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PROBABILISTIC ASPECTS OF INFINITE TREES AND SURFACES*

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We present a simple construction of a probability measure on rooted infinite planar trees as a limit of a sequence of uniform measures on finite trees. We compute the conditional probability measure on the set of trees containing a given finite tree and use this to determine the distribution of the number of vertices at a given distance from the root, and thereby the Hausdorff dimension associated with this measure. The construction can be generalised to other ensembles of infinite discrete structures. We indicate, in particular, how it can be adapted in a straight forward manner to obtain a probability measure on infinite planar surfaces by using a certain correspondence between quadrangulated surfaces and so-called well labelled trees. The Hausdorff dimension of this measure turns out to be 4. Details of these latter results will appear elsewhere.

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1. Introduction

Suppose we have a canonical ensemble of manifolds, or other geometric objects endowed with a metric, whose statistical distribution is given by a uniform probability measure μ_V on manifolds of fixed total volume V. Denoting by $N_r(M)$ the volume of a spherical shell of radius r and thickness 1 cantered at some distinguished point of the manifold M, the *Hausdorff dimension* $d_{\rm H}$ of the ensemble is commonly characterised by

$$\langle N_r\rangle_V \sim r^{d_{\rm H}-1} \quad \text{for } 1 << r << V^{\frac{1}{d_{\rm H}}}\ , \tag{1}$$

where $\langle\cdot\rangle_V$ denotes the average over manifolds with respect to μ_V, see *e.g.* [1] Ch. 4. Clearly, in order to fix $d_{\rm H}$ uniquely by (1), one must consider in some

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way or other the limiting behaviour as $V \to \infty$. There are, indeed, several different ways to accomplish this. One way is to consider a grand canonical version of the ensemble in question and relating $d_{\rm H}$ to the critical exponent of the mass, defined as the exponential decay rate of a suitable two-point function. This has been implemented for random trees in [2], determining $d_{\rm H}$ to be 2, and for random planar triangulations (two-dimensional pure gravity) in [3], where it was argued that $d_{\rm H} = 4$. Another possible route is to consider the limiting behaviour of the radius R_V, considered as a random variable on the ensemble of manifolds with total volume V, as $V \to \infty$. Showing that the limit

$$\lim_{V\to\infty} V^{-\frac{1}{d_{\rm H}}} R_V = R$$

exists as a finite random variable on some probability space can then be taken as a manifestation of (1). For the case of trees it is well known [12] that with $d_{\rm H} = 2$ the limit above exists and is proportional to the maximal height of the so-called Brownian excursion. For the case of quadrangulated surfaces the existence of the limit with $d_{\rm H} = 4$ has recently been demonstrated in [9].

In both of the preceding interpretations it is implicitly assumed that (1) holds also for $r \sim V^{\frac{1}{d_{\rm H}}}$, and the obtained value of $d_{\rm H}$ can be seen as characterising the asymptotic behaviour of the global volume of manifolds as a function of their linear extent. A more straight forward and local interpretation of (1) is to consider the limit μ_∞ of the finite volume measures as $V \to \infty$ realized on a space of manifolds with infinite volume, if possible, and defining $d_{\rm H}$ by

$$\langle N_r \rangle_\infty \sim r^{d_{\rm H}-1} \quad \text{as } r \to \infty \,. \tag{2}$$

In [4] such a limiting measure for triangulated planar surfaces was constructed, and in [5] it was established that

$$N_r \sim r^3 \quad \text{as } r \to \infty \,,$$

up to possible logarithmic corrections, for almost all triangulations with respect to this measure. Although, strictly speaking, this does not imply (2), it clearly carries similar information.

In this paper we shall adapt the method of [4] to the case of trees and construct by simple combinatoric arguments a limiting uniform probability measure on infinite trees and prove (2) with $d_{\rm H} = 2$. A different, but closely related, discussion of this limiting measure appears in [11]. We also indicate how the method can be simply generalised to so-called well labelled trees and how this leads to a proof of (2) with $d_{\rm H} = 4$ for quadrangulated planar surfaces.

This article is organised as follows. In Section 2 we give a simple illustration of the general method by applying it to infinite random walks in $\mathbb{Z}^d$ emerging from some fixed point and showing how it produces the well known uniform probability measure on such walks. In Section 3 we construct the uniform measure on infinite rooted planar trees and compute the distribution of the number of points at a given distance from the root. In particular, it follows that the branches attached to some fixed finite tree are independently distributed, and that the Hausdorff dimension of the measure is 2. In Section 4 we define the notion of a well labelled planar tree and explain how the results of the previous section extend to the class of such trees. Moreover, the implications of this result for infinite planar quadrangulated surfaces are explained.

2. Random walks in $\mathbb{Z}^d$

Let Ω be the set of simple random walks starting at 0 in $\mathbb{Z}^d$, that is

$$\Omega = \left(\bigcup_{N=0}^{\infty} \Omega_N \right) \cup \Omega_\infty \ ,$$

where Ω_∞ is the set of infinite sequences $(\omega_0, \omega_1, \dots)$ in $\mathbb{Z}^d$ such that $|\omega_{i+1} - \omega_i| = 1$ and $\omega_0 = 0$, and similarly Ω_N is the set of finite sequences $(\omega_0, \dots, \omega_N)$ subject to the same relations, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{Z}^d$.

For each $\omega \in \Omega$ and each non-negative integer r we let $B_r(\omega)$ denote the first r steps of ω or, more precisely,

$$B_r(\omega) = (\omega_0, \dots, \omega_s) \ ,$$

where $s = r$, if $\omega \in \Omega_\infty$ or if $\omega \in \Omega_N$ and $N \geq r$, and $s = N$ otherwise. Clearly, if $\omega, \omega' \in \Omega$, then $\omega = \omega'$ if and only if $B_r(\omega) = B_r(\omega')$ for all r, whereas, if $\omega \neq \omega'$, there exists a largest $r = R$ such that $B_R(\omega) = B_R(\omega')$ in which case we define the distance between ω and ω' to be $d(\omega, \omega') = \frac{1}{R+1}$. It is a trivial matter to verify that this defines a metric d on Ω. Alternatively, we may write

$$d(\omega, \omega') = \inf \left\{ \frac{1}{r+1} \mid B_r(\omega) = B_r(\omega') \ , \ r \in \mathbb{N}_0 \right\} \ ,$$

where $\mathbb{N}_0$ denotes the set of non-negative integers.

A first thing to note is that Ω *is a compact metric space*: To see this, let ω^n, $n \in \mathbb{N}$, be any sequence in Ω. For each $r \in \mathbb{N}_0$ the set Ω_r is finite. Hence, there is a subsequence ω^{n_i}, $i \in \mathbb{N}$, such that $B_r(\omega^{n_i})$ is constant as a function of i. Applying a standard diagonal argument we can choose this subsequence such that $B_i(\omega^{n_j}) = B_i(\omega^{n_i})$ for all $i \leq j$. It follows that this sequence determines a unique element $\omega \in \Omega$ such that $B_i(\omega) = B_i(\omega^{n_i})$ for all $i \in \mathbb{N}$. Hence, in particular, $\omega^{n_i} \to \omega$ as $i \to \infty$ which proves the claim.

We remark that, under the canonical identifications

$$\Omega_N = \{\pm 1, \ldots, \pm d\}^N \,, \qquad \Omega_\infty = \{\pm 1, \ldots, \pm d\}^{\mathbb{N}} \,,$$

it is clear that the topology defined by the metric d on Ω_N or Ω_∞ is the standard product topology. In particular, these spaces are compact by Tychonoff's theorem. However, the set $\bigcup_{N=0}^\infty \Omega_N$ of finite walks is not compact, being a discrete, dense subset of Ω whose boundary is Ω_∞.

Let now ν_N be the uniform probability measure on $\Omega_N \subset \Omega$, *i.e.* the normalised counting measure on Ω_N. In the limit $N \to \infty$ these measures define a uniform probability measure on Ω_∞, stated more precisely as follows.

Theorem 2.1 *Considering ν_N as a measure on Ω we have*

$$\nu_N \to \nu \qquad \text{as } N \to \infty \,,$$

where ν is a Borel probability measure concentrated on Ω_∞. We call ν the uniform probability measure on Ω_∞.

Here convergence is understood in the weak sense, that is

$$\int_\Omega f d\nu_N \to \int_\Omega f d\nu$$

as $N \to \infty$ for all bounded continuous functions f on Ω.

Proof: We shall make use here and in the proof of the corresponding result for trees in the next section of a standard fact concerning weak convergence of measures that can be found *e.g.* in [6], and which may be stated as follows. Suppose $\rho_N, N \in \mathbb{N}$, is a sequence of probability measures on a metric space M and suppose that $\mathcal{U}$ is a family of both open and closed subsets of M such that

(i) any finite intersection of sets in $\mathcal{U}$ belongs to $\mathcal{U}$,

(ii) any open subset of M may be written as a finite or countable union of sets from $\mathcal{U}$,

(iii) the sequence $\rho_N(A)$, $N \in \mathbb{N}$, is convergent for all sets $A \in \mathcal{U}$.

Then the sequence ρ_N, $N \in \mathbb{N}$, is weakly convergent provided it is *tight*, that is if for each $\varepsilon > 0$ there exists a *compact* subset C of M such that

$$\rho_N(M \setminus C) < \varepsilon \quad \text{for all } N . \tag{3}$$

In fact, the last condition ensures by Theorem 6.1 of [6] that there exists a convergent subsequence ρ_{N_i}, $i \in \mathbb{N}$. Calling its limit ρ we get from Theorem 2.1 in [6] that $\rho_{N_i}(A) \to \rho(A)$ as $i \to \infty$ for all $A \in \mathcal{U}$, since A has empty boundary. Combining this with *(iii)* above we conclude that $\rho_N(A) \to \rho(A)$ as $N \to \infty$ for all $A \in \mathcal{U}$ and Theorem 2.2 in [6] then implies $\rho_N \to \rho$.

In the present case the tightness property is automatic since our metric space Ω is compact. Thus we only need to provide a family $\mathcal{U}$ with the desired properties. For this purpose let $\mathcal{B}_s(\omega_0)$ be the open ball in Ω of radius $s > 0$ centred at $\omega_0 \in \Omega$,

$$\mathcal{B}_s(\omega_0) = \{\omega \in \Omega \mid d(\omega, \omega_0) < s\} .$$

Since the positive values assumed by d form a discrete set (the inverse natural numbers) it is clear that these balls are both open and closed. We claim that

$$\mathcal{U} = \left\{ \mathcal{B}_{\frac{1}{r}}(\omega) \mid r \in \mathbb{N},\ \omega \in \bigcup_{N=0}^{\infty} \Omega_N \right\}$$

satisfies properties *(i)*–*(iii)* above. Property *(i)* follows from the easily verifiable fact that any two balls in Ω are either disjoint or one is contained in the other. Property *(ii)* follows since the set $\bigcup_{N=0}^{\infty} \Omega_N$ of finite walks is a dense countable subset of Ω. Finally, to verify property *(iii)*, let $\omega_0 \in \Omega_{N_0}$. For $1 \leq r \leq N_0$ one has

$$\mathcal{B}_{\frac{1}{r}}(\omega_0) \cap \Omega_N = \{\omega \in \Omega_N \mid B_r(\omega) = B_r(\omega_0)\} .$$

If $r \leq N_0$ and $N \geq N_0$ this gives

$$\nu_N(\mathcal{B}_{\frac{1}{r}}(\omega_0)) = (2d)^{-r} ,$$

whereas, if $r, N > N_0$, we have $\mathcal{B}_{\frac{1}{r}}(\omega_0) \cap \Omega_N = \emptyset$. In both cases $\nu_N(\mathcal{B}_{\frac{1}{r}}(\omega_0))$ converges as $N \to \infty$ and the proof is complete.

Remark 2.2 Under the identification $\Omega_\infty = \{\pm 1, \ldots, \pm d\}^{\mathbb{N}}$ we see that $\mathcal{B}_{\frac{1}{r}}(\omega_0)$, for $r \leq N_0$, is identified with the cylinder set consisting of sequences whose first $r+1$ elements coincide with those of ω_0. The last part of the proof above shows that

$$\nu(\mathcal{B}_{\frac{1}{r}}(\omega_0)) = (2d)^{-r} ,$$

from which we conclude, not surprisingly, that

$$\nu = \otimes_{i=1}^{\infty} \nu^{(i)} \ ,$$

where $\nu^{(i)}$ denotes the uniform probability measure on the i'th copy of $\{\pm 1, \ldots, \pm d\}$.

3. Random planar trees

In the following we shall use the notation Γ for the set of rooted planar trees, where the root vertex is assumed to have order 1. This last condition is not essential for the following, but will turn out convenient. Trees are here allowed to be infinite, but vertices are of finite order. The adjective planar means that trees are assumed embedded into the plane $\mathbb{R}^2$ such that no links intersect except possibly at vertices, and we identify trees that can be mapped onto each other by an orientation preserving homeomorphism of the plane. A more precise combinatorial definition is as follows. Once a fixed orientation of $\mathbb{R}^2$ is chosen the set of vertices at (graph) distance i from the root in a given rooted planar tree τ has a natural ordering. This can be obtained *e.g.* by choosing a righthanded coordinate system for $\mathbb{R}^2$ and mapping the tree into $\mathbb{R}^2$ such that the vertices at distance i from the root are mapped onto the vertical line through $(i, 0)$ and then ordering according to their second coordinate. We call this ordered set $D_i = (x_{i1}, \ldots, x_{in_i})$. In particular, $D_0 = x_0$ consists of the root x_0 alone and $D_1 = x_1$ consists of the unique vertex x_1 connected to x_0. The links in the tree are specified by mappings $\phi_i : D_i \to D_{i-1}$ preserving the ordering, *i.e.* the links in τ are given by $(x_{ik}, x_{i-1\phi_i(k)})$, $1 \leq k \leq n_i$. It is clear that any (finite or infinite) sequence $(D_0, D_1, D_2, \ldots)$ of finite non-empty pairwise disjoint ordered sets, where D_0, D_1 are one-point sets, together with orientation preserving maps $(\phi_1, \phi_2, \ldots)$ as above uniquely specifies a rooted planar tree. Moreover, two such trees given by $\{(D_0, D_1, D_2, \ldots), (\phi_1, \phi_2, \ldots)\}$ and $\{(D'_0, D'_1, D'_2, \ldots), (\phi'_1, \phi'_2, \ldots\})$, respectively, are identical if and only if there exist order preserving bijective maps $\psi_i : D_i \to D'_i$ such that $\phi'_i = \psi_{i-1} \circ \phi_i \circ \psi_i^{-1}$ for all i.

We have

$$\Gamma = \left(\bigcup_{N=1}^{\infty} \Gamma_N \right) \cup \Gamma_\infty \ ,$$

where Γ_N consists of trees with maximal vertex distance from the root equal to N, *i.e.* $D_i = \emptyset$ for $i > N$ but $D_N \neq \emptyset$, and Γ_∞ consists of infinite trees. We say that $\tau \in \Gamma_N$ has radius $\rho(\tau) = N$ and $\rho(\tau) = +\infty$ for $\tau \in \Gamma\infty$. Furthermore, for a finite tree τ, we denote by $|\tau|$ the number of links in τ, and otherwise set $|\tau| = +\infty$.

For $r \in \mathbb{N}_0$ and $\tau \in \Gamma$ with distance classes $D_i(\tau)$ we define the ball $B_r(\tau)$ of radius r in τ to be the subtree of τ generated by $D_0(\tau), \dots, D_r(\tau)$, if $r < \rho(\tau)$, and equal to τ otherwise. By analogy with the random walk case, see also [4, 11], we define a metric on Γ by

$$d(\tau, \tau') = \inf \left\{ \frac{1}{r+1} \mid B_r(\tau) = B_r(\tau') \ , \ r \in \mathbb{N}_0 \right\}$$

and corresponding balls

$$\mathcal{B}_s(\tau_0) = \{\tau \in \Gamma \mid d(\tau, \tau_0) < s\} \ .$$

Remark 3.1 The following facts are easy to verify:

- The set $\bigcup_{N=0}^{\infty} \Gamma_N$ of finite trees is a countable dense subset of Γ.
- For $s > 0$ and $\tau_0 \in \Gamma$ the ball $\mathcal{B}_s(\tau_0)$ is both open and closed and
$$\tau \in \mathcal{B}_s(\tau_0) \Rightarrow \mathcal{B}_s(\tau) = \mathcal{B}_s(\tau_0).$$
In particular, two balls are either disjoint or one is contained in the other.
- Γ is not compact: Let τ_n be the (unique) tree of radius 2 with $n+2$ vertices. Then $d(\tau_n, \tau_m) = 1$ for $n \neq m$ and hence τ_n, $n \in \mathbb{N}$, has no convergent subsequence.

  On the other hand, the subset $\Gamma^{(M)}$ consisting of trees with vertex degrees bounded by $M < +\infty$ is seen to be compact by the same argument as for random walks given in the previous section.

As a substitute for compactness we shall use the following result.

Proposition 3.2 *Let K_r, $r \in \mathbb{N}$, be a sequence of positive numbers. Then the subset*

$$C = \bigcap_{r=1}^{\infty} \{\tau \in \Gamma \mid |B_r(\tau)| \leq K_r\}$$

of Γ is compact.

Proof: Let τ_n, $n \in \mathbb{N}$, be any sequence in C. For each $r \in \mathbb{N}$ the set $\{\tau \in \Gamma_r \mid |\tau| \leq K_r\}$ is finite. Hence there exists a subsequence τ_{n_i}, $i \in \mathbb{N}$, such that $B_r(\tau_{n_i})$ is constant as a function of i. Applying a diagonal argument we may choose this subsequence such that $B_i(\tau_{n_j}) = B_i(\tau_{n_i})$ for all $i \leq j$. It follows that this subsequence determines a unique tree $\tau \in C$ such that

$B_i(\tau) = B_i(\tau_{n_i})$ for all $i \in \mathbb{N}$. In particular, $\tau_{n_i} \to \tau$ as $i \to \infty$, which completes the proof.

Let now μ_N be the normalised uniform measure on

$$\Gamma'_N = \{\tau \in \Gamma \mid |\tau| = N\} \ , \quad N \in \mathbb{N}.$$

It is well known, see *e.g.* [1] that the number of trees in Γ'_N is given by

$$C_N = \frac{(2N-2)!}{N!(N-1)!}$$

such that

$$\mu_N(\tau) = C_N^{-1} \text{ for } \tau \in \Gamma'_N \ , \quad \mu_N(\Gamma \setminus \Gamma'_N) = 0 \ .$$

Theorem 3.3 *Considering μ_N as a measure on Γ we have*

$$\mu_N \to \mu \quad \textit{as} \quad N \to \infty \ ,$$

where μ is a Borel probability measure concentrated on Γ_∞. We call μ the uniform probability measure on Γ_∞.

Proof: By Remark 3.1 the family of balls

$$\mathcal{V} = \left\{\mathcal{B}_{\frac{1}{r}}(\tau) \mid r \in \mathbb{N},\ |\tau| < +\infty\right\}$$

fulfils *(i)* and *(ii)* in the proof of Theorem 2.1. Hence it will suffice to prove that the sequence μ_N, $N \in \mathbb{N}$, is tight and that $\mu_N(A)$ converges as $N \to \infty$ for $A \in \mathcal{V}$.

We first prove tightness. By Proposition 3.2 it is sufficient to show that for each $\varepsilon > 0$ and $r \in \mathbb{N}$ there exists $K_r > 0$ such that

$$\mu_N(\{\tau \in \Gamma \mid |B_r(\tau)| > K_r\}) < \varepsilon \tag{4}$$

for all N. Replacing ε by $\varepsilon/2^r$ in (4) and choosing K_r correspondingly, Proposition 3.2 gives the desired compact set C fulfilling (3).

We proceed to show (4) by induction on r.

The case $r = 1$ is trivial so let us consider first $r = 2$, in which case $|B_r(\tau)|$ equals the valency of the neighbour x_1 of the root x_0. Clearly, there is a bijective correspondence between trees τ with $|B_2(\tau)| = k+1$ and elements in Γ^k, given by mapping τ onto the ordered set $(\tau_1, \dots, \tau_k)$ of branches at x_1 different from the root link (x_0, x_1). Since $|\tau| = |\tau_1| + \cdots + |\tau_k| + 1$ we get

$$\begin{aligned} &\mu_N(\{\tau \in \Gamma \mid |B_2(\tau)| = k+1\}) \\ &= C_N^{-1} \sum_{N_1+\cdots+N_k=N-1} \prod_{i=1}^{k} C_{N_i} \ \leq \ k \sum_{\substack{N_1+\cdots+N_k=N-1 \\ N_1 \geq (N-1)/k}} C_N^{-1} \prod_{i=1}^{k} C_{N_i} \ . \end{aligned} \tag{5}$$

Using the well known facts (see *e.g.* [1]) that

$$C_N \sim N^{-3/2} 4^N$$

and

$$Z(x) = \sum_{N=1}^{\infty} C_N t^N = \frac{1}{2}\left(1 - \sqrt{1-4x}\right) \ , \quad x \leq \frac{1}{4} \ ,$$

we obtain

$$\mu_N(\{\tau \in \Gamma \mid |B_2(\tau)| = k+1\}) \leq c \cdot k^{5/2} Z\left(\tfrac{1}{4}\right)^{k-1} = 2c \cdot k^{5/2} 2^{-k} \ , \qquad (6)$$

where the constant c is independent of N. Clearly, (4) follows for $r = 2$ by choosing K_2 large enough.

Now assume (4) holds for a given $r \geq 2$. For any $K > 0$ we then have

$$\begin{aligned} &\mu_N(\{\tau \in \Gamma \ \mid \ |B_{r+1}(\tau)| > K\}) \\ &\leq \varepsilon + \mu_N(\{\tau \in \Gamma \ \mid \ |B_{r+1}(\tau)| > K, \ |B_r(\tau)| \leq K_r\}) \ . \end{aligned}$$

Since the set of different balls $B_r(\tau)$ with $|B_r(\tau)| \leq K_r$ is finite, it suffices to show that

$$\mu_N(\{\tau \in \Gamma \mid |B_{r+1}(\tau)| > K, \ B_r(\tau) = \tau_0\}) \to 0 \qquad (7)$$

as $K \to \infty$, uniformly in N for any fixed $\tau_0 \in \Gamma_r$. This is obtained in a similar fashion as for $r = 2$:

Let $R = \sharp D_r(\tau_0)$ be the number of vertices in τ_0 at maximal distance r from the root. Any $\tau \in \Gamma$ with $B_r(\tau) = \tau_0$ and $|B_{r+1}(\tau)| = K$ is obtained by attaching a sequence $(\tau_1, \ldots, \tau_k)$ of trees in Γ at their roots to (some of) those R vertices in τ_0, where $k = K - |\tau_0|$. Since there are

$$\binom{k+R-1}{R-1} \leq \frac{K^{R-1}}{(R-1)!} \qquad (8)$$

different ways of attaching a given sequence in prescribed order, and $|\tau| = |\tau_0| + |\tau_1| + \cdots + |\tau_k|$, we get

$$\begin{aligned} &\mu_N(\{\tau \in \Gamma \mid |B_{r+1}(\tau)| \ = \ K, \ B_r(\tau) = \tau_0\}) \\ &\leq \frac{K^{R-1}}{(R-1)!} C_N^{-1} \sum_{N_1+\cdots+N_k = N-|\tau_0|} \prod_{i=1}^{k} C_{N_i} \leq c' \cdot K^{R+3/2} 2^{-K} \ , \end{aligned}$$

where the last inequality follows in the same way as (6), the constant c' being independent of N and K. Clearly this finishes the proof of (7), and hence of the tightness of $\mu_N, \ N \in \mathbb{N}$.

It remains to prove that $\mu_N(\mathcal{B}_{\frac{1}{r}}(\tau_1))$ converges as $N \to \infty$ for all $r \in \mathbb{N}$ and finite $\tau_1 \in \Gamma$. For this purpose, assume that $\mathcal{B}_{\frac{1}{r}}(\tau_1) \neq \emptyset$ and set $\tau_0 = B_r(\tau_1) \in \Gamma_r$. Then

$$\mathcal{B}_{\frac{1}{r}}(\tau_1) = \{\tau \in \Gamma \mid B_r(\tau) = \tau_0\} \,.$$

Let $v_1, \ldots, v_R$ denote the vertices in τ_0 at maximal distance r from the root. Then any $\tau \in \mathcal{B}_{\frac{1}{r}}(\tau_1)$ is obtained by attaching a sequence $(\tau_1, \ldots, \tau_R)$ of R trees in Γ to τ_0 such that the root link of τ_i is identified with the link in τ_0 with endpoint v_i. This gives

$$\mu_N(\mathcal{B}_{\frac{1}{r}}(\tau_1) = C_N^{-1} \sum_{N_1+\cdots+N_k=N+R-N_0} \prod_{i=1}^{R} C_{N_i} \,, \tag{9}$$

where $N_i = |\tau_i|$. Each term in this sum has $N_i \geq (N+R-N_0)/R$ for some $i = 1, \ldots, R$. For fixed $A > 0$ we can bound the sum of terms for which in addition $N_j \geq A$, for some $j \neq i$, by

$$\begin{aligned}
&R^2 \sum_{\substack{N_1+\cdots+N_R=N+R-N_0 \\ N_1 \geq (N+R-N_0)/R,\ N_2 \geq A}} C_N^{-1} \prod_{i=1}^{R} C_{N_i} \\
&\leq \mathrm{cst} \cdot R^2 \cdot 4^{N_0-R} \sum_{\substack{N_3,\ldots,N_R \geq 1 \\ N_2 \geq A}} \left(\frac{NR}{N+R-N_0}\right)^{3/2} N_2^{-3/2} \prod_{i=3}^{R} C_{N_i} 4^{-N_i} \\
&\leq \mathrm{cst} \cdot A^{-1/2} Z\left(\tfrac{1}{4}\right)^{R-2} \leq \mathrm{cst} \cdot A^{-1/2} \,,
\end{aligned} \tag{10}$$

where the constants depend on τ_0 but are independent of A and N.

The remaining contribution to $\mu_N(\mathcal{B}_{\frac{1}{r}}(\tau_1))$ can, for $(N+R-N_0)/R > A$, be written as

$$\sum_{i=1}^{R} \sum_{\substack{N_1+\cdots+N_R=N+R-N_0 \\ N_j \leq A,\ j \neq i}} C_N^{-1} \prod_{i=1}^{R} C_{N_i} \overset{N\to\infty}{\longrightarrow} R \left(\sum_{n=1}^{A} C_N 4^{-n}\right)^{R-1} 4^{R-N_0} \,.$$

Letting $A \to \infty$ and using (10), we finally get

$$\mu_N(\{\tau \in \Gamma \mid B_r(\tau) = \tau_0\}) \overset{N\to\infty}{\longrightarrow} R \cdot Z(\tfrac{1}{4})^{R-1} 4^{R-N_0} = 2R \cdot 2^R \cdot 4^{-N_0} \tag{11}$$

proving the claimed convergence.

Thus we have established the existence of μ. That μ is concentrated on Γ_∞ is clear, since the set of finite trees is countable, each of its elements having vanishing μ-measure. This finishes the proof of Theorem 3.3.

As noted in the proof, any ball of radius $r \in \mathbb{N}$ in Γ is of the form

$$A(\tau_0) = \{\tau \in \Gamma \mid B_r(\tau) = \tau_0\} ,$$

where $\tau_0 \in \Gamma_r$, and can be identified homeomorphically with Γ^R, where $R = R(\tau_0)$ is the number of vertices at maximal distance r from the root. We have already computed the μ- volume of $A(\tau_0)$ in Eq. (11) above. In the same way, the vanishing of the expressions in (10), as $A \to \infty$, yields a closed form of the conditional probability measure $d\mu(\tau_1, \dots, \tau_R | A(\tau_0))$ on $A(\tau_0)$. This may be expressed as follows.

Corollary 3.4 *For $\tau_0 \in \Gamma_r$ we have*

$$\mu(A(\tau_0)) = R \cdot 2^{R+1} \cdot 4^{-|\tau_0|} \tag{12}$$

and

$$d\mu(\tau_1, \dots, \tau_R | A(\tau_0)) = \mu(A(\tau_0))^{-1} \sum_{i=1}^{R} d\mu(\tau_i) \prod_{j \neq i} d\rho(\tau_j) , \tag{13}$$

where the measure ρ is concentrated on finite trees and defined by

$$\rho(\tau) = 4^{-|\tau|} ,$$

i.e. ρ is the (unnormalised) grand canonical measure on $\Gamma \backslash \Gamma_\infty$ at the critical point $\frac{1}{4}$ in the language of [1].

Remark 3.5 It follows from (13) and the fact that ρ is concentrated on finite trees whereas μ is concentrated on infinite trees that with probability 1 only one of the branches $\tau_1, \dots, \tau_R$ is infinite. The probability that a given branch τ_i is infinite is R^{-1}, and the probability measure when conditioned on the set

$$A_i(\tau_0) = \{\tau \in A(\tau_0) | \tau_i \text{ is infinite}\}$$

is given by

$$d\mu(\tau_1, \dots, \tau_R | A_i(\tau_0)) = 4^{|\tau_0|} 2^{-R-1} d\mu(\tau_i) \prod_{j \neq i} d\rho(\tau_j) . \tag{14}$$

In particular, $\tau_1, \dots, \tau_R$ are independently distributed.

Furthermore, it follows that the measure μ is concentrated on the set of trees τ which contain exactly one simple infinite path s, the *spine* of τ, originating at the root x_0, and τ is obtained by attaching finite branches to the vertices of the spine.

As an application we compute next the probability distribution of the number

$$d_r(\tau) = \sharp D_r(\tau)$$

of vertices at distance r from the root.

Proposition 3.6 *For $r \geq 2$ and $d \in \mathbb{N}$ we have*

$$\mu(\{\tau \mid d_r(\tau) = K\}) = \frac{K}{r^2}\left(1 - \frac{1}{r}\right)^{K-1}, \tag{15}$$

and

$$\mu(\{\tau \mid d_r(\tau) \geq K\}) = \left(1 + \frac{K-1}{r}\right)\left(1 - \frac{1}{r}\right)^{K-1}. \tag{16}$$

Proof: The second formula is an immediate consequence of the first one. In order to prove (15), we first note that the number of trees $\tau_0 \in \Gamma_r$ with prescribed values $d_i(\tau_0) = K_i,\ i = 0, 1, \ldots, r$, equals

$$\binom{K_1 + K_0 - 1}{K_0 - 1}\binom{K_2 + K_1 - 1}{K_1 - 1} \cdot \ldots \cdot \binom{K_r + K_{r-1} - 1}{K_{r-1} - 1}.$$

Here $K_0 = K_1 = 1$ and $K_r = R(\tau_0)$. It follows from (12) that

$$\begin{aligned}&\mu(\{\tau \mid d_r(\tau) = K\}) \\ &= K2^{-K-1} \sum_{K_2,\ldots,K_{r-1}} \binom{K_3 + K_2 - 1}{K_2 - 1} \cdots \binom{K + K_{r-1} - 1}{K_{r-1} - 1} 4^{-(K_2+\ldots+K_{r-1})}\end{aligned}$$

which also holds for $r = 2$ provided the empty sum is set to 1. Applying repeatedly the binomial formula

$$\sum_{K=1}^{\infty} \binom{L + K - 1}{K - 1} x^K = \frac{x}{(1-x)^{L+1}},$$

this expression reduces to

$$\frac{K}{2}\frac{z_2}{1 - z_2} \cdot \frac{z_3}{1 - z_3} \cdot \ldots \cdot \frac{z_{r-1}}{1 - z_{r-1}}(2z_r)^K, \tag{17}$$

where the numbers $z_2, z_3, \ldots$ are defined recursively by

$$z_2 = \frac{1}{4}, \qquad z_{i+1} = \frac{1}{4(1 - z_i)}.$$

The solution of this recursion relation is seen to be

$$z_i = \frac{1}{2} - \frac{1}{2i}\ ,$$

which gives

$$\frac{z_2}{1-z_2} \cdot \frac{z_3}{1-z_3} \cdot \ldots \cdot \frac{z_{r-1}}{1-z_{r-1}} = \frac{2}{r(r-1)}\ .$$

Inserting this expression into (17) yields Eq. (15).

From (15) we obtain the average size of the spherical shells $D_r(\tau)$ and balls $B_r(\tau)$.

Theorem 3.7 *For $r \geq 1$ we have*

$$\langle d_r \rangle_\mu = 2r - 1\ , \tag{18}$$

and

$$\langle |B_r| \rangle_\mu = r^2\ , \tag{19}$$

where $\langle \cdot \rangle_\mu$ denotes the average value with respect to μ.

Proof: The case $r = 1$ is obvious. By (15) we get for $r \geq 2$

$$\langle d_r \rangle_\mu = \sum_{K=1}^{\infty} \frac{K^2}{r^2} \left(1 - \frac{1}{r}\right)^{K-1} = r^{-2}\left(r^2 + \left(1 - \frac{1}{r}\right) 2r^3\right) = 2r - 1\ .$$

The second identity (19) follows by summing the former from 1 to r.

As a consequence we finally have

Corollary 3.8 *The Hausdorff dimension of μ is $d_{\mathrm{H}} = 2$.*

4. Random planar quadrangulations

In this section we briefly outline an extension of the preceding results to random planar quadrangulations via so-called well labelled trees. Detailed results can be found in [8].

A well labelled planar tree is a rooted planar tree τ whose vertices are labelled by positive integers such that the root has label 1 and such that labels of neighbouring vertices differ by ± 1 or 0. In [9, 10] a bijective map between finite well labelled planar trees and finite quadrangulations of the 2-sphere with a marked oriented link has been constructed. In this correspondence the root link is not assumed to have order 1. Instead, the trees are assumed to have a marked oriented link, the *root link*, whose ends are called the *first and second root vertex*, respectively. It suffices for our purposes to note the following properties of this map:

(a) The vertices in a finite well labelled tree and the vertices in the corresponding quadrangulated surface except one marked vertex (the initial vertex of the marked link) can be canonically identified.

(b) The label of a vertex in a well labelled tree equals the distance in the corresponding quadrangulation from the vertex to the marked vertex.

(c) The mapping is local in such a way that it can be extended to infinite well labelled trees with the property that each label occurs only finitely many times. The image of such an infinite well labelled tree is then an infinite quadrangulation of a domain in the plane, see [8].

The generating function for the number $D_N^{(1)}$ of planar quadrangulated surfaces with N quadrangles, and hence of the number of well labelled trees with N links, has been computed in [14] and is given by

$$W^{(1)}(x) = \sum_{N=0}^{\infty} D_N x^N = \frac{18x - 1 + (1-12x)^{\frac{3}{2}}}{54x^2}\,, \quad \text{for } x \le \tfrac{1}{12}\,, \tag{20}$$

which gives

$$D_N^{(1)} = 2 \cdot 3^N \frac{(2N)!}{N!(N+2)!}\,. \tag{21}$$

Note that we have included in $W^{(1)}(x)$ the contribution $D_0 = 1$ from the tree with only one vertex.

We also introduce the notion of a *k-labelled* tree, differing from that of a well labelled tree only in that the first root is assumed to have label k instead of 1. Let $D_N^{(k)}$ denote the number of k-labelled trees with N links and $W^{(k)}$ the corresponding generating function. It is then easy to see that all $D_N^{(k)}$ have the same asymptotic behaviour as $N \to \infty$ given by (21)

$$D_N^{(k)} \sim N^{-\frac{5}{2}} 12^N\,. \tag{22}$$

The function $W^{(k)}(x)$ can be related to the generating function $Z^{(k)}(x)$ for the number of k-labelled trees with N links and first root vertex of order 1 by decomposing a tree at its first root into trees with first root vertex of order 1. This gives

$$W^{(k)}(x) = \frac{1}{1 - Z^{(k)}(x)}\,, \quad x \le \frac{1}{12}\,. \tag{23}$$

We shall only need the value of $Z^{(k)}$ (and $W^{(k)}$) at the critical point $x = \frac{1}{12}$. By abuse of notation we set

$$Z^{(k)} \doteq Z^{(k)}\left(\frac{1}{12}\right)\,.$$

For $k=1$ Eqs. (20) and (23) give $Z^{(1)}=\frac{1}{4}$. On the other hand, one has the recursion relation

$$Z^{(k)}(x)=\frac{x}{1-Z^{(k-1)}}+\frac{x}{1-Z^{(k)}}+\frac{x}{1-Z^{(k+1)}}\,,\quad k\geq 1\,,\tag{24}$$

which is obtained by decomposing the sum over trees defining $Z^{(k)}$ according to the order and label of their second root x_1, and where the first term should be omitted for $k=1$. This determines $Z^{(k)}$ for all $k\in\mathbb{N}$ and one easily verifies that the solution is

$$Z^{(k)}=\frac{1}{2}-\frac{1}{k(k+3)}\tag{25}$$

Remark 4.1 The relation (24) has also been considered in [7] and its solution found for arbitrary $x\leq\frac{1}{12}$ with $Z^{(1)}(x)$ given by (20) and (23). We shall, however, not need this more general result for our purposes.

One can now turn the space $\mathcal{W}^{(k)}$ of k-labelled trees into a metric space in a similar way as done previously for Γ and, using the results (22) and (25), the proof of the existence theorem for the measure μ in the previous section can be carried over to the present situation with only modest changes, see [8]. We formulate the result as follows.

Theorem 4.2 *Let $\mu_N^{(k)}$ denote the uniform probability measure on the set of k-labelled trees with N links. Considering this as a measure on $\mathcal{W}^{(k)}$ we have*

$$\mu_N^{(k)}\to\mu^{(k)}\quad \textit{as } N\to\infty\,,$$

where $\mu^{(k)}$ is a Borel probability measure concentrated on the space $\mathcal{W}_\infty^{(k)}$ of infinite k-labelled trees. We call $\mu^{(k)}$ the uniform probability measure on $\mathcal{W}_\infty^{(k)}$.

Likewise one obtains a factorised form of conditional probability measures analogous to, but slightly more involved than, that of Corollary 3.4. In particular, one finds that the measure is supported on trees with exactly one simple infinite path, the spine, originating at the first root vertex, and that the branches attached to the spine are independently distributed.

As mentioned at the beginning of this section, in order to interprete $\mu^{(1)}$ as a measure on quadrangulated surfaces, it is crucial that it is supported on well labelled trees in which each label occurs only finitely many times. Letting N_k denote the number of occurrences of label k, one of the main results of [8] is that the average value of N_k is finite and fulfils

$$\langle N_k\rangle_{\mu^{(1)}}\sim k^3\,.\tag{26}$$

In particular, this establishes that N_k is indeed finite almost surely for each k. Moreover, as a consequence of property (b) of the mapping from well labelled trees to quadrangulated surfaces, the average value $\langle N_k \rangle_{\mu^{(1)}}$ is nothing but the average value of the number of vertices at distance k from the marked vertex, and hence (26) shows that $d_{\mathrm{H}} = 4$.

As a final result of [8] we mention that trees with exactly one infinite spine and finitely many occurrences of each label correspond to quadrangulated planar surfaces with exactly one boundary component. Similarly, it was found in [4] that the measure constructed there for triangulated surfaces is supported on infinite surfaces with exactly one boundary component.

5. Concluding remarks

Random trees are frequently discussed by probabilists in terms of stochastic processes, the so-called Galton–Watson processes. Thus, the uniform measure μ_N on planar trees with N links considered in Section 3 may be viewed as the measure obtained by conditioning the Galton–Watson process with geometric offspring distribution

$$p_n = 2^{-n-1}\,, \quad n \in \mathbb{N}_0\,,$$

on the event that the total progeny equals N. The limiting behaviour of a general Galton–Watson process conditioned in this way, as $N \to \infty$, has been studied in [13] but from a different point of view than here. It would be interesting to apply the methods used in this article to more general ensembles of random planar trees.

Similarly, the measures $\mu_N^{(k)}$ may be viewed as originating from a multi-type Galton–Watson process and one may envisage various corresponding generalisations of the results in Section 4.

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