ENSEMBLE OF CAUSAL TREES*

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We discuss the geometry of trees endowed with a causal structure using the conventional framework of equilibrium statistical mechanics. We show how this ensemble is related to popular growing network models. In particular we demonstrate that on a class of *afine* attachment kernels the two models are identical but they can differ substantially for other choice of weights. We show that causal trees exhibit condensation even for asymptotically linear kernels. We derive general formulae describing the degree distribution, the ancestor–descendant correlation and the probability that a randomly chosen node lives at a given geodesic distance from the root. It is shown that the Hausdorff dimension d_{H} of the causal networks is generically infinite.

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1. Introduction

The study of networks is becoming increasingly popular (for a recent review see *e.g.* [1] and also [2]). The main reason for that is an emergence of great wealth of data on Internet, WWW, science citation networks, cell metabolism networks and so on. Most of those networks (if not all) exhibit features that are not explained by the classical theory of random graphs due to Erdos and Renyi [3]. Perhaps the most prominent among those features is the power-like degree distribution. Degree of a vertex is the number of links connected to it. While classical theory predicts a Poissonian distribution for the degree of a vertex in a random graph, in many of naturally occurring networks this distribution was found to have power-like tails. One way to understand this is based on an ancient observation "For unto every one that hath shall be given, and he shall have abundance: but from him that hath not shall be taken away even that which he hath." [4]: a popular web page is more

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probable to attract more links to it, a frequently cited paper is more likely to get more citations and so on. In more modern context this principle was formulated in [5] and adopted to the description of networks in [6] and [7]. This is a *diachronic* approach concentrating on the description of growing networks. This is very natural as most of the networks we encounter are a result of some growth process. The simple models studied in the literature try to capture the essential features of the growth mechanism.

One can look at the networks in a different way that is also quite natural as this is the approach taken in statistical mechanics and probability theory. In this *synchronic* view we treat each network as a single element of a *statistical ensemble* [8]. The ensemble is defined by specifying the "phase space" that is the class of graphs belonging to it and the weight (or probability) for each graph in the ensemble. The probabilities can be assigned *ad hoc* or, what is more interesting, derived from other principles. In particular it is clear that each growing network model defines also a statistical ensemble. The ensemble consists of all the graphs that can be constructed by the specified growth process and the probability assigned to each graph is the probability of constructing given graph. Thus the growth mechanism implicitly defines the probability for each graph. We find it worthwhile to study what kind of ensembles can be obtained from the growing network models [9]. The motivation for this is twofold. First using another "toolbox" one can obtain more insight into original models. Indeed we are in position to make some general statements about the correlation functions in growing random networks (GN) models. Secondly while, as stated, natural networks are usually grown, often we may not have an access to the growth history and we are effectively left with the statistical ensemble approach.

In this contribution which is mostly based on [9] we will describe a statistical ensemble that incorporates the causal structure inherent in the GN models. We will show how it relates to the GN models and derive some results on correlation functions.

2. Causal trees

2.1. Definition

First we review very quickly the growing random network model [7, 10]. In the simplest version we start with a single vertex and then at each stage we attach a new vertex to one of the already existing ones. The probability for attaching the new v_{n+1} vertex to some old one v_i depends *only* on the degree n_i of the node v_i and is proportional to A_{n_i}, which is called *attachment kernel*.

It is clear that this process produces a rooted, labeled tree, each node being labeled by the time at which it was inserted into the network and

the first node being the root. It is also clear that not all the labelings are possible: the label of the father must be smaller then the label of its child. We will call trees that satisfy this condition *causal*.

In order to define the weights we must calculate the probability for constructing a given tree T. To obtain a node t with degree n we must have attached a new node to the node with degree $n - 1$. The probability of this happening at the time t is:

$$P(n|t) = \frac{A_{n-1}}{\sum_{i \in T(t-1)} A_{n_i}}, \tag{1}$$

where $T(t - 1)$ is the tree at the partial stage of the construction: just before attaching the new node t. Unfortunately the normalizing factor in the denominator in general depends on the exact structure of the tree $T(t)$. In consequence the overall probability of building a tree will depend on the way it was constructed, in particular on the labeling. This problem is exemplified in the figure 1. This situation, apart from being impossible to work with, is quite unnatural. What we would like is to have a weight that depends only on the nodes degree and do factorize:

$$\rho(T) = \rho(n_1, \ldots, n_N) = \prod_{i=1}^{N} q_{n_i}. \tag{2}$$

It turns out that there exists a class of GN models that is compatible with the above requirement. Those are the models with *afine* attachment kernels *i.e.* of the form:

$$A_n = n + \omega, \qquad \omega > -1, \tag{3}$$

where ω is a constant. For such kernels the normalization factor depends only on the size of the tree:

$$\sum_i A_{n_i} = \sum_i n_i + \sum_i \omega = 2N - 2 + N\omega. \tag{4}$$

For this class of attachment kernels the choice

$$q_n = q_1 \prod_{k=1}^{n-1} A_k \tag{5}$$

leads to a model *identical* with the original GN model. For other kernels we will still define our model by the formulas (2) and (5). In this situation we can only expect some form of asymptotic or qualitative agreement between the models if any. In fact, as we will show later, the two models can differ significantly even for very simple non-afine kernels.
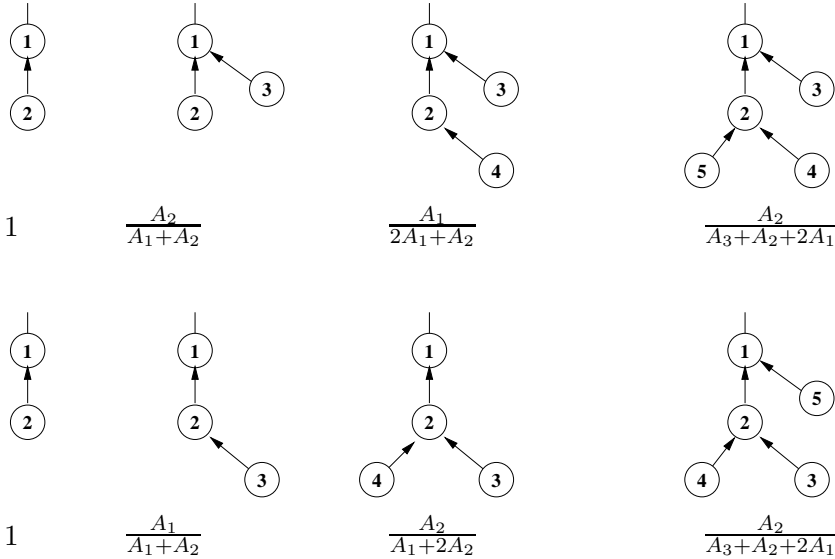
Fig. 1. Two ways of constructing the same (non labeled) tree. The labels below each tree show the probability of obtaining this tree from the precedent one.

2.2. Recursion relation

Most of the properties of the ensemble can be derived from the canonical partition function:

$$z_N = \sum_T \frac{L(T)}{N!} \rho(T) \,, \tag{6}$$

where we sum over all the *non labeled* trees and $L(T)$ is the number of distinct causal labelings of the given tree T.

We start by deriving a recursion relation for $L(T)$. It is convenient at this stage to change to the *planted* ensemble. This amount to attaching to the root vertex an additional link: *a stem*. This does not change in any way the properties of the ensemble in the large N limit but makes the calculations easier. From k planted trees $T_1, \ldots, T_k$ we can construct a new tree $T = T_1 \otimes \cdots \otimes T_k$ by joining together the stems at a new node (see figure 2). The number of causal labelings of the resulting tree is:

$$L(T_1 \oplus \cdots \oplus T_k) = \frac{N!}{N_1! \cdots N_k!} \frac{1}{k!} L(T_1) \cdots L(T_k) \tag{7}$$

with $N_1 + \cdots + N_k = N$. One has to give $N + 1$ labels to the nodes of the compound tree. However, the smallest label must be attached to the root. The remaining N labels are arbitrarily distributed among the trees. This is
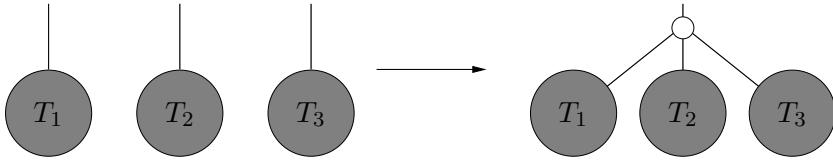
Fig. 2. Operation $T_1, \ldots, T_k \to T_1 \otimes \cdots \otimes T_k$.

the origin of the multinomial factor. Permuting the trees T_i does not change the compound tree. This explains the presence of the factor $1/k!$.

Because of the property (2) the weights of the new tree obviously factorize as:

$$\rho(T_1 \oplus \cdots \oplus T_k) = p_{k+1}\rho(T_1) \cdots \rho(T_k). \tag{8}$$

The partition function Z_{N+1} can be constructed by summing the trees of size smaller or equal to N (see figure 3):
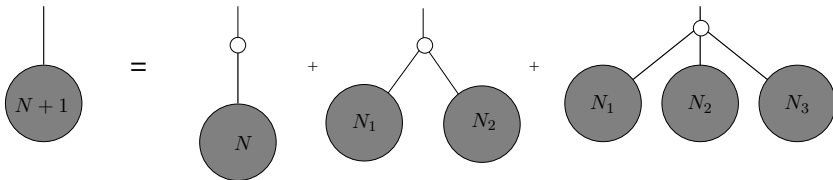

Fig. 3. Recursion relation for z_{N+1}.

$$
\begin{aligned}
z_{N+1} &= \frac{1}{(N+1)!} \sum_{k=1}^{\infty} \sum_{T_1,\ldots,T_k} \delta_{N_1+\cdots+N_k,N} \\
&\times \rho(T_1 \oplus \cdots \oplus T_k) L(T_1 \oplus \cdots \oplus T_k).
\end{aligned} \tag{9}
$$

Inserting (7) and (8) and rearranging the terms we arrive at:

$$Z_{N+1} = \frac{1}{N+1} \sum_{k=1}^{\infty} \frac{p_{k+1}}{k!} \sum_{N_1,\ldots,N_k} \delta_{N_1+\cdots+N_k,N} \prod_{i=1}^{k} z_{N_k}. \tag{10}$$

Adding $z_1 = q_1$ and summing both sides of equation (10) we get

$$\sum_N N z_N \mathrm{e}^{-N\mu} = \mathrm{e}^{-\mu} \left(\sum_{k=0}^{\infty} \frac{q_{k+1}}{k!} Z(\mu)^k \right), \tag{11}$$

where

$$Z(\mu) = \sum_N z_N e^{-\mu N} \tag{12}$$

is the grand-canonical partition function. Finally

$$Z'(\mu) = -e^{-\mu} F(Z) \,, \tag{13}$$

where

$$F(Z) = \sum_{k=1}^{\infty} \frac{q_k}{(k-1)!} Z^{k-1} \,. \tag{14}$$

Equation (13) can be integrated to give

$$e^{-\mu(Z)} = G(Z) \equiv \int_0^Z \frac{dx}{F(x)} \,. \tag{15}$$

The function $G(Z)$ is a positive monotonically growing function of Z, bounded from above (one can ignore the trivial case where all q_n except q_1 and q_2 are zero). Hence μ is bounded from below: $Z(\mu)$ has a singularity at some $\mu = \bar{\mu}$. Denote by $\bar{x}$ the radius of convergence of the series $F(Z)$. The critical value of μ is given by

$$\bar{\mu} = -\log G(\bar{x}) \,. \tag{16}$$

This formula holds also when the radius of convergence $\bar{x}$ is infinite, since all terms in the series (14) are positive and the integral in (16) is convergent in all cases of interest: $G(\infty) < \infty$. Please note that $\bar{\mu}$ is a "free energy" density:

$$\bar{\mu} = \lim_{N \to \infty} \frac{1}{N} \log z_N \,. \tag{17}$$

2.3. Degree distribution

The vertex degree distribution is calculated using:

$$\pi_n = q_n \frac{\partial \bar{\mu}}{\partial q_n} \tag{18}$$

which gives

$$\pi_n = \frac{1}{G(\bar{x})} \frac{q_n}{(n-1)!} \int_0^{\bar{x}} \frac{dx}{F(x)^2} x^{n-1} \,. \tag{19}$$

Again, this formula is also valid when $\bar{x} = \infty$.

Summing over n and using the definitions of F and G one easily checks that π_n is normalized to unity, as it should be. One further finds

$$\sum_n n\,\pi_n = 2 - \underbrace{\frac{\bar{x}}{F(\bar{x})G(\bar{x})}}_{c}. \tag{20}$$

On a tree, the r.h.s. should equal 2. This is the case when $F(x)$ diverges at $x = \bar{x}$ and hence $c = 0$. Otherwise one encounters a pathology (anomaly), which looks similar to that appearing in some maximally random tree models (and in the so-called balls-in-boxes model, see [8, 11]), where in the large N limit one misses singular node(s) contributing term(s) of the type

$$N^{-1}\delta(n - cN). \tag{21}$$

Such nonuniformly behaving terms disappear if one first takes the $N \to \infty$ limit in (20). It will be shown later that the average distance between nodes is finite when $F(\bar{x}) < \infty$. This means that singular node(s) — with unbounded connectivity — are indeed expected to show up.

2.4. Condensation

In order to check if the described anomaly really signals an appearance of a singular vertex we have studied causal trees with the weights

$$q_n = \begin{cases} 1 & n \leq d, \\ (n-d)! & n > d, \end{cases} \tag{22}$$

where d is an integer greater or equal to two, derived using (5) from the *delayed* linear attachment kernel

$$A_n = \begin{cases} 1 & n \leq d, \\ n - d + 1 & n > d. \end{cases} \tag{23}$$

In this case the coefficients of the power series (14) behave like:

$$\frac{(k-d)!}{(k-1)!} \sim \frac{1}{k^{d-1}} \qquad \text{for } k \to \infty. \tag{24}$$

The term c in (19) is obviously not zero and its value is found to be $c \approx 0.584692$. In the figure 4 we have plotted the result of simulations of the model of causal trees with 1000 and 4000 nodes (circles and diamonds). One can clearly see peaks corresponding to the singular node. The vertical
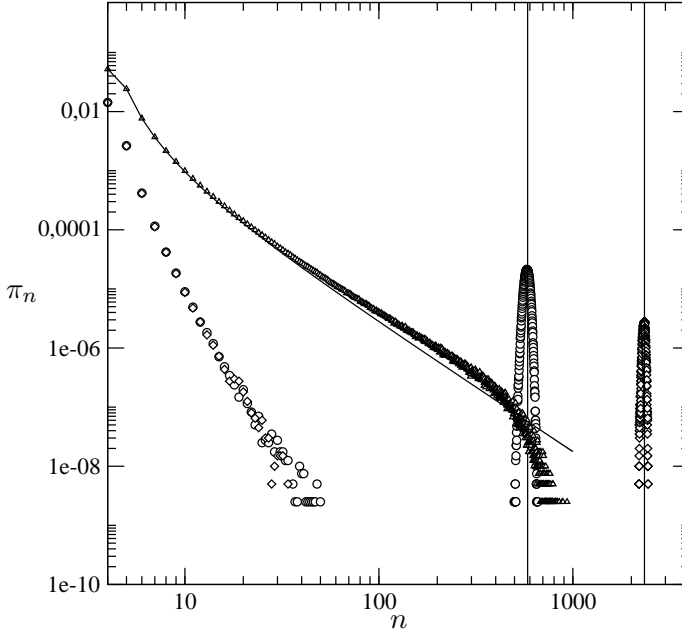
Fig. 4. Vertex degree distribution for causal trees (1000 nodes (circles), 4000 nodes (diamonds)) and GN model (4000 nodes (triangles)) in each case $d = 5$.

lines mark the positions of those peaks predicted from (21). The perfect agreement confirms our statement that the "link deficiency" in (20) signals the existence of a singular node.

It is interesting to ask what is the shape of the vertex degree distribution for the growing random model which has exactly the same kernel (23). Of course, in this case the two models are not identical because the kernel is not afine. The resulting degree distribution for this model is plotted in the figure 4 (triangles) [12]. The continuous line is the approximate solution taken from [10]. The observed discrepancy is due to finite size effects. As we can see the distributions in causal trees and growing networks differ greatly, in particular there is no singular vertex in the GN model for this choice of kernel.

2.5. Degree correlations

Now, we turn to the calculation of the ancestor–descendant degree correlation. It is obvious that an ancestor plays the role of the root of the subgraph involving all its descendants. One can read from (10) the degree

distribution of the root:

$$z_l(N) = \frac{1}{N}\frac{q_l}{(l-1)!} \sum_{N_1,\ldots,N_{l-1}} \delta_{N_1+\cdots+N_{l-1},N-1} \prod_1^{l-1} Z_{N_i} \,. \tag{25}$$

Going over to the grand-canonical ensemble one finds:

$$\frac{\mathrm{d}Z_l(\mu)}{\mathrm{d}\mu} = -\mathrm{e}^{-\mu}\frac{q_l}{(l-1)!}Z^{l-1}(\mu) \tag{26}$$

which, taking (13) into account and after integration yields

$$Z_l\big(\mu(Z)\big) = \frac{q_l}{(l-1)!}\int_0^Z \mathrm{d}x\,\frac{x^{l-1}}{F(x)} \,. \tag{27}$$

Using similar arguments one writes the weight of graphs where the root has the degree l and its daughter the degree k as

$$z_{kl}(N) = \frac{q_l}{N(l-2)!} \sum_{N_1,\ldots,N_{l-1}} \delta_{N_1+\cdots+N_{l-1},N-1}$$
$$\times \prod_{i=1}^{l-2} Z_{N_i} z_k(N_{l-1}) \,. \tag{28}$$

Hence

$$\frac{\mathrm{d}Z_{kl}(\mu)}{\mathrm{d}\mu} = -\mathrm{e}^{-\mu}\frac{q_l}{(l-2)!}Z^{l-2}(\mu)Z_k(\mu) \,. \tag{29}$$

Integrating the above equation one finally obtains

$$Z_{kl}\big(\mu(Z)\big) = \frac{q_l}{(l-2)!}\frac{q_k}{(k-1)!}\int_0^Z \mathrm{d}x_2\,\frac{x_2^{l-2}}{F(x_2)}\int_0^{x_2} \mathrm{d}x_1\,\frac{x_1^{k-1}}{F(x_1)} \tag{30}$$

which is the conditional probability, up to normalization, that a descendant has the degree k when the ancestor's degree is l. The normalization is determined summing over k on the r.h.s. above, with the result $(l-1)Z_l(\mu)$.

2.6. Fractal dimension

Repeating over and over the iteration process leading to Eq. (30) one gets

$$
Z_{k_1 k_2 \ldots k_r}(\mu(Z)) = \prod_{j=2}^{r} \frac{q_{k_j}}{(k_j - 2)!} \frac{q_{k_1}}{(k_1 - 1)!} \int_0^Z \mathrm{d}x_r \frac{x_r^{k_r - 2}}{F(x_r)}
$$

$$
\times \int_0^{x_r} \mathrm{d}x_{r-1} \frac{x_{r-1}^{k_{r-1}-2}}{F(x_{r-1})} \cdots \int_0^{x_2} \mathrm{d}x_1 \frac{x_1^{k_1 - 1}}{F(x_1)} . \tag{31}
$$

Summing over node degrees $k_1, k_2, \cdots, k_r$ one obtains the weight of all graphs with a point separated by r links from the root, *i.e.* the two-point correlation function $C(r, \mu)$ introduced in Sect. 1.2:

$$
C(r, \mu(Z)) = \int_0^Z \mathrm{d}x_r \frac{F'(x_r)}{F(x_r)} \int_0^{x_r} \mathrm{d}x_{r-1} \frac{F'(x_{r-1})}{F(x_{r-1})} \cdots
$$

$$
\times \int_0^{x_3} \mathrm{d}x_2 \frac{F'(x_2)}{F(x_2)} \int_0^{x_2} \mathrm{d}x_1 . \tag{32}
$$

For finite $\bar{x}$, replacing the upper limit of integration over x_1 by $\bar{x}$ and performing all integrations, one gets

$$
C(r, \mu(Z)) \leq \bar{x} \frac{\left(\ln F(Z)\right)^{r-1}}{(r-1)!} . \tag{33}
$$

Hence, the tail of $C(r, \mu)$ falls at least as fast as a Poissonian. Consequently $\langle r \rangle_\mu$ grows at most like $\ln F(Z)$. Assuming that $F(z)$ has at the most a power singularity at $z = \bar{x}$ one concludes that

$$
\langle r \rangle_\mu \leq \mathrm{const} \ln \frac{1}{\delta \mu} \tag{34}
$$

and therefore

$$
\langle r \rangle_N \leq \mathrm{const} \ln N \tag{35}
$$

since $\delta \mu$ scales like N^{-1}. The argument is rather heuristic, but suggestive (see also the examples in the section IIF of Ref. [9]). It appears that generically the causal trees have the *small-world* property $d_\mathrm{H} = \infty$, contrary to the maximum entropy trees whose generic fractal dimension is finite [8, 13, 14]. This phenomenon is easy to understand intuitively: the causal structure

suppresses long branches. This can be seen by noting that along a branch from the root to the leaf no label permutations are possible, hence a tree with a few long branches admits much less causal labelings then a "short fat" one.

3. Summary

We have studied a statistical ensemble of tree graphs endowed with a causal structure. We have derived some general formulas describing the degree distribution, the ancestor-descendant correlation, and the probability that a node lives at a given geodesic distance from the root. Using these last results, we have shown that our causal networks have generically the small-world property *i.e.* their Hausdorff dimension is infinite.

We have shown that our model coincides with the growing random model for an afine class of attachment kernels. Outside this class however the models can wildly differ. In particular we have demonstrated that while condensation of links can be observed in causal trees it is not to be seen in their growing network analogue (*i.e.* for the same weights).

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