deviations of empirical spectral characteristics from the spectral correlations of purely stochastic matrices can be used as a source of inferring the important correlations, not so visible when investigated by other methods. We shall first formulate the basics of matrix probability theory, and then we shall discuss a sample application in the case of a financial covariance matrix, a key ingredient of any theory of investment and/or financial risk management.

Let us assume, that we want to study statistical properties of infinite random matrices. We are interested in the spectral properties of $N \times N$ matrix X, (in the limit $N \to \infty$), which is drawn from a matricial measure

$$dX \;\; \exp - N \mathrm{Tr} V(X) \tag{66}$$

with a potential $V(X)$ (in general not necessarily polynomial). We shall restrict ourselves to real symmetric matrices for the moment, since their spectrum is real. The average spectral density of the matrix X is defined as

$$\rho(\lambda) = \frac{1}{N} \langle \mathrm{Tr} \delta(\lambda - X) \rangle = \frac{1}{N} \left\langle \sum_i \delta(\lambda - \lambda_i) \right\rangle , \tag{67}$$

where $\langle ... \rangle$ means averaging over the ensemble (66). Using the standard folklore, that the spectral properties are related to the discontinuities of the Green's function we may introduce

$$\mathcal{G}(z) = \frac{1}{N} \left\langle \mathrm{Tr} \frac{1}{z - X} \right\rangle , \tag{68}$$

where z is a complex variable. Due to the known properties of the distributions

$$\lim_{\varepsilon \to 0} \frac{1}{\lambda \pm i\varepsilon} = PV \frac{1}{\lambda} \mp i\pi\delta(\lambda) \tag{69}$$

we see that the imaginary part of the Green's function reconstructs spectral density (67)

$$-\frac{1}{\pi} \lim_{\varepsilon \to 0} \mathrm{Im} \; \mathcal{G}(z)|_{z=\lambda+i\varepsilon} = \rho(\lambda) . \tag{70}$$

The natural (from the point of view of the physicist) Green's function shall serve us as an auxiliary construction explaining the crucial concepts of the theory of matrix (noncommutative) probability theory. Let us define a functional inverse of the Green's function (sometimes called a Blue's function [32]), *i.e.* $\mathcal{G}[\mathcal{B}(z)] = z$. The fundamental object in noncommutative probability theory, so-called R function or R-transform, is defined as

$$\mathcal{R}(z) = \mathcal{B}(z) - \frac{1}{z} . \tag{71}$$

With the help of the R-transform we shall now uncover several astonishing analogies between the classical and matricial probability theory.

We shall start from the analog of the central limit theorem. It reads [30]: The spectral distributions of independent variables $\hat{X}_i$,

$$\hat{S}_K = \frac{1}{\sqrt{K}}(\hat{X}_1 + \ldots + \hat{X}_K) \tag{72}$$

each with arbitrary probability measure with zero mean and finite variance $\langle \mathrm{Tr}\hat{X}_i^2 \rangle = \sigma^2$, converges towards the distribution with R-transform $\mathcal{R}(z) = \sigma^2 z$.

Let us now find the exact form of this limiting distribution. Since $\mathcal{R}(z) = \sigma^2 z$, $\mathcal{B}(z) = \sigma^2 z + 1/z$, so its functional inverse fulfills

$$z = \sigma^2 \mathcal{G}(z) + 1/\mathcal{G}(z)\,. \tag{73}$$

The solution of this quadratic equation (with proper asymptotics $\mathcal{G}(z) \to 1/z$ for large z) is

$$\mathcal{G}(z) = \frac{z - \sqrt{z^2 - 4\sigma^2}}{2\sigma^2} \tag{74}$$

so the spectral density, supported by the cut of the square root, is

$$\rho(\lambda) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - \lambda^2}\,. \tag{75}$$

This is the famous Wigner semi-circle [35] (actually, semi-ellipse) ensemble. The omni-presence of this ensemble in various physical applications finds a natural explanation — it is a consequence of the central limit theorem for non-commuting random variables. Thus the Wigner ensemble is a noncommutative analog of the Gaussian distribution. Indeed, one can show, that the measure (66) corresponding to Green's function (74) is $V(X) = \sigma^{-2}X^2$.

Let us look in more detail, what "independence" means for two identical matrix valued ensembles, *e.g.* of the Gaussian type, with zero mean and unit variance. We are interested in finding the discontinuities of the Green's function

$$\mathcal{G}_{1+2}(z) \sim \int D\hat{X}_1 D\hat{X}_2 \mathrm{e}^{-N\mathrm{Tr}\hat{X}_1^2}\mathrm{e}^{-N\mathrm{Tr}\hat{X}_2^2}\mathrm{Tr}\frac{1}{z - (\hat{X}_1 + \hat{X}_2)}\,. \tag{76}$$

In principle, this requires a solution of the convolution, with matrix-valued, noncommuting entries! Here we can see how the R-transform operates. This is the transform, which imposes the additive property for the all cumulants:

all spectral cumulants obey $k_i(X_1 + X_2) = k_i(X_1) + k_i(X_2)$, for all $i = 1, 2, \ldots, \infty$ [30, 36].

Mathematicians call such a property "freeness", hence the name free random variables. The R-transform is an analog of the logarithm of the characteristic function (32) in the classical probability theory, and fulfills the addition law [30]

$$\mathcal{R}_{1+2}(z) = \mathcal{R}_1(z) + \mathcal{R}_2(z)\,. \tag{77}$$

Note that we keep the notation underlying the similarities between the classical and non-commutative (matricial) probability calculus. In the above example, the matrix valued convolution of two Gaussian ensembles with a unit variance gives again a Gaussian ensemble, with the spectrum (semicircle) rescaled by $\sqrt{2}$. Technically, it comes from the fact that $\mathcal{R}_{1+2}(z) = \mathcal{R}_1(z) + \mathcal{R}_2(z) = z + z = 2z$. This is like the usual convolution of two Gaussian probability distribution, forming also a Gaussian but with a variance rescaled by a factor $\sqrt{2}$.

At this moment one can start to really appreciate the power of the noncommutative approach to probability. For large matrices $\hat{X}$ and $\hat{Y}$ (exact results hold in the $N = \infty$ limit), the knowledge of their spectra is usually sufficient for predicting the spectrum of the sum $\hat{X} + \hat{Y}$.

The noncommutative calculus allows also to generalize the additive law for non-hermitian matrices [37, 38], and even formulate the multiplicative law, *i.e.* infer the knowledge of all moments of the spectral function of the product of $\hat{X}\hat{Y}$, knowing only the spectra of $\hat{X}$ and $\hat{Y}$ separately (so-called S-transform) [30]. As such, it offers a powerful shortcut in analyzing stochastic properties of large ensembles of data. Moreover, the larger the sets the better, since finite size effects scale at least as $1/N$.

Let us check the possibility of appearance of power-like spectra in noncommutative probability theory. Motivated by the construction in classical probability, we pose the following problem: What is the most general form of the spectral distribution of random matrix ensemble, which is stable under matrix convolution, *i.e.* has the same functional form as the original distributions, modulo shift and rescaling? Surprisingly, non-commutative probability theory follows from the Lévy–Khinchine theorem of stability in classical probability. In general, the needed $\mathcal{R}(z)$ behaves like $z^{\alpha-1}$, where $\alpha \in (0, 2]$. More precisely, the list is exhausted by the following R-transforms [39]:

(i) $\mathcal{R}(z) = \mathrm{e}^{i\pi\phi} z^{\alpha-1}$, where $\alpha \in (1, 2]$, $\phi \in [\alpha - 2, 0]$

(ii) $\mathcal{R}(z) = \mathrm{e}^{i\pi\phi} z^{\alpha-1}$, where $\alpha \in (0, 1)$, $\phi \in [1, 1 + \alpha]$

(iii) $\mathcal{R}(z) = a + b \log z$, where a, b are complex and $\Im a \geq 0$ and $b \geq -\frac{1}{\pi}\Im a$.

Note that the stability index α is restricted to precisely the same values as in the one-dimensional case (38). The asymptotic form of the spectra is power-like, *i.e.* $\rho(\lambda) \sim 1/\lambda^{\alpha-1}$. Singular case *(iii)* corresponds, in a symmetric case ($b = 0$), to the Cauchy distribution. Note that the case *(i)* with $\alpha = 2$ corresponds to the Gaussian ensemble. For spectral distributions, several other analogies to Lévy distributions hold. In particular, there is a one-to-one correspondence for spectral analogs of ranges, asymmetries and shifts. Spectral distributions exhibit also duality laws ($\alpha \to 1/\alpha$), like their classical counterparts [40, 41]

To convince the reader, how useful the formalism of non-commutative probability theory could be for the analysis of financial data, let us reconsider the example from the previous chapter.

We analyze a time series of prices of N companies, measured at equal sequence of T intervals. The returns (here relative daily changes of prices) could be recast into $N \times T$ matrix X. This matrix defines the empirical $N \times N$ covariance matrix $\tilde{C}$ (60). This matrix forms today a cornerstone of every methodology of measuring the market risk [22].

We can now confront the empirical data, assuming the extreme scenario, that the covariance matrix is completely noisy (no-information), *i.e.* $X = \hat{X}$ is stochastic, belonging to *e.g.* a random matrix ensemble. By central limit theorems, we can consider either matricial Gaussian or matricial Levy–Khinchin stability basins. From technical point of view, the problem of finding spectral distribution for covariance matrix reduces to convolution of a square $T \times T$ matrix $\hat{X}^2$ and a "deterministic" diagonal projector P, with the first N elements equal to 1, and the remaining $(T - N)$ set to zero. Exact formula, corresponding to $T, N \to \infty$, $N/T = a$ fixed comes from a "back-of the envelope" calculation [42]. For symmetric Lévy distributions, for completely random matrices, the Green's function is given by

$$\tilde{\mathcal{G}}(z) = 1/z[1 + f(z)]\,, \tag{78}$$

where $f(z)$ is a multivalued solution of a transcendental equation

$$(1 + f)(f + a)\frac{1}{f^{2/\alpha}} = z\,. \tag{79}$$

In the case $\alpha = 2$, equation is algebraic (quadratic), and the spectrum is localized on a finite interval. In all other cases the range of the spectrum is infinite, with the large eigenvalue distribution scaling as $1/\lambda^{\alpha+1}$.

A reader familiar with methods of multivariate statistical analysis immediately recognizes, that the case $\alpha = 2$ corresponds to the spectral distribution of celebrated Wishart distribution. Indeed, the normalized solution of a quadratic equation (*i.e.* (79) with $\alpha = 2$) leads to the spectral function

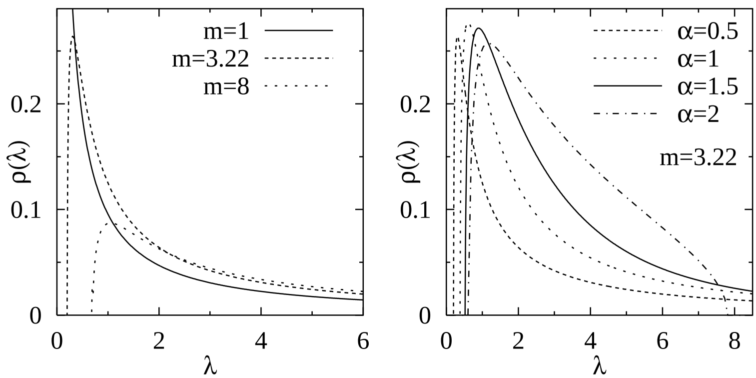

Fig. 4. Spectral densities of the covariance matrix of free random Lévy matrices with the stability index $\alpha = 1/2$ and different values of the asymmetry parameter by $m = T/N = 1/a$ (left figure); and with the given asymmetry parameter $m = T/N = 3.22$ and different values of the stability index α (right figure).

(46) mentioned already. This result was rediscovered several times in the context of various physical applications, with the help of various random matrix techniques [43].

We would like to stress, how natural and fundamental is this result from the point of view of non-commutative probability and central limit theorems.

From this point of view, it is also puzzling how late the random matrices (in our language matricial probabilities) were used for the analysis of financial data. The breakthrough came in 1999, when two groups [44, 45] have analyzed the spectral characteristics of empirical covariances, calculated for all companies belonging to Standard and Poor 500 index, which remained listed from 1991 till 1996. The spectrum of the empirical covariance matrix constructed from this matrix was then confronted with the analytically known spectrum of a covariance matrix constructed solely from the maximal-entropy (Gaussian) ensemble with the same number of rows and columns.

The unexpected (for many) results showed, that the majority of the spectrum of empirical covariance matrices is populated by noise!

In the case of a Gaussian disorder, 94% of empirical eigenvalues were consistent with random matrix spectra [44]. Only few largest eigenvalues did not match the pattern, reflecting the appearance of large clusters of companies, generally corresponding to the sectorization of the market and market itself [23]. The analysis done with the power law ($\alpha = 1.5$) not only confirmed the dominance of stochastic effects, but even interpreted the clusters as possible large stochastic events [46]. It also pointed at the dangers of using the covariance matrix (which assumes implicitly the finite dispersion) in the case when power laws are present.

The random matrix analysis posed therefore a fundamental question for quantitative finances. If empirical covariance matrices are so "noisy", why there are so valuable for practitioners? Every industrial application of risk measurement depends heavily on covariance matrix formulation. The Markowitz's theory of diversification of investment portfolios depends crucially on the information included in the covariance matrix [25]. If indeed the lower part of the covariance matrix spectrum has practically no information, the effects of noise would strongly contaminate the optimal choice of the diversification, resulting in the dangerous underestimation of the risk of the portfolio.

Bouchaud and others [47] suggested a way out, simply filtering out the noisy part of the correlation matrix and repeating the Markowitz analysis with refined matrix. This resulted in a better approximation of the risk.

Their analysis did not answer however the fundamental question. If the original matrix is noisy, *i.e.* has almost no information, how come the covariance matrices form the pillars of quantitative finance?

We tried to answer this question in the previous section, shedding some light on a rather nontrivial relation between the true covariance matrix C and its estimator $\tilde{C}$. The relation between the Green's functions $\mathcal{G}$ and $\tilde{\mathcal{G}}$ was obtained in the framework of Random Matrix Theory. Some other recent papers using tools of random matrix theory for investigating the properties of covariance matrices are [48–51].

We would like to point out at this moment, that matrix probability theory seems to be ideally suited tool for better understanding the role of covariance matrix and a way of quantitatively assessing the role of the noise, important correlations and the stability of the analysis. In our opinion, the full power of random matrix techniques was not recognized yet by the quantitative finance community.

Finally, we would like to point out an exciting possibility of introducing the dynamics formulated in the matrix probability language. The simplest dynamics of price (S) movement of the asset is canonically [17] described by the stochastic equation

$$dS = S_{t+dt} - S_t = (\mu dt + \sigma d\eta) S_t \,, \tag{80}$$

where the deterministic evolution is governed by the interest rate (drift) μ and the stochastic term is represented by the Wiener measure $d\eta$, multiplied by dispersion (called in finance volatility) σ. The Wiener measure could be realized as $\sqrt{dt}N(0,1)$, where $N(0,1)$ is a Gaussian with zero mean and unit variance. Therefore $\langle d\eta \rangle = 0$ and $\langle (d\eta)^2 \rangle = dt$, reflecting the random walk character of the process. Since the process is multiplicative, the resulting Fokker–Planck equation is a heat equation with respect to the log S, solved by the log normal distribution. Note, that (80) has the *same* content as already written equations (9),(30) for wealth and prices, respectively.

One is tempted to write a similar stochastic equation for the *vector* of prices. The standard extension [52] reads

$$S_{t+dt,i} = (1 + \mu_i dt + \sqrt{dt} A_{ij} \eta_j) S_{t,i} \,, \tag{81}$$

where the noise *vector* η_i obeys $\langle \eta_i \eta_j \rangle \sim \delta_{ij}$ and A_{ij} is the square root of the correlation matrix.

Note however, that one may write a different equation, but now for the *matrix* analog of the Wiener measure. It is not difficult to see, that the role of the white noise is now played by Gaussian ensemble of random matrices, resulting into the matrix evolution for the whole *vector* of prices. Taking the finite time step, we get

$$S_{t+dt,i} = (\delta_{ij} + \mu_{ij} dt + \sigma \sqrt{dt} X_{ij}) S_{t,j} \,, \tag{82}$$

where μ is a deterministic matrix and X is a real Gaussian matrix and not a vector. Diffusion takes then place in the space of matrices. Finite time evolution results in the infinite product of large, *non-commuting* matrices, ordered along the diffusive path, similarly like the chronological operators do for the time evolution of non-commuting Hamiltonians. Here, however, the evolution is dissipative (spectrum is complex). Surprisingly, random matrix techniques [53] allow to analyze the changes of the spectrum of such stock market evolution operators as a function of time t, similarly as in the case of a single asset, where the lognormal packet spreads according to the heat equation.

This approach, basically equivalent to one of the matrix generalizations of the Ito-like processes, may allow to study the time properties of the spectra of large sets of financial data. Moreover, the method seems not to be restricted to the Gaussian world, due to the mathematical power of matricial probability calculus and the matrix valued stochastic differential equations may turn out to be a powerful tool of time series analysis of large sets of data. This "matrix econophysics" (as a witticism, or maybe "wittencism", we may use abbreviation M-econophysics to paraphrase M-theory) may also give a rather precise meaning of "quantum economy", a vague term often encounter in the literature. In the language of a matrix-valued probability calculus, the "quantum nature" comes from the fact, that basic objects of the probability calculus are operators, represented as large, non-commuting matrices, represented in economy by arrays of data. The relevant observables in this language are related to the statistical properties of their spectra.

10. Econophysics or econoscience?

In the course of the presentation, we only briefly analyzed some selected methods related to the description of real complex systems such as economic or financial markets. The idea was to give the reader not familiar with this field some sort of a sampler, hopefully an appetizer. We did not mention at all several intriguing attempts to describe financial crashes using the insight from physics [54]. Neither did we mention promising attempts to use the concepts of cascades and/or turbulence for explaining the observed correlations and multifractality in high frequency time series [55]. We omitted natural, from the point of view of the physicist, modifications of the option theories [3]. Our presentation of macroeconomic applications was restricted to simple patterns of wealth distribution, and we ignored the whole dynamics of this process. We did not discuss several other issues, usually covered by econophysics conferences [56, 57].

At this moment, instead of continuing the list of our sins, let us come back to the titular question — how "solid" is econophysics as a science? We would like to point at few dangers, which in our opinion, every econophysicist has to take into account.

1. First, we believe that laws of physics do not change in time. Certainly, this is not true for most of the laws of economy. Most dramatic are the financial markets. Technical developments (computers, Internet) or legal regulations have a major impact on the field.

2. Second, "the material points", *i.e.* agents are not passive — they are thinking entities, and sometimes they are very smart. This invalidates immediately the "stationarity" principle. Methods and strategies evolve continuously in time, and the "quasistationarity" is rather due to the traditional conservatism of financial institutions. Abandoning this conservatism leads to the situation, where more adequate are concepts of biological evolutionism mixed with elements of the game theory. Indeed, this lead is seriously studied nowadays [58, 59]. Taking into account the complexity of the system, the speed at which the systems may evolve and the multidimensional space of the systems, whose topology may more reflect the virtual network of connections than real geographic distances [60, 61], the need of such studies is obvious. As recently pointed [62], economy may evolve into cyberscience. Then, the role of the methods of physics will be reduced, and physics will serve as a source of complementary methodology with respects to the methods of biology, mathematics, psychology and computer science.

3. Even assuming the methods of physics are applicable at certain time horizons, econophysics may not be immediately successful in the sense

of making an impact on economic or financial markets. What seems to be absolutely crucial is that not only physicists should be convinced that they understand "markets", they have also to convince about that the "market makers". This requires several ingredients. The first is the *quality* of the research. The second is the continuous verification of models/theories with the data. The third is the *close* cooperation between the physicists and economists and financial advisors.

All these three ingredients are often difficult to fulfill. The semantic discrepancies, much too carelessly (also by us) usage of physicists' slang (like quantum economy, gauge theory, stock market Hamiltonian, spin-glass portfolio *etc.*), some mutual gaps in education, sometimes lack of crucial data *etc.*, may trigger the situation, where econophysics may start to evolve in "splendid isolation" from the mainstream of economy.

All these dangers may slow down, the however *unavoidable* on long run, (in our opinion), impact of methods of physics on economy and financial markets. Historical definition of economy, as an *art* of "optimal allocation of scarce resources to given ends", needs to be replaced by the *science* of "economic agents — processors of information" [62].

We do hope, that this review at least partially convinced the sceptical reader, that the concepts of statistical physics can enrich this science, hopefully making even a major impact at the fundamental level.

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