THE DIRAC OPERATOR COUPLED TO 2D QUANTUM GRAVITY*

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We implement fermions on 2D dynamical random triangulation and determine the spectrum of the Dirac operator. We study the dependence of the spectrum on the hopping parameter and use finite size analysis to determine critical exponents. The results for regular, for Euclidean and for Lorentzian lattices are briefly presented.

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1. Introduction

In this paper we study properties of the Dirac operator on 2D: regular, Euclidean and Lorentzian lattices. An explicit construction of the Dirac–Wilson and Majorana–Wilson operators on a randomly triangulated plane was introduced in [1]. However this construction was based on a global frame. Such an approach works only for lattices embedded in a flat background. It is possible to generalize the construction by introduction a field of local frames [2–7]. This makes possible to put fermions on a curved manifold with any topology.

The paper is organized as follows: first we recall the definition of the model of fermions interacting with gravity. Then we show the analytical result obtained for regular lattice. Next we present results of Monte-Carlo simulations for Euclidean and for Lorentzian lattices.

2. The model

To construct a model of fermions interacting with gravity one has to properly define fermion field on a curved space. The basic difficulty comes

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from the fact that on such a space generally there is no global frame. Therefore it is impossible to introduce global directions one can associate gamma matrices with. This difficulty can be overcome by introduction a set of local orthonormal frames. This makes possible to define a spin connection that can be used to parallel transport and to calculate derivatives of spinors. When the curved space is represented by a dynamical triangulation this leads to the following Dirac–Wilson action [5]:

$$S = -2K \sum_{\langle ij \rangle} \bar{\Psi}_i \mathcal{H}_{ij} \Psi_j + \sum_i \bar{\Psi}_i \Psi_i \; = \; \sum_{i,j} \bar{\Psi}_i \mathcal{D}_{ij} \Psi_j \, . \tag{1}$$

The fermionic fields Ψ_i are located in the centers of triangles, the sum $\langle ij \rangle$ goes over oriented pairs of neighboring triangles, and K is a hopping parameter. The hopping operator $\mathcal{H}_{ij}$ and the Dirac–Wilson operator $\mathcal{D}_{ij}$ are defined by use of relative positions of local frames in triangles i and j [5]. To get a partition function of the model one has to integrate over fermion field and sum over dynamical triangulations from some class $\mathcal{T}$:

$$\mathcal{Z} = \sum_{T \in \mathcal{T}} \int \prod_i d\bar{\Psi}_i d\Psi_i \; \mathrm{e}^{-S_T} = \sum_{T \in \mathcal{T}} \mathcal{Z}_T \, . \tag{2}$$

The integral over fermion field on a given triangulation defines the partition function $\mathcal{Z}_T$, which provides a weight of triangulation T in the ensemble. The definition of $\mathcal{T}$ depends on a model. We consider here three different cases. At first we consider only flat regular lattice build of equilateral triangles. There is no summation over triangulations. In the second case fermions interact with Euclidean gravity. The sum goes over all triangulations of spherical topology. In the third case we consider Lorentzian, torodoidal lattices with causal structure.

It is convenient to introduce Majorana–Wilson $\widehat{\mathcal{D}}$ operator defined as:

$$\widehat{\mathcal{D}} = i\epsilon \mathcal{D} \, , \tag{3}$$

where ϵ is antisymmetric tensor. It can be shown, that Majorana–Wilson operator is represented by hermitian matrix [6]. Therefore all its eigenvalues are real. We analyze scaling of the lowest eigenvalue to deduce some properties of the geometry.

3. Regular lattice

On the beginning let us consider Dirac operator on a regular flat lattice build of equilateral triangles. The spectra of Dirac–Wilson and Majorana–Wilson operators can be calculated analytically. Diagonalization of action matrix (1) leads to the following eigenvalues:

$$\lambda = 1 \pm K\sqrt{3}\sqrt{w \pm i\sqrt{4-(w-1)^2}}\,, \tag{4}$$

where:

$$w = \cos(p_1) + \cos(p_2) + \cos(p_1 - p_2)\,. \tag{5}$$

The distribution of eigenvalues on a finite lattice with periodic boundary conditions is shown in figure 1. In this case momenta admit the values $p_{1,2} = \frac{2\pi}{L}(k_{1,2} + \frac{1}{2})$, where $k_{1,2}$ are integer numbers and L is linear extent of the lattice. Adding 1/2 to the $k_{1,2}$ corresponds to antiperiodic boundary conditions. Spectrum in figure 1 was evaluated for the hopping parameter $K = 1/3$. For this value the oval shape formed by eigenvalues touches with

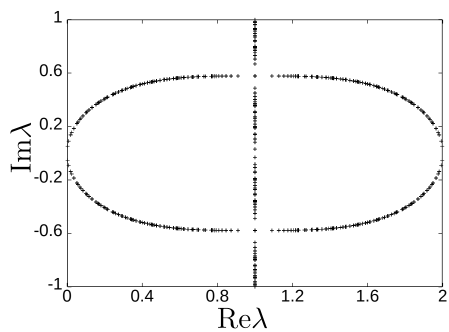

Fig. 1. Eigenvalues of the Dirac–Wilson operator on a 2D regular lattice build of 800 equilateral triangles with periodic boundary conditions. The hopping parameter is equal to $K = 1/3$.

the left edge the origin $(0, 0)$. The smallest modules of eigenvalues correspond to momentum $p_1 = p_2 = \pi/L$ for which $w = 1 + 2\cos(\pi/L)$. When the lattice size goes to infinity $L \to \infty$, parameter $w \to 3$ and the smallest eigenvalue approaches origin. For antiperiodic boundary conditions zero eigenvalue appears in the spectrum for every lattice size when $p_1 = p_2 = 0$ and $w = 3$. Similarly one can find eigenvalues of the Majorana–Wilson operator $\widehat{\mathcal{D}}$:

$$\widehat{\lambda} = \pm\sqrt{1 + K^2(w+6) \pm \sqrt{(K^2(w-3)+4)^2 + 36K^2 - 16}}\,. \tag{6}$$

It can be shown that for real K and $w \in [-3, 3]$ function (6) is equal zero only for $K = 1/3$ and $w = 3$. Thus critical value of the hopping parameter K on a regular lattice equals $K_{\rm cr} = 1/3$. When $K \neq K_{\rm cr}$ spectrum of the Majorana–Wilson operator is truncated up to some point — the smallest eigenvalue that appears in the spectrum. This is a mass gap M. The scaling of the M at critical point is given by:

$$M \simeq \frac{\pi}{3L}\,. \tag{7}$$

4. Euclidean lattice

Let us now replace regular lattice with a set of fluctuating dynamical lattices. In this chapter we consider interaction of fermions with Euclidean gravity. Therefore the class $\mathcal{T}$ in (2) is a set of all spherical triangulations. The weight for each triangulation is given by $\mathcal{Z}_T$. A typical spectrum of the Dirac–Wilson operator is shown in figure 2 (left). When one changes hopping parameter K the spectrum rescales around point $(1,0)$ without noticeable shape change. One can find a pseudocritical value $K = K_*$ for

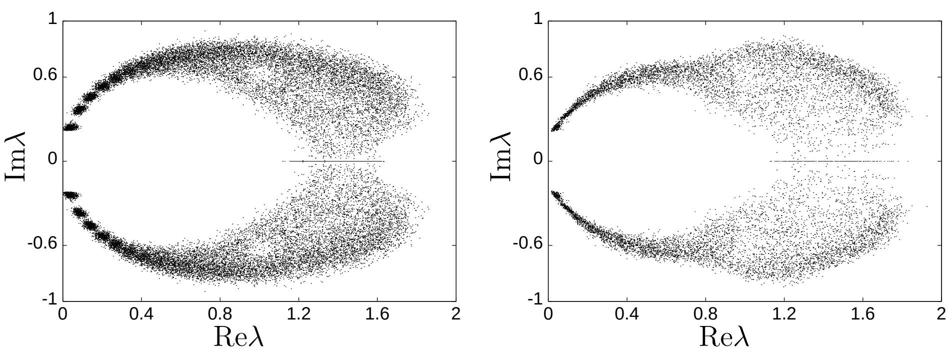

Fig. 2. The average spectrum of the Dirac–Wilson operator on: Euclidean lattice with $N = 64$ triangles for the hoping parameter $K = 0.364$ (left figure), Lorentzian lattice with $N = 64$ triangles and the hopping parameter $K = 0.3486$ (right figure).

which the claw-shaped left end of the spectrum crosses imaginary axis. For this value the smallest (*i.e.* the smallest module) eigenvalues, which appear in the spectrum are minimal. Similar effect can be seen when studying spectrum of the Majorana–Wilson operator. Let us define a mass gap M as a mass center of the smallest eigenvalue density distribution $\rho_0(\lambda)$:

$$M = \int d\lambda \lambda \rho_0(\lambda)\,. \tag{8}$$

The function $M(K)$ has minimum M_* at pseudocritical hopping parameter value K_*. The values K_* and the smallest mass gaps M_* for different lattice sizes N are shown in figure 3. We fit standard scaling functions to data points:

$$M_*(N) \;=\; \frac{b}{N^{\frac{1}{d_{\mathrm{H}}}}}\left(1 + \frac{t}{N}\right)\,, \tag{9}$$

$$K_*(N) = K_\infty - \frac{a}{N^\kappa}\,. \tag{10}$$

Exponent d_{H} is a lattice Hausdorff dimension. A typical linear extent of the

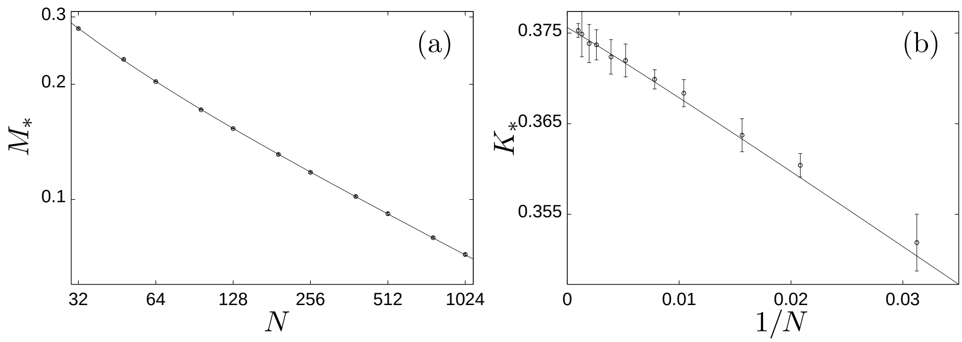

Fig. 3. Minimal mass gap M_* (a) and pseudocritical value K_* (b) scaling with lattice size N. The curves represent the best fit to the (9) and (10) formula: $1/d_{\mathrm{H}} = 0.348(4)$, $b = 0.80(2)$, $t = 5.7(5)$ and $K_\infty = 0.3756(16)$, $\kappa = 1.3(30)$, $a = 0.9(5)$.

lattice scales as $L = N^{1/d_{\mathrm{H}}}$. When a particle mass is equal to zero L sets the typical correlation length. Its inverse defines a mass M. Thus we have $M \sim 1/L = 1/N^{1/d_{\mathrm{H}}}$. Additional finite size correction t/N gives significant improvement of the fit. We get the following parameter values: $d_{\mathrm{H}} = 2.87(3)$, $b = 0.80(2)$, $t = 5.7(5)$. The best fit of the formula (10) leads to values $K_\infty = 0.3756(16)$, $\kappa = 1.03(30)$ and $a = 0.9(5)$. The corresponding curves are plotted in figure 3. The critical hopping parameter K_∞ is in agreement with analytically calculated value $K_{\mathrm{cr}} = 0.37461\ldots$ [5].

5. Lorentzian lattice

In case of 2D Lorentzian gravity the sum in (2) goes over all triangulations which have a time sliced structure [8, 9]. Each time slice consists of a random number of vertices V_t on a circle. Vertices on a neighboring slices t and $t+1$ are connected by $V_t + V_{t+1}$ time-like links making $V_t + V_{t+1}$ triangles. Each triangle has one space-like edge on a constant time line and two time-like edges. From technical reasons each lattice is periodic in temporal direction. Index t runs periodically in a range $t = 1, \ldots T$. Typical Dirac–Wilson operator spectrum is shown in figure 2 (right). Comparing this with analogous spectrum on Euclidean lattice some differences can be seen for large eigenvalues. Those differences come from the fact that Euclidean and Lorentzian lattices have distinct properties on small length scales. The behavior of the small eigenvalues is in both cases similar. When the hopping parameter K changes, the spectrum rescales around $(1,0)$ point like in Euclidean case. Repeating procedure described in the previous chapter we can examine scaling of the smallest eigenvalues of the Majorana–Wilson operator. We consider here gravity interacting with two kinds of field, one with central charge $c = 1/2$, and the other with central charge $c = 4$. We use the properties of the Majorana–Wilson operator spectrum to probe the fractal properties of the geometry. The dependence of the smallest mass gap on a lattice size is shown in figure 4. We fit the scaling formula:

$$M_*(N) = aN^{-1/d_{\mathrm{H}}} \tag{11}$$

and get the parameter values: $a = 2.04(4)$ and $d_{\mathrm{H}} = 2.11(5)$ for $c = 1/2$ and $a = 2.88$ and $d_{\mathrm{H}} = 1.77$ for $c = 4$.

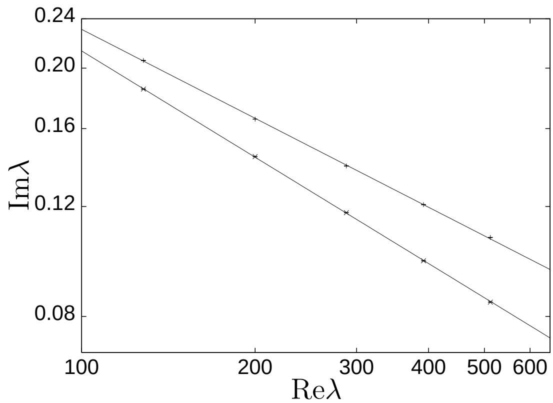

Fig. 4. Minimal mass gap M_* of fermionic particle as a function of lattice volume N for lattices with conformal charge $c = 1/2$ (upper line) and $c = 4$ (lower line). The fit $M_*(N) = aN^{-1/d_{\mathrm{H}}}$ gives $a = 2.04(4)$ and $d_{\mathrm{H}} = 2.11(5)$ for $c = 1/2$ and $a = 2.88(4)$ and $d_{\mathrm{H}} = 1.77(3)$ for $c = 4$.

6. Summary

We performed Monte-Carlo simulations to get a spectrum of the Dirac–Wilson operator for fermions interacting with 2D Euclidean gravity and with 2D Lorentzian gravity. The influence of lattice dynamic on fermions can be seen in a shape of the spectrum of Dirac operator but also in a critical value of the hopping parameter. This value is different than the one calculated for regular lattice. We examine the behavior of the smallest eigenvalues. This eigenvalues define a mass gap related to inverse of the correlation length and to long-range fermion excitations. Critical exponent controlling the mass gap scaling reveals fractal properties of the geometry.

Hausdorff dimension values calculated for: Euclidean gravity, Lorentzian gravity interacting with central charge $c = 1/2$ and with charge $c = 4$, equal respectively: $d_{\mathrm{H}} = 2.87(3)$, $d_{\mathrm{H}} = 2.11(5)$ and $d_{\mathrm{H}} = 1.77(3)$. That seems to be in agreement with values 3, 2 and 3/2. The Hausdorff dimension 3 for Euclidean gravity is predicted by theoretical calculations for fermionic particle immersed in a fractal Euclidean gravity background. The value 2 results from canonical dimension of the Lorentzian gravity and reflects the fact that Lorentzian geometry below $c = 1$ barrier does not develop fractal structure. The last value 3/2 is connected with scaling of the lowest momenta on a bubble which arises on a Lorentzian lattice above $c = 1$ barrier [7].

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