3D LORENTZIAN QUANTUM GRAVITY FROM THE ASYMMETRIC ABAB MATRIX MODEL*

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The asymmetric ABAB-matrix model describes the transfer matrix of three-dimensional Lorentzian quantum gravity. We study perturbatively the scaling of the ABAB-matrix model in the neighborhood of it's symmetric solution and deduce the associated renormalization of three-dimensional Lorentzian quantum gravity.

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1. Introduction

Matrix models have been very useful in the study of the "quantum geometry" of two-dimensional quantum gravity. In [1] this program was extended to three-dimensional quantum gravity. It was shown how the so-called ABAB two-matrix model describes the transfer matrix of tree-dimensional quantum gravity. More precisely a non-perturbative, background independent definition of quantum gravity, which emphasizes the causal structure of space-time and which allows rotations between Lorentzian and Euclidean signature, was proposed in [2,3], generalizing an explicit solvable two-dimensional model with these features [4]. In the model, which has an UV lattice cut off a which should be taken to zero in the continuum limit, one can define the concept of proper time. In the Euclidean sector the corresponding evolution operator is defined in terms of the transfer matrix $\hat{T}$ describing the transition between quantum states at (proper) time $n \cdot a$ and (proper) time $(n+1) \cdot a$. The transfer matrix is related to the quantum Hamiltonian of the system by

$$\hat{T} = \mathrm{e}^{-a\hat{H}}. \tag{1.1}$$

The ABAB model is defined by the

$$\begin{aligned} Z(\alpha_1, \alpha_2, \beta) &= \mathrm{e}^{-N^2 F(\alpha_1, \alpha_2, \beta)} \\ &= \int \mathrm{d}A\mathrm{d}B\, \mathrm{e}^{-N \mathrm{tr}\,(\frac{1}{2}(A^2+B^2) - \frac{\alpha_1}{4}A^4 - \frac{\alpha_2}{4}B^4 - \frac{\beta}{2}\mathrm{ABAB})}. \end{aligned} \tag{1.2}$$

Under the assumption discussed in [1] the free energy $F(\alpha_1, \alpha_2, \beta)$ is related to the matrix elements of the transfer matrix $\hat{T}$ in a way reviewed in the next section. The matrix model (1.2) has a scaling limit for $\alpha_1 = \alpha_2$ which was analyzed in [5]. This allowed us in [1] to determine the corresponding phase diagram for the three-dimensional quantum gravity model and to map the bare coupling constants of the gravity model to the matrix model coupling constants $\alpha_1 = \alpha_2$ and β [7]. However, in order to study details of the scaling relevant to three-dimensional quantum gravity we have to study the matrix model for $\alpha_1 \neq \alpha_2$. In the scaling limit of interest for us both α_1 and α_2 will scale to a critical value α_c, but independently. Since we are interested only in the behavior of the theory near the symmetric solution we need only the perturbative expansion around this solution rather than the complete solution in the asymmetric case[1].

[1] While writing this article the asymmetric ABAB matrix model has been solved by Paul Zinn-Justin [8]. The behavior close to the symmetric line $\alpha_1 = \alpha_2$ is the same as the one reported here and to extract it one has to expand the elliptic functions which appear in the solution, an effort comparable to the one used here.

The rest of this article is organized as follows. In Sec. 2 we review shortly the non-perturbative definition of three-dimensional Lorentzian quantum gravity [2,3] and its relation to the ABAB matrix model. In Sec. 3 we review the machinery needed to solve the ABAB matrix model for a symmetric choice of coupling constants [5]. In Sec. 4 we discuss the solution of the general ABAB matrix model, and in Sec. 5 we expand around the symmetric critical point relevant for three-dimensional quantum gravity. In Sec. 6 we discuss how to extract information about the transfer matrix of 3D gravity, knowing the free energy of the asymmetric ABAB matrix model.

2. Quantum gravity and the ABAB matrix model

Simplicial Lorentzian quantum gravity is defined in the following way: the spatial hypersurfaces of constant proper-time are two-dimensional equilateral triangulations. Such triangulations define uniquely a two-dimensional geometry. It is known that this class of geometries describes correctly the quantum aspects of two-dimensional Euclidean gravity. It is also known that the description of two-dimensional Euclidean quantum gravity in terms of the class of (generalized) triangulations is quite robust. In [2] we used this universality in the following way: the two-dimensional geometry of the spatial hypersurfaces is represented by quadrangulations and it was shown that it is possible to connect any such pair of quadrangulations by a set of three-dimensional "simplexes". More precisely, let a be the lattice spacing separating two neighboring spatial hypersurfaces a (proper)-time t and $t + a$. Then each square at t is connected to a vertex at $t + a$ and each square at $t + a$ is connected to a vertex at proper-time t. A further needed three-dimensional building block is a tetrahedron connecting a spatial link at t to a spatial link at $t+a$. The proper-time propagator for (regularized) three-dimensional quantum gravity between two spatial hypersurfaces separated by a proper time $T = n \cdot a$ is obtained by inserting $n-1$ intermediate spatial hypersurfaces and summing over all possible geometries constructed as described above, The weight of each geometry is given by the Einstein action, here conveniently the Regge action for piecewise linear geometries. The naive continuum limit is obtained by scaling the lattice spacing $a \to 0$ while keeping $T = n \cdot a$ fixed. However, different scaling relations between T and a might in principle be possible[2].

Let g_t and g_{t+a} be spatial two-geometries at t and $t+a$, *i.e.* two quadrangulations and let $\langle g_{t+a}|\hat{T}|g_t\rangle$ be the transition amplitude or proper time

[2] In two-dimensional *Euclidean* quantum gravity the proper-time T scales anomalously and one has to keep $n\sqrt{a}$ fixed. This is in contrast to the situation in two-dimensional *Lorentzian* quantum gravity as defined in [4] where the proper-time T scales canonically. The relation between the two models is well understood [9].

propagator from t to $t+a$. By definition, $\hat{T}$ is the transfer matrix in the sense of Euclidean lattice theory, and it satisfies the axioms of a transfer matrix [3]. In the case where the spatial topology is that of S^2 it was argued in [1] that the continuum limit could be obtained as the large N scaling limit of the matrix model (1.2). Let N_t and N_{t+a} denote the number of squares in the quadrangulations associated with g_t and g_{t+a}. The two-volumes of the corresponding geometries are thus $N_t a^2$ and $N_{t+a}a^2$, respectively, and the relation to $F(\alpha_1, \alpha_2, \beta)$ defined by (1.2) is

$$F(\alpha_1, \alpha_2, \beta) = \sum_{g_t, g_{t+a}} \mathrm{e}^{-z_t N_t - z_{t+a} N_{t+a}} \langle g_{t+a} | \hat{T} | g_t \rangle, \tag{2.1}$$

where z_t and z_{t+a} are dimensionless boundary cosmological constants. The naive relation between the matrix model coupling constants and the bare gravitational and cosmological coupling constants G_N and Λ of three-dimensional gravity is as follows:

$$\alpha_1 = \mathrm{e}^{\tilde{k} - \tilde{\lambda} - z_t}, \qquad \alpha_2 = \mathrm{e}^{\tilde{k} - \tilde{\lambda} - z_{t+a}}, \qquad \beta = \mathrm{e}^{-(\frac{1}{2}\tilde{\lambda} + \tilde{k})}, \tag{2.2}$$

where

$$\tilde{k} = \frac{a}{4\pi G_N} \left(-\pi + 2\cos^{-1}\frac{1}{3} + \sin^{-1}\frac{2\sqrt{2}}{3} \right), \quad \tilde{\lambda} = \frac{a^3 \Lambda}{24\sqrt{2} G_N}. \tag{2.3}$$

In this paper we shall discuss the non-perturbative renormalization of the coupling constants. In principle we are interested in the limit $z_t = z_{t+a} = 0$, *i.e.* $\alpha_1 = \alpha_2$. However, in order to be able to extract the information about the scaling of the boundary cosmological constants we have to keep z_t and z_{t+a} different from zero at intermediate steps. Thus these *boundary cosmological constants* should be viewed as source terms for boundary area operator.

3. The symmetric case: $\alpha_1 = \alpha_2 = \alpha$

Let us for later convenience shortly review the technique for solving the matrix model (1.2) used in [5] (based on earlier results [6]).

By a character expansion of the term $e^{\frac{1}{2}\beta \mathrm{tr\,ABAB}}$ one can write

$$Z(\alpha_1, \alpha_2, \beta) \sim \sum_{\{h\}} \left(\frac{M\beta}{2} \right)^{\sum h_i - \frac{N(N-1)}{2}} c_{\{h\}} \, R_{\{h\}}(\alpha_1) R_{\{h\}}(\alpha_2), \tag{3.1}$$

where the sum is over the representations of $GL(M)$, characterized by the shifted highest weights $h_i = m_i + M - i$, $(i = 1, \ldots, M)$, where the m_i are the standard highest weights and where the large M limit of the coefficient $c_{\{h\}}$ is

$$\log c_{\{h\}} = -\sum_i \frac{h_i}{2}\Big(\log\frac{h_i}{2}-1\Big) - \frac{1}{2}\log\Delta(h), \qquad \Delta(h) = \prod_{i<j}(h_i-h_j). \quad (3.2)$$

Finally, if $\chi_{\{h\}}$ denotes the character associated with $\{h\}$,

$$R_{\{h\}}(\alpha) = \int \mathrm{d}A\ \chi_{\{h\}}(A)\ \exp M\Big(-\frac{1}{2}\mathrm{tr}\,A^2 + \frac{\alpha}{4}\mathrm{tr}\,A^4\Big). \quad (3.3)$$

It is now possible to perform a double saddle point expansion of (3.1) and (3.3). In order to describe the formalism let us introduce the notation

$$\Re f(z) \equiv \frac{f(z+i0)+f(z-i0)}{2}, \qquad \Im f(z) = \frac{f(z+i0)-f(z-i0)}{2}. \quad (3.4)$$

This notation is useful when $f(z)$ has cuts. The saddle point expansion assumes the existence of an eigenvalue density $\tilde{\rho}(\lambda)$, or equivalently a resolvent associated with the matrix integral (3.3):

$$\omega(\lambda) = \frac{1}{M}\sum_k \frac{1}{\lambda-\lambda_k}, \qquad -\pi i\tilde{\rho}(\lambda) = \Im\omega(\lambda), \quad (3.5)$$

and (after rescaling $h \to Mh$) a density of highest weights $\rho(h)$, or the corresponding "resolvent" $H(h)$:

$$H(h) = \frac{1}{M}\sum_k \frac{1}{h-h_k}, \qquad -\pi i\rho(h) = \Im H(h). \quad (3.6)$$

In Ref. [5] the double saddle point expansion is analyzed in the case $\alpha_1 = \alpha_2 = \alpha$. The density $\rho(h)$ was assumed to be different from zero in the interval $[0, h_2[$, and equal to 1 in the interval $[0, h_1]$. Further, for a given eigenvalue distribution λ_k of the matrix A coming from the saddle point of (3.3) one can define a function $L(h)$, with same cut as H by

$$\Re L(h_j) = \frac{2}{M}\frac{\partial}{\partial h_j}\log\chi_{\{h\}}(A(\lambda_k)). \quad (3.7)$$

The analysis of [5] shows that $L(h) = H(h) + F(h)$ where $F(h)$ is analytic on the cut of $H(h)$ but has an additional cut $[h_3, \infty[$ where

$$2\Re L(h) = \log\frac{h}{\alpha} + H(h). \quad (3.8)$$

It can now be shown that the function $D(h) = 2L(h) - H(h) - 3\log h + \log(h - h_1)$ only has square root type cuts on $[h_1, h_2]$ and $[h_3, \infty[$ and on these cuts satisfies the following equations:

$$\Re D(h) = \log \frac{h - h_1}{\beta h^2}, \quad h \in I_0 = [h_1, h_2]\,, \tag{3.9}$$

$$\Re D(h) = \log \frac{h - h_1}{\alpha h^2}, \quad h \in I_1 = [h_3, \infty[. \tag{3.10}$$

Eqs. (3.9)–(3.10) constitute a standard Hilbert problem and the inversion formula is unique [10]. The function holomorphic in the plane with cuts I_0 and I_1 is given by

$$\begin{aligned} D(h) = \log \frac{h - h_1}{\beta h^2} - \frac{\log \beta/\alpha}{i\pi} r(h) \int_{h_3}^{\infty} \mathrm{d}h \, \frac{1}{(h - h')r(h')} \\ + r(h) \int_{-\infty}^{h_1} \mathrm{d}h' \frac{1}{(h - h')r(h')} - 2r(h) \int_{-\infty}^{0} \mathrm{d}h' \frac{1}{(h - h')r(h')}\,, \end{aligned} \tag{3.11}$$

where

$$r(h) = \sqrt{(h - h_1)(h - h_2)(h - h_3)} \tag{3.12}$$

and where we have chosen the cut structure shown in Fig. 1. Following [10] the meaning of $r(h')$ on the cut is $r(h' + i0)$, *i.e.* the function on the "left side" of the cut. The integrals can be expressed in terms of standard elliptic functions. However, we do not need the explicit expressions here.

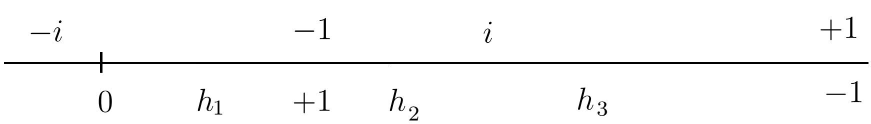

Fig. 1. The cut structure of $r(h)$ in the complex h-plane.

From $D(h)$ we can derive the expression for $\rho(h)$ which is

$$\rho(h) = -\frac{\Im H(h)}{i\pi} = -\frac{\Im D(h)}{i\pi}, \quad h \in]h_1, h_2[. \tag{3.13}$$

We have (h always $h+i0$ if ambiguities) :

$$\rho(h) = \frac{-r(h)}{\pi}\int_{-\infty}^{h_1}\frac{\mathrm{d}h'}{(h-h')ir(h')} - \frac{(-2r(h))}{\pi}\int_{-\infty}^{0}\frac{\mathrm{d}h'}{(h-h')ir(h')} + \frac{\log\beta/\alpha}{\pi^2}(-r(h))\int_{h_3}^{\infty}\frac{\mathrm{d}h}{(h-h')r(h')}. \tag{3.14}$$

Note that the derivative of $D(h)$ and $\rho(h)$ after h_i are elementary functions of h. For instance, differentiating after h_3 we have

$$\frac{\partial D(h)}{\partial h_3} = \frac{ir(h)}{h-h_3}\frac{\partial W}{\partial h_3} = \frac{ir(h)F_3}{2(h-h_3)} = -i\pi\frac{\partial\rho(h)}{\partial h_3}, \tag{3.15}$$

where the last equality is valid for $h\in I_0$ and where $W(h_1,h_2,h_3)$ and $F_3(h_1,h_2,h_3)$ are defined below (Eqs. (3.19) and (3.22)). Thus we can write

$$D(h;h_3+\delta) = D(h;h_3) + \delta\,\frac{ir(h)F_3(h_3)}{2(h-h_3)} + \delta^2\frac{ir(h)}{4(h-h_3)}\left(F_3'(h_3) + \frac{F_3(h_3)}{2(h-h_3)}\right) + \cdots \tag{3.16}$$

and similarly for $\rho(h)$. The function $F_3(h_1,h_2,h_3)$ is a sum of elliptic integrals.

3.1. Boundary conditions for KZ

The starting formula for $D(h)$ is (3.11). The general large h behavior of this function is

$$c_1h^{1/2} - \log(-\alpha h) + c_2h^{-1/2} + O(1/h). \tag{3.17}$$

However, according to the analysis in [5] $c_1=0$ and $c_2=-(-\alpha)^{-1/2}$. This gives two boundary conditions for the constants h_1,h_2,h_3 which appear in $r(h)$ and thus in $D(h)$. The coefficients c_1 and c_2 can be identified by expanding the integrand in (3.11) in powers of $1/h$ and one obtains the boundary conditions:

$$c_1 = iW(h_1,h_2,h_3) = 0, \qquad c_2 = i\Omega = \frac{i}{\sqrt{\alpha}}, \tag{3.18}$$

where we have used the first of the equations (3.18) to simplify the second, and where

$$W(h_1,h_2,h_3) = \frac{\log\beta/\alpha}{\pi}\int_{h_3}^{\infty}\frac{\mathrm{d}h'}{r(h')} + \int_{-\infty}^{h_1}\frac{\mathrm{d}h'}{ir(h')} - 2\int_{-\infty}^{0}\frac{\mathrm{d}h'}{ir(h')}\,, \tag{3.19}$$

$$\Omega(h_1,h_2,h_3) = -\frac{\log\beta/\alpha}{\pi}\int_{h_1}^{h_2}\frac{h'\mathrm{d}h'}{r(h')} + \int_{h_2}^{h_3}\frac{h'\mathrm{d}h'}{ir(h')} + 2\int_{0}^{h_1}\frac{h'\mathrm{d}h'}{ir(h')}\,. \tag{3.20}$$

We will not need the explicit expressions for the integrals.

The final boundary condition is

$$\int_{h_1}^{h_2}\mathrm{d}h\;\rho(h) = 1 - h_1, \tag{3.21}$$

where $\rho(h)$ is given by (3.14).

For a given choice of α and β we have a solution (h_1, h_2, h_3) of the three boundary conditions (3.18) and (3.21) and this in principle gives a solution of the matrix integral. Using the parametrization in terms of (h_1, h_2, h_3) we see that the model is defined on a two-dimensional hyper-surface in a three-dimensional parameter space. Singularities of the map between variables (α, β) and some parametrization of this surface (say in terms of (h_2, h_3) after eliminating h_1) represent critical points (lines) of the model.

3.2. The critical line

The generic behavior of $D(h)$ when $h \to h_3$ is clearly $(h-h_3)^{1/2}$, simply coming from the term $r(h)$ in the representation (3.11). However, this behavior can change to $(h-h_3)^{3/2}$ along a curve $\alpha_c(\beta)$ in the (β,α) coupling constant plane. According to [5] this is the critical line of phase A of the ABAB matrix model and according to [1] this is there the continuum limit of 3D gravity should be found. Similarly the criticality in the B phase is derived from the behavior or $D(h)$ when $h \to h_2$, when a generic behavior $(h-h_2)^{1/2}$ changes to $(h-h_2)^{3/2}$.

We now study the change of (h_1, h_2, h_3) as α and β change infinitesimally. For simplicity we first present the result when α/β is constant.

Let us first identify the coefficient of $\sqrt{h-h_3}$ in $D(h)$. Using (3.15) in the expression for $D(h)$ the coefficient can be written as

$$ir_0(h_3)F_3(h_1,h_2,h_3), \qquad r_0(h) \equiv \sqrt{(h-h_1)(h-h_2)}, \tag{3.22}$$

One has

$$\frac{\partial W(h_1,h_2,h_3)}{\partial h_3} = \frac{1}{2}F_3(h_1,h_2,h_3). \tag{3.23}$$

We define F_1 and F_2 similarly to F_3 and have relations like (3.23). From the boundary conditions (3.18) it follows that the variation of h_1,h_2,h_3 as α,β change with the ratio α/β fixed satisfy

$$F_1\delta h_1 + F_2\delta h_2 + F_3\delta h_3 = 0\,, \tag{3.24}$$

$$\tilde{g}_1F_1\delta h_1+\tilde{g}_2F_2\delta h_2+\tilde{g}_3F_3\delta h_3 = 2\frac{\delta\alpha}{\alpha^{3/2}}\,, \tag{3.25}$$

where $\tilde{g}_i = 2h_i - \sum_{j=1}^{3} h_3$.

The final boundary condition involves the density. Since $\rho(h_1)=1$ and $\rho(h_2)=0$ the variation of (3.21) just becomes:

$$\int_{h_1}^{h_2} \mathrm{d}h\,\Big(\frac{\partial\rho}{\partial h_1}\delta h_1 + \frac{\partial\rho}{\partial h_2}\delta h_2 + \frac{\partial\rho}{\partial h_3}\delta h_3\Big) = 0, \tag{3.26}$$

and after some partial integrations (3.26) can be written as

$$E_1F_1\delta h_1 + E_2F_2\delta h_2 + E_3F_3\delta h_3 = 0\,, \tag{3.27}$$

where

$$\int_{h_1}^{h_2} \frac{\mathrm{d}h\; r(h)}{h_i - h} = E_i, \qquad i = 1,2,3 \tag{3.28}$$

are elliptic integrals. It is easy to repeat the derivation in the case where the ratio α/β is not assumed constant. The only change is that the RHS of (3.24), (3.25) and (3.27) change to include contributions where all entries are proportional to $\delta\alpha$ and $\delta\beta$ times coefficients A_i and B_i which are (calculable) functions of h_1,h_2,h_3.

Summarizing, the complete set of equations thus reads

$$\begin{pmatrix} F_1 & F_2 & F_3 \\ \tilde{g}_1F_1 & \tilde{g}_2F_2 & \tilde{g}_3F_3 \\ E_1F_1 & E_2F_2 & E_3F_3 \end{pmatrix} \begin{pmatrix} \delta h_1 \\ \delta h_2 \\ \delta h_3 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}\delta\alpha + \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}\delta\beta. \tag{3.29}$$

It is easy to see that the Jacobian of the transformation (3.29) is proportional to $F_1F_2F_3$ and in general is non-zero, except at points where $F_3=0$ (phase A) or where $F_2=0$ (phase B).

Consequently in general the solution to (3.29) will be of the form:

$$\delta h_i \sim \delta\alpha, \delta\beta \quad i = 1, 2, 3. \tag{3.30}$$

However the critical line of phase A is characterized by $F_3(h_1^c, h_2^c, h_3^c) = 0$, determining a critical line on the surface in the (h_1, h_2, h_3) parameter space. Let us now discuss a situation when we want to study infinitesimal displacement from this line. In this case we have two possible situations. For a particular choice of $(\delta\alpha, \delta\beta)$ the rank of (3.29) is two and δh_3 can be only determined from higher-order terms. This means that following this particular direction in the parameter space we still have the behavior given by (3.30). This special direction corresponds to a displacement **along** the critical line. For any other direction in $(\delta\alpha, \delta\beta)$ plane we have

$$\delta h_1,\ \delta h_2 \sim \delta\alpha, \delta\beta \qquad (\delta h_3)^2 \sim \delta\alpha, \delta\beta. \tag{3.31}$$

We shall come back to this discussion later.

Criticality of type B can be discussed along the same lines, except that in this case $F_2(h_1^c, h_2^c, h_3^c) = 0$ and we should exchange δh_3 with δh_2. It is important to go to the asymmetric case to understand the physical difference between the two phases.

4. The asymmetric case $\alpha_1 \neq \alpha_2$

As mentioned the construction of the transfer matrix requires that we perturb away from $\alpha_1 = \alpha_2$. Let us discuss the general structure of the matrix model with $\alpha_1 \neq \alpha_2$ (as explained above we will only need an infinitesimal perturbation away from $\alpha_1 = \alpha_2$ in the continuum limit)[3]. The main difference in the analysis of the matrix model with $\alpha_1 \neq \alpha_2$ compared to the situation $\alpha_1 = \alpha_2 = \alpha$ is that the saddle point solution involves two eigenvalue densities $\tilde{\rho}_1(\lambda)$ and $\tilde{\rho}_2(\lambda)$ corresponding to the two one-matrix integrals (3.3) with $\alpha = \alpha_1$ and $\alpha = \alpha_2$. Similarly, we will have two functions $L_1(h)$ and $L_2(h)$ corresponding to (3.7) since the eigenvalue densities $\tilde{\rho}_1(\lambda)$ and $\tilde{\rho}_2(\lambda)$ appear via the matrix $A(\lambda)$ in (3.7). On other hand we have only one density $\rho(h)$ coming from the saddle point of (3.1). In order to solve the saddle point equations it is natural to follow the same strategy as in [5] and make an educated guess about the analytic structure of the functions involved and then show the self-consistency of the solution. Since we have two functions $L_i(h)$, associated with the same $\rho(h)$ but different $\tilde{\rho}(\lambda)$'s, and the cut of $L(h)$ from $[h_3, \infty[$ can be traced to the saddle point equation for $\tilde{\rho}(\lambda)$ (see [5] for a discussion), it is natural to assume that $L_1(h)$ and $L_2(h)$

[3] As already mentioned an explicit solution of the asymmetric ABAB model has been published while this manuscript was being completed [8].

have a cut from $[0,h_2]$ (with $\rho(h)=1$ in $[0,h_1]$), and that they have separate cuts $[h_3^{(1)},\infty[$ and $[h_3^{(2)},\infty[$. In the case where $\beta=0$ this structure is indeed realized [4].

We can now write down the generalization of (3.9)–(3.10)

$$\Re[D_1(h)] \;=\; \log\frac{h-h_1}{h^2\beta} + \Re[f(h)] \qquad \forall h \in I_0 \equiv [h_1,h_2]\,, \tag{4.1}$$

$$\Re[D_1(h)] \;=\; \log\frac{h-h_1}{h^2\alpha_1} \qquad \forall h \in I_1 \equiv [h_3^{(1)},\infty[\,, \tag{4.2}$$

and

$$\Re[D_2(h)] \;=\; \log\frac{h-h_1}{h^2\beta} - \Re[f(h)] \qquad \forall h \in I_0 \equiv [h_1,h_2]\,, \tag{4.3}$$

$$\Re[D_2(h)] \;=\; \log\frac{h-h_1}{h^2\alpha_2} \qquad \forall h \in I_2 \equiv [h_3^{(2)},\infty[\,. \tag{4.4}$$

In (4.1)–(4.4) the D's are related to the L's and the function H as below Eq. (3.8):

$$D_i(h) = 2L_i(h) - H(h) - 3\log h + \log(h-h_1), \qquad i=1,2, \tag{4.5}$$

where the subtractions of the log's are made to ensure that the functions D_i's have square root cuts. As in the case of a single α, we assume that $L_i(h) = F_i(h) + H(h)$ where $F_i(h)$ is analytic on the cut I_0 of $H(h)$. The function

$$f(z) \equiv F_1(z) - F_2(z) = L_1(z) - L_2(z) \tag{4.6}$$

is at this point unknown, but we can write $\Re[f(z)]=f(z)$ on I_0. If we assume $f(z)$ is known on I_0, Eqs. (4.1)–(4.2) and (4.3)–(4.4) are standard singular integral equations of the Hilbert type and can readily be solved and one can write

$$D_1(z) \;=\; D_1^{kz}(z) + r_1(z)\oint_{I_0}\frac{dt}{2\pi i}\frac{f(t)}{(z-t)r_1(t)}\,, \tag{4.7}$$

$$D_2(z) \;=\; D_2^{kz}(z) - r_2(z)\oint_{I_0}\frac{dt}{2\pi i}\frac{f(t)}{(z-t)r_2(t)}\,, \tag{4.8}$$

where $D_{1,2}^{kz}(z)$ are given by formula (3.11) with $\alpha, h_3 = \alpha_1, h_3^{(1)}$ and $\alpha, h_3 = \alpha_2, h_3^{(2)}$, respectively.

[4] And this is also cut-structure assumed in the solution of the asymmetric ABAB matrix model reported in [8].

We have

$$\Im[D_k(h)] = -i\pi\rho(h), \qquad k = 1, 2\,. \tag{4.9}$$

Therefore, the "imaginary" parts of Eqs. (4.7) and (4.8) are

$$i\pi\rho(h) \;=\; i\pi\rho_1^{kz}(h) - r_1(h) \fint_{I_0} \frac{dt}{\pi i} \frac{f(t)}{(t-h)r_1(t)}\,, \qquad \forall h \in I_0 \tag{4.10}$$

$$i\pi\rho(h) \;=\; i\pi\rho_2^{kz}(h) + r_2(h) \fint_{I_0} \frac{dt}{\pi i} \frac{f(t)}{(t-h)r_2(t)}\,, \qquad \forall h \in I_0 \tag{4.11}$$

where $\fint$ is the principal value of the integral. Eqs. (4.10)–(4.11) determine $f(z)$ and $\rho(h)$ in terms of the densities ρ_1^{kz}, ρ_2^{kz} (Eq. (4.17)), corresponding to $\alpha = \alpha_1$, $h_3 = h_3^{(1)}$ and $\alpha = \alpha_2$, $h_3 = h_3^{(2)}$.

We can obtain an equation for $f(z)$ by subtracting (4.10) and (4.11):

$$i\pi(\rho_1^{kz}(h) - \rho_2^{kz}(h)) = \fint_{I_0} \frac{dt\; f(t)}{\pi i(t-h)} \left(\frac{r_1(h)}{r_1(t)} + \frac{r_2(h)}{r_2(t)} \right)\,. \tag{4.12}$$

4.1. Uniqueness of the solution

Let us discuss the solution of (4.12). It is a singular integral equation. In order to bring it into a standard form of singular integral equations, and for convenience of later applications, we introduce the notation

$$\Delta\rho^{kz}(h) \equiv \frac{i\pi}{2} \Big(\rho_2^{kz}(h) - \rho_1^{kz}(h)\Big), \tag{4.13}$$

and

$$(t-h)\,k(h,t) \equiv \frac{1}{2} \left(\sqrt{\frac{h - h_3^{(1)}}{t - h_3^{(1)}}} + \sqrt{\frac{h - h_3^{(2)}}{t - h_3^{(2)}}} - 2\sqrt{\frac{h - h_3}{t - h_3}} \right)\,, \tag{4.14}$$

where h_3 in following always will refer to the value

$$h_3 = \frac{1}{2}(h_3^{(1)} + h_3^{(2)}), \tag{4.15}$$

and the function $r(h)$ will refer to (3.12) with h_3 given by (4.15).

The function $k(h,t)$ is regular at $h = t$. Let us further introduce

$$\Delta\rho_r(h) \equiv \frac{\Delta\rho^{kz}(h)}{2r(h)}, \qquad f_r(t) \equiv \frac{f(t)}{r(t)}\,. \tag{4.16}$$

We can now write Eq. (4.12) as

$$\fint_{I_0} \frac{dt}{\pi i} \; \frac{f_r(t)}{t-h} + \int_{I_0} \frac{dt}{\pi i} \; k(h,t) f_r(t) = \Delta\rho_r(h) \,, \tag{4.17}$$

where only the first integral is singular. The so-called *dominant part* of this singular integral equation is given by

$$\fint_{I_0} \frac{dt}{\pi i} \frac{\tilde{f}_r(t)}{t-h} = \Delta\rho_r(h) \,. \tag{4.18}$$

This equation has precisely one zero mode:

$$\tilde{f}_r(t) = \frac{1}{r_0(t)} \fint_{I_0} \frac{dh}{\pi i} \; \frac{r_0(h)\Delta\rho_r(h)}{h-t} + \frac{C}{r_0(t)} \,, \tag{4.19}$$

where $r_0(t) = \sqrt{(t-h_1)(t-h_2)}$. Expressed in terms of $f(t)$ we have

$$\tilde{f}(t) = \sqrt{t-h_3} \left(\fint_{I_0} \frac{dh}{\pi i} \; \frac{\Delta\rho(h)}{(h-t)\sqrt{h-h_3}} + C \right) \,. \tag{4.20}$$

By moving the k-term in Eq. (4.17) to the rhs, we can repeat the steps leading to (4.20), and moving the k-term back to the lhs we finally obtain:

$$f(t) + \int_{I_0} \mathrm{d}s N(t,s)\, f(s) = \sqrt{t-h_3} \left(\fint_{I_0} \frac{dh}{\pi i} \frac{\Delta\rho(h)}{(h-t)\sqrt{h-h_3}} + C \right) \,, \tag{4.21}$$

where the kernel $N(t,s)$ is a *Fredholm kernel*:

$$N(t,s) = -\frac{\sqrt{t-h_3}}{r(s)} \fint_{I_0} \frac{\mathrm{d}h}{\pi^2} \; \frac{k(h,s) r_0(h)}{h-t} \,. \tag{4.22}$$

In general the solution to the Fredholm equation (4.21) will be unique [10]. *We thus have a one-parameter family of solutions* $f_C(t)$.

In order to determine the *four* parameters $h_1, h_2, h_3^{(1)}, h_3^{(2)}$ and the constant C we need *four* boundary conditions and one more condition, which in this case will be the normalization condition for ρ:

$$h_1 + \int_{h_1}^{h_2} \rho(t) dt = 1 \,. \tag{4.23}$$

The boundary conditions are again obtained by the requirement that large h asymptotics of $D_j(h)$ contains no $h^{1/2}$ term while the coefficient of the $h^{-1/2}$ term is $(-\alpha_j)^{-1/2}$. As we show below, consistency of the boundary conditions to the order studied in this paper permits a simple elimination of the parameter C. Thus (3.18) is replaced by the four equations:

$$W^{(j)} + (-1)^j \int_{I_0} \frac{dt}{\pi} \frac{f(t)}{r_j(t)} \;=\; 0\,, \tag{4.24}$$

$$\Omega^{(j)} + (-1)^j \int_{I_0} \frac{dt}{\pi} \frac{f(t)t}{r_j(t)} \;=\; \frac{1}{\sqrt{\alpha_j}}\,, \tag{4.25}$$

where $j = 1, 2$, $W^{(j)} = W^{(j)}(h_1, h_2, h_3^{(j)})$ and $\Omega^{(j)} = \Omega^{(j)}(h_1, h_2, h_3^{(j)})$. In Eqs. (4.24) and (4.25) the function $f(t)$ is the solution to (4.21) for a particular value of C.

The set of boundary conditions means that in the asymmetric case a theory is defined on a three-dimensional hyper-surface in the four-dimensional parameter space $(h_1, h_2, h_3^{(1)}, h_3^{(2)})$. Singular points of the map between parameters $(\alpha^{(1)}, \alpha^{(2)}, \beta)$ and some particular parametrization of this hyper-surface (say in terms of $(h_2, h_3^{(1)}, h_3^{(2)})$, after eliminating h_1) will correspond to critical points of the theory.

5. Expanding around $\alpha_1 = \alpha_2$

We do not need to solve the asymmetric ABAB model explicitly. As explained above, in the context of 3d quantum gravity we need only to study the infinitesimal variation around the symmetric point $\alpha_1 = \alpha_2 = \bar{\alpha}$, $\beta = \bar{\beta}$ to which corresponds the h_i values $\bar{h}_1, \bar{h}_2, \bar{h}_3$.

Let us introduce the notation

$$\alpha \equiv \sqrt{\alpha_1 \alpha_2} = \bar{\alpha} - \delta\alpha, \qquad h_3 = \frac{h_3^{(1)} + h_3^{(2)}}{2}\,. \tag{5.1}$$

We now assume that α_1 is close to α_2, *i.e.* the two branch points $h_3^{(1)}$ and $h_3^{(2)}$ are close, and expand in the difference. It is convenient to write

$$h_3 = \bar{h}_3 + \Delta, \qquad h_3^{(1)} = \bar{h}_3 + \delta_1 = h_3 + \delta, \qquad h_3^{(2)} = \bar{h}_3 + \delta_2 = h_3 - \delta, \tag{5.2}$$

where

$$\Delta = \frac{\delta_1 + \delta_2}{2}, \qquad \delta = \frac{\delta_1 - \delta_2}{2}, \tag{5.3}$$

as well as

$$\log\alpha^{(1)} = \log\alpha + \epsilon, \qquad \log\alpha^{(2)} = \log\alpha - \epsilon. \tag{5.4}$$

In the following we will study the the relation between ϵ, δ and Δ imposed by the boundary conditions (4.24) and (4.25), since this is what determines the continuum physics.

5.1. Finding $f(t)$ and $D_{1,2}(h)$

Recall the integral equation (4.21) which determines $f(t)$. The function $k(t,h)$ has an expansion

$$k(t,h) = \delta^2 k_1(t,h) + \delta^4 k_2(t,h) + \cdots\,. \tag{5.5}$$

Thus knowing $\Delta\rho(h)$ allows us to calculate $f(t)$ perturbatively in δ. We can also expand $\Delta\rho(h)$ in δ and ε. This makes the integration on the LHS of Eq. (4.21) possible order by order in terms of elementary functions. The expansion of $\Delta\rho(h)$ is based on the following two observations: first we have

$$D^{kz}(h;\alpha+\varepsilon) = D^{kz}(h;\alpha) + \varepsilon\mathcal{G}(h), \tag{5.6}$$

$$\mathcal{G}(h) = -\frac{\varepsilon}{i\pi}\, r(h) \int_{h_3}^{\infty} \mathrm{d}h\, \frac{1}{(h-h')r(h')}. \tag{5.7}$$

Next $D^{kz}(h; h_3+\delta)$ and $\mathcal{G}(h)$ have expansions of the form (3.16). Thus $\Delta\rho^{kz}(h)$ has an expansion

$$\Delta\rho^{kz}(h) = (a_1\delta + b_1\varepsilon) + (a_2\delta + b_2\varepsilon)\delta^2 + \cdots\,, \tag{5.8}$$

where only b_1 is not an elementary function of h. Explicitly we have

$$\Delta\rho^{kz}(h) = \left(-\delta\frac{r(h)}{h-h_3}F_3(h_3) - \epsilon\mathcal{G}(h,h_3)\right) + \cdots\,. \tag{5.9}$$

Here and in the following we use the shorthand notation $F_3(h_3)$ for the function $F_3(h_1,h_2,h_3,\alpha,\bar{\beta})$. The integral in (4.21) can now be performed and one obtains

$$f(t) = \left(C'\sqrt{t-h_3} + \delta\frac{r_0(h_3)}{\sqrt{t-h_3}}F_3(h_3) - \epsilon\right) + \cdots\,, \tag{5.10}$$

where $C' = C + \delta F_3(h_3) + \epsilon\frac{1}{i\pi}\int_{h_3}^{\infty}\frac{dh'}{r(h')}$.

Finally we can insert $f(t)$ in (4.7) and (4.8) to obtain $D_{1,2}(h)$. To order in ε and δ^2 we obtain

$$
\begin{aligned}
&D_j(h; h_1, h_2, h_3^j; \alpha_j, \bar{\beta}) = D^{kz}(h; h_1, h_2, h_3; \alpha, \bar{\beta}) \\
&\mp\varepsilon \pm C'\left(\sqrt{h-h_3} + O(\delta)\right) \pm \delta\, \frac{ir_0(h_3)F_3(h_3)}{2\sqrt{h-h_3}} \\
&+\frac{\delta^2}{4}\frac{ir_0(h)}{\sqrt{h-h_3}}\left(\frac{\mathrm{d}F_3(h_3)}{\mathrm{d}h_3} + \frac{F_3(h_3)}{2(h-h_3)} + \frac{F_3(h_3)}{2(h_3-h_1)} + \frac{F_3(h_3)}{2(h_3-h_2)}\right)\ldots \quad (5.11)
\end{aligned}
$$

Notice that at this order ε only appears as a constant to ensure the behavior $\log(h-h_1)/h^2\alpha_j$ at the cut of $D_j(h)$.

5.2. The critical surface

The form of the boundary conditions (4.23), (4.24) and (4.25) to the order δ^2 can easily be obtained from the expansion above. From (4.9) we get

$$\frac{1}{2}(\Im[D_1(h) + D_2(h)] = -i\pi\rho(h), \quad h \in I_0, \tag{5.12}$$

which implies that

$$\rho(h) = \rho^{kz}(h; h_1, h_2, h_3; \alpha, \bar{\beta}) + O(\delta^2) \tag{5.13}$$

and the form of the $O(\delta^2)$ corrections can be explicitly obtained from (5.11).

The generic large h behavior of $D_j(h)$ is, as in the symmetric case,

$$D_j(h) \sim iW_j h^{1/2} - \log \alpha_j h + i\Omega_j h^{-1/2} + O(1/h)\,. \tag{5.14}$$

By the same argument we must have $W_j = 0$ and $\Omega_j = 1/\sqrt{\alpha_j}$. From the boundary conditions it follows that $C' = 0$ to this order since otherwise it would be impossible to satisfy the large h asymptotics for both $D_j(h)$. In effect we get

$$
\begin{aligned}
W_j = W \;&=\; W^{kz}(h_1, h_2, h_3, \alpha, \bar{\beta}) \qquad (5.15)\\
&+\frac{\delta^2}{4}\left(\frac{\mathrm{d}F_3(h_3)}{\mathrm{d}h_3} + \frac{F_3(h_3)}{2(h_3-h_1)} + \frac{F_3(h_3)}{2(h_3-h_2)}\right)\cdots = 0.
\end{aligned}
$$

In a similar way we obtain explicit expressions for Ω_j to this order in δ.

As in the symmetric case we expect the critical behavior to be signaled by the change in the generic behavior of $D_j(h)$ when $h \to h_3^{(j)} = h_3 \pm \delta$ (phase A) or when $h \to h_2$ (phase B). At a first glance the expansion (5.11) looks singular when $h \to h_3$. It is however a simple exercise to prove that the generic behavior

$$D_j(h) = C_j \left(h - h_3^{(j)}\right)^{1/2} + O\left(\left(h - h_3^{(j)}\right)^{3/2}\right) \tag{5.16}$$

is preserved and singular terms can be absorbed into the expansion of $h_3^{(j)}$ around h_3. The coefficients C_j have a non-trivial dependence on j (the expansion in δ starts with $O(\delta)$) and consequently there are two possible hypersurfaces $C_j = 0$ in the parameter space. These two hypersurfaces cross along a line in the parameter space, corresponding to $\delta = 0$.

For $h \to h_2$ situation is different. Here the generic behavior is

$$D_j(h) = \tilde{C}_j (h - h_2)^{1/2} + O\left((h - h_2)^{3/2}\right). \tag{5.17}$$

To the order presented in this paper we find that $\tilde{C}_j = \tilde{C}$ is a symmetric function of δ and in consequence we have a unique critical hyper-surface.

To illustrate the difference between the two phases let us consider the effect of an infinitesimal shift in α_i and $\bar{\beta}$ on $h_1, h_2, h_3^{(i)}$. Let us consider the neighborhood of a symmetric solution where $\delta h_1,\ \delta h_2,\ \delta$ and Δ are all of the same order. It is convenient to parametrize the displacement by $\delta\alpha$ and ϵ

$$\begin{pmatrix} F_1 & F_2 & F_3 & 0 \\ \tilde{g}_1 F_1 & \tilde{g}_2 F_2 & \tilde{g}_3 F_3 & 0 \\ E_1 F_1 & E_2 F_2 & E_3 F_3 & 0 \\ 0 & 0 & 0 & \tilde{g}_4 F_3 \end{pmatrix} \begin{pmatrix} \delta h_1 \\ \delta h_2 \\ \Delta \\ \delta \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ 0 \end{pmatrix} \delta\alpha + \begin{pmatrix} 0 \\ 0 \\ 0 \\ A_4 \end{pmatrix} \epsilon + \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{pmatrix} \delta\bar{\beta}. \tag{5.18}$$

which should be compared with (3.29) for the symmetric case. The Jacobian of this transformation is proportional to $F_1 F_2 F_3^2$ and is in general nonzero, except at critical points where either $F_3 = 0$ (phase A) or $F_2 = 0$ (phase B).

In phase A we can choose a particular direction in $(\delta\alpha, \delta\bar{\beta}, \epsilon)$ space along which the rank of (5.18) is two. The corresponding value of ϵ must be zero in this case and the direction follows the one-dimensional critical line of the symmetric case. For any other direction in $(\delta\alpha, \delta\bar{\beta}, \epsilon)$ space we need higher-order terms in the expansion and we get

$$\begin{aligned} \delta h_1,\ \delta h_2 &\sim \delta\alpha, \delta\beta, \\ \Delta^2 + \delta^2 &\sim a\delta\alpha + b\delta\bar{\beta} = \delta\tilde{\alpha}, \\ \Delta \cdot \delta &\sim \epsilon. \end{aligned} \tag{5.19}$$

These relations can be diagonalized to give

$$(\Delta \pm \delta)^2 \sim A\delta\tilde{\alpha} \pm B\epsilon \tag{5.20}$$

representing the critical behavior related to the approach to two critical surfaces discussed above.

The situation is different in phase B. In this case we can chose a two-parameter family of directions in the $(\delta\alpha, \delta\bar{\beta}, \epsilon)$ space, where the rank of (5.18) is three. This two-dimensional object corresponds to displacements along the critical surface of the B phase. For a displacement not in this plane we need again higher order terms proportional to δh_2^2. The generic situation in this case is

$$\delta h_1,\ \Delta,\ \delta \sim \delta\alpha, \delta\bar{\beta}, \epsilon, \qquad \delta h_2^2 \sim \delta\alpha, \delta\bar{\beta}, \epsilon. \tag{5.21}$$

5.3. The scaling relations and renormalization of 3D gravity

We want to relate the scaling limit of the matrix model discussed above to the continuum limit of the discretized 3D gravity.

(Euclidean) quantum field theories can be defined as the scaling limit of suitable discretized statistical theories. The continuum coupling constants are then defined by a specific approach to a critical point of the statistical theory. Different approaches to the critical point might lead to different coupling constants or even different continuum theories. In our case we want to show that it is possible to approach a critical point in such away that the canonical scaling expected from a theory of 3D gravity is reproduced.

A natural parametrization of the asymmetric model, as discussed above, is in terms of four parameters $h_1, h_2, h_3^{(i)}$ and the theory is defined on a three-dimensional hyper-surface in this parameter space. Let us consider a line going through a specific point $h_1^c, h_2^c, h_3^{(i)} = h_3^c$ on a critical surface of the model and parametrized by a dimension-full parameter a. This curve can be mapped on a corresponding curve in the (α_i, β) space and we are interested in a behavior

$$\begin{aligned} \log\alpha_i &= \log\alpha_c + \frac{c_1 a}{G_N} + Z_i a^2 + \frac{c_2 \Lambda}{G_N} a^3\,, \\ \log\beta &= \log\beta_c + \frac{c_3 a}{G_N} + \frac{c_4 \Lambda}{G_N} a^3 \end{aligned} \tag{5.22}$$

for $a \to 0$. In (5.22) $\log\alpha_c,\ \log\beta_c$ represent an additive renormalization of the inverse gravitational constant and G_N is a renormalized gravitational constant. Similarly Z_i will have an interpretation of the renormalized surface cosmological constants and Λ of the cosmological constant.

In the context of 3D gravity $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$ is only a technical devise in order to obtain information about the transfer matrix. Thus we are led to consider critical points with $\alpha_1 = \alpha_2$. These fall in two classes: phase A and phase B, which look quite different viewed from the larger surface of the asymmetric ABAB matrix model, as described above. As discussed in [1] only phase A is relevant for conventional canonical quantum gravity. In this way we are naturally led to consider a critical point lying on the part of the critical line corresponding to phase A. As discussed in the preceding section this line is an intersection of two critical surfaces, where a particular combination $\Delta \pm \delta$ vanishes. Only on this line we may expect both the in- and out- states of the transfer matrix to have a non-trivial scaling in the continuum limit.

As shown above a generic behavior for infinitesimal deviations around this point is given by (5.19). This is clearly not consistent with the scaling (5.22). The only possible direction consistent with the lowest-order scaling is exactly along a critical line $\alpha_i = \alpha_c(\beta)$. This means that the ratio c_1/c_3 is a uniquely defined function of a position on a critical line. To this order the asymmetry $\alpha_1 - \alpha_2 = 0$.

Changing a position of a point along the critical line corresponds in this interpretation to a finite change of the renormalized gravitational constant, or in other words to a change of the scale parameter a. Consequently this should be irrelevant for the continuum properties of the theory and we should get the same continuum limit independently of the particular choice of a point on the critical line $\alpha_i = \alpha_c(\beta)$ as long as we stay in phase A. The critical line itself in the neighborhood of a critical point admits a parametrization in terms of the scale parameter a. By a suitable redefinition of the scale parameter we can simplify (5.22) to

$$\delta\beta = \beta - \beta_c = \frac{a\beta_c}{G_N} \tag{5.23}$$

shifting the a dependence of the curve to the β dependence.

The scaling limit of a theory is defined by the approach to a critical point along the scaling curve $\alpha_i = \alpha_i(\delta\beta)$ with $\delta\beta$ parametrized by (5.23). The scaling curve $\alpha_i = \alpha_i(\delta\beta)$ touches the critical line $\alpha_i = a_c(\beta)$ at $\beta = \beta_c$, corresponding to $\delta\beta = 0$. At this point the tangents of the two curves have to coincide in order for (5.19) and (5.22) to be consistent, and the curve $\alpha_i = \alpha_i(\Delta\beta)$ deviates from the critical line $\alpha_i = \alpha_c(\beta)$ only by higher order terms

$$\delta\alpha_i = \alpha_i(\delta\beta) - \alpha_c(\delta\beta) \propto Z_i a^2. \tag{5.24}$$

The deviation of the scaling curve from the critical line will determine the scaling limit of the theory through the singular behavior of the free energy $F(\alpha_i, \beta)$. Let us split $\delta\alpha_i$ into a symmetric part $\delta\alpha$ and asymmetric part ϵ. From (5.20) we get

$$(\Delta \pm \delta)^2 \propto A\delta\alpha \pm B\epsilon. \tag{5.25}$$

This quantity becomes singular when $A\delta\alpha \pm B\epsilon = 0$, corresponding to the two critical surfaces in the A phase. Let us recall that we discuss here a critical behavior of the quantity $F(\alpha_i, \beta)$ closely related to the transfer matrix. The two possible critical limits correspond to either the in- or out-state of the transfer matrix becoming critical. However, we are interested in a limit which *ensures* that the in- and the out-states become critical simultaneously. This can only be done if $\delta\alpha$ and ϵ scale differently, an attractive possibility being

$$\begin{aligned} \delta\alpha &\propto a^2 \,, \\ \epsilon &\propto a^3 \,, \end{aligned} \tag{5.26}$$

and consequently

$$\begin{aligned} \Delta &\propto a \,, \\ \delta &\propto a^2 \,. \end{aligned} \tag{5.27}$$

Notice that this means that asymmetry ϵ contributes at exactly the same level as the cosmological term Λ. The scaling curve corresponding to this particular choice would deviate from the symmetry plane $\alpha_1 = \alpha_2$ only by $O(a^3)$ terms. As we shall discuss in the next section this scaling is also attractive when expanding the transfer matrix in powers of a in order to extract the Hamiltonian of 3D quantum gravity.

We should note here that the limit $Z_i \to 0$ is singular. This corresponds to a symmetric deviation and we expect only a contribution from the cosmological term in the scaling analysis. Since in this limit $\delta = 0$ we necessarily have

$$\Delta^2 \propto a^3 \Lambda G_N. \tag{5.28}$$

We discussed this limit in [7].

6. The transfer matrix

The transfer matrix contains the information necessary to derive the Hamiltonian of 3D gravity. From the free energy of the asymmetric ABAB model we can extract some information about the transfer matrix as is clear from formula (2.1). Let us be more precise about this (see [1] for a detailed discussion). The free energy of the asymmetric ABAB matrix model involves according to (2.1) a summation over the individual geometric states $|g\rangle$ which label in- and out-states. However, one can use the free energy to extract information about the the areas $N_{\rm in}$ and $N_{\rm out}$ (the number of squares in the in- and out-quadrangulations) of the in- and out-states $|g_{\rm in}\rangle$ and $|g_{\rm out}\rangle$. We expect this quantity to capture the essential part of physical information about the time evolution of a two-dimensional universe (*cf. e.g.* [12]). Let us consider the state

$$|N\rangle = \frac{1}{\sqrt{\mathcal{N}(N)}} \sum_{g_t} \delta_{N,N(g_t)} |g_t\rangle \,, \tag{6.1}$$

where $\mathcal{N}(N)$ is the number of quadrangulations of given area N. The norm of such state is

$$\langle N'|N\rangle = \delta_{N,N'} \,, \tag{6.2}$$

since states $|g_1\rangle$ and $|g_2\rangle$ with different quandrangulations are orthogonal. The number of quadrangulations constructed from N squares grows exponentially as

$$\mathcal{N}(N) = N^{-7/2}\, \mathrm{e}^{\mu_0 N}(1 + O(1/N^2)), \tag{6.3}$$

where μ_0 is known. The sum 2.1 can now be written as

$$F(\alpha_1,\alpha_2,\beta) = \sum_{N_t,N_{t+a}} \mathrm{e}^{-z_t N_t - z_{t+a} N_{t+a}} \langle N_{t+a}|\hat{T}|N_t\rangle \sqrt{\mathcal{N}(N_t)\mathcal{N}(N_{t+a})}. \tag{6.4}$$

The exponential part of $\sqrt{\mathcal{N}(N_t)(N_{t+a})}$ can be absorbed in additive renormalizations of the boundary cosmological constants z_t and z_{t+a} (*i.e.* additive renormalizations of $\log\alpha_i$, recall (2.1)). It follows that in the scaling limit, *i.e.* for large N where we can use (6.3), the Laplace transforms of the matrix elements $\langle N_1|\hat{T}|N_2\rangle$ are equal to the "7/2" fractional derivative[5] of the free energy $F(\alpha_1,\alpha_2,\beta)$:

$$\sum_{N_t,N_{t+a}} \mathrm{e}^{-z_t N_t - z_{t+a} N_{t+a}} \langle N_{t+a}|\hat{T}|N_t\rangle = \left(\frac{\partial}{\partial z_t}\frac{\partial}{\partial z_{t+a}}\right)^{7/4} F(\alpha_1,\alpha_2,\beta)\,, \tag{6.5}$$

where $\frac{\partial}{\partial z} = \frac{\partial}{\partial \log\alpha}$ as is clear from (2.1).

The scaling limit, and thus the continuum physics, is determined by the singular part of the free energy. The leading behavior of of this singular part when we approach a critical point as described in the previous Section, is given by $F(\alpha_1,\alpha_2,\beta) \propto (\delta\alpha)^{5/2}$. It is now straightforward to apply (6.5) and one finds

$$\left(\frac{\partial}{\partial \log\alpha_1}\right)^{7/4} \left(\frac{\partial}{\partial \log\alpha_2}\right)^{7/4} F(\alpha_1,\alpha_2,\beta)\alpha \approx \frac{1}{\delta\alpha}. \tag{6.6}$$

This is exactly the leading-order behavior we expect for the transfer matrix when $a \to 0$ from (1.1):

$$\langle N_1|\mathrm{e}^{-a\hat{H}}|N_2\rangle \to \langle N_1|\hat{I}|N_2\rangle = \delta_{N_1,N_2}, \tag{6.7}$$

[5] There are standard ways to define the concept of a fractional derivative, see for instance [11].

and the Laplace transform of δ_{N_1,N_2} is (for large N's)

$$\frac{1}{\delta\alpha_1 + \delta\alpha_2} = \frac{1}{a^2(Z_1 + Z_2)}(1 + O(a)). \tag{6.8}$$

It would be very interesting to expand to next order in a and thus obtain information of $\hat{H}$. These terms come from the $O(a^3)$ terms discussed in the previous Section. Of course it would only give us information about the matrix elements related to the states of the form (6.1), but as discussed in detail in [1] we expect that this is the only information relevant in the continuum limit of 3D gravity if the topology of space is spherical.

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