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NONCOMMUTATIVE COORDINATE TRANSFORMATIONS AND THE SEIBERG–WITTEN MAP *

Jesper Møller Grimstrup

Science Institute, University of Iceland
Dunhaga 3, 107 Reykjavik, Iceland
e-mail: `jesper@raunvis.hi.is`

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Noncommutative conformal transformations are constructed on noncommutative $\mathbb{R}^4$ and used to derive the Seiberg–Witten differential equation.

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1. Introduction

One of the surprises arising in noncommutative gauge theory is the existence of a map between noncommutative and commutative gauge theories. The so-called Seiberg–Witten map was first deduced from the observation that different regularization schemes (point-splitting *vs* Pauli-Villars) in the field theory limit of open string theory lead either to a commutative or a noncommutative gauge theory [1] and thus suggest an equivalence between them. Later, the Seiberg–Witten map has been extensively studied. One of the most interesting approaches is set within the Kontsevich star product formalism [2]. Here, the Seiberg–Witten map is found to constitute (part of) an equivalence map between equivalent star products [3–6]. In particular, this study shows that the Seiberg–Witten map is an integral part of any noncommutative gauge theory obtained through deformation quantization of a Poisson manifold. The solution is constructive and can, at least in the Abelian case, be used to compute the SW-map for the gauge field.

On noncommutative $\mathbb{R}^4_\theta$ characterized by a constant noncommutativity parameter θ, the Seiberg–Witten map has been constructed using various other techniques. Switching to a BRST-setting the map can be constructed perturbatively using a cohomological approach as well as the initial

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assumption of gauge equivalence [7, 8] (see also [9] and references therein for a treatment within both the BRST and BV formalisms). Further, the Seiberg–Witten differential equation may be derived using an ansatz of covariant coordinate transformations [10–12]. The latter approach is closely related to that of [5] in the sense that it gives a physical interpretation to the application of (a quantum version of) Moser's Lemma [13]. It differs, however, by its applicability to all fields and its straight forward treatment of the non-Abelian case.

The Seiberg–Witten map has been exploited to formulate noncommutative gauge theories perturbatively in θ [14–16]. Recently, a noncommutative version of the Standard Model was proposed using Seiberg–Witten maps for all fields [17, 18]. Such theories involve a genuine coupling constant of negative dimension and thus jeopardize renormalization. This question was studied in [19, 20] for the noncommutative Maxwell theory and in [21, 22] it was shown that θ-expanded models must involve all possible counterterms linear in θ permitted by power-counting. At higher orders in θ things become more involved; the conclusion seems to be that such theories should be regarded as effective theories only.

In the present paper we reinvestigate the derivation of the Seiberg–Witten map presented in [10]. In particular, we demonstrate that the map can be constructed without additional information about the action of the given theory.

The note is organized as follows: In Section 2 we give a definition of a Seiberg–Witten map. In Section 3 we introduce coordinate transformations on $\mathbb{R}^4_\theta$ and demonstrate in Section 4 that particle (*i.e.* active) coordinate transformations compatible with the gauge symmetry must exist on physical grounds. This leads to the Seiberg–Witten differential equation for the noncommutative gauge field. In Section 5 we expand everything around $\theta = 0$ to obtain the gauge equivalence condition. Finally, we summarize in Section 6.

2. The Seiberg–Witten map

Briefly stated, a SW-map is a map between gauge theories defined on a noncommutative algebra and its commutative counterpart which maps gauge equivalent classes onto gauge equivalent classes in a manner that allows a *perturbation* of the theory in the noncommutativity parameter characterizing the noncommutative algebra.

More specific, let a general noncommutative algebra $\mathcal{A}_\theta$ be characterized by a noncommutativity parameter θ. For $\theta = 0$ the algebra coincides with the manifold $\mathcal{M}$. We assume that a gauge theory can be formulated on $\mathcal{A}_\theta$ involving the gauge field A and possible matter fields Ψ. We denote by δ_Λ the (infinitesimal) noncommutative gauge transformations involving the gauge parameter Λ. For $\theta = 0$ the corresponding gauge field, matter field and gauge parameter are denoted by a, ψ and λ. By a SW-map we understand a map

$$\rho : \mathcal{A}_\theta \to \mathcal{M}\,, \tag{2.1}$$

where (I) $\rho(A)$ takes the form of an expansion of the noncommutative field in terms of the deformation parameter θ and the corresponding commutative field and gauge parameter

$$\rho\big(A\big) = a + \mathcal{O}(\theta)\,, \qquad \rho\big(\Psi\big) = \psi + \mathcal{O}(\theta)\,. \tag{2.2}$$

Usually, one omits the $'\rho\,'$ and simply write $A(a,\theta)$ *etc.* (II) the expansion is constructed to respect the gauge symmetry in the sense that the *gauge equivalence condition* must hold:

$$A(a,\theta) + \delta_\Lambda A(a,\theta) = A(a + \delta_\lambda a, \theta)\,, \tag{2.3}$$

where λ is a commutative gauge parameter related to Λ. For the matter fields the gauge equivalence condition reads

$$\Psi(\psi, a, \theta) + \delta_\Lambda \Psi(\psi, a, \theta) = \Psi(\psi + \delta_\lambda \psi, a + \delta_l a, \theta)\,. \tag{2.4}$$

The map is generated by differential equations

$$\delta A = \delta\theta \mathcal{X}\,, \quad \delta\Psi = \delta\theta \mathcal{X}' \quad etc\,. \tag{2.5}$$

It is important to realize that the Seiberg–Witten map is a map from the noncommutative to the commutative algebra and not the reverse, as it is sometimes stated in the literature.

3. Noncommutative conformal transformations

We are interested in gauge theory defined on noncommutative $\mathbb{R}^4$ which is understood to be the algebra $\mathbb{R}^4_\theta$ of Schwarz class functions on ordinary $\mathbb{R}^4$ equipped with the $\star$-product

$$(f \star g)(x) = \int d^4y \int \frac{d^4k}{(4\pi)^4} f\left(x + \frac{1}{2}\,\theta \cdot k\right) g(x+y) \mathrm{e}^{\mathrm{i}k\cdot y}\,, \tag{3.1}$$

where $\theta^{\mu\nu}$ is an anti-symmetric constant tensor of dimension [length]^2. In a field theory context $\theta^{\mu\nu}$ represents a background field. The noncommutative gauge field A_μ transforms under infinitesimal noncommutative gauge transformations according to

$$\delta^g_\lambda A_\mu \;=\; \partial_\mu \lambda - \mathrm{i}[A_\mu, \lambda]_\star = D_\mu \lambda\,, \tag{3.2}$$

where the $\star$-commutator $[\cdot,\cdot]_\star$ is the ordinary commutator equipped with the $\star$-product. The background field $\theta^{\mu\nu}$ clearly does not transform under gauge transformations:

$$\delta^g_{\hat{\lambda}}\theta^{\mu\nu} \;=\; 0\,. \tag{3.3}$$

Conformal transformations on $\mathbb{R}^4_\theta$ are naturally divided into three different kinds:

1. *Observer transformations* (denoted δ^{obs}): The transformations of the reference frame. Amounts to a transformation of *both* dynamical and background fields. Such transformations always constitute an invariance of the theory.

2. *Particle transformations* (denoted δ^{par}): The transformation of dynamical fields only.

3. *Inverse particle transformations* (denoted $\delta^{-\mathrm{par}}$): The transformation of background fields only.

In the absence of background fields the first two kinds of conformal transformations coincide. In the general case one has

$$\delta^{\mathrm{obs}} = \delta^{\mathrm{par}} + \delta^{-\mathrm{par}}\,. \tag{3.4}$$

Infinitesimal observer conformal transformations of the noncommutative gauge field A_μ and the background field $\theta^{\mu\nu}$ are easily constructed from conformal transformations of the commutative gauge field using the 'quantization' $x^\mu\cdot\mathcal{O} \rightarrow \frac{1}{2}\{x^\mu,\mathcal{O}\}_\star$ where the $\star$-anti-commutator $\{\cdot,\cdot\}_\star$ is the ordinary anti-commutator equipped with the $\star$-product. We define

$$\delta^{\mathrm{obs}}_f A_\mu \;=\; \tfrac{1}{2}\{f^\alpha,\partial_\alpha A_\mu\}_\star + \tfrac{1}{2}\{\partial_\mu f^\alpha, A_\alpha\}_\star\,, \tag{3.5}$$

where f^α is a Killing vector. Further, we let the background field $\theta^{\mu\nu}$ transform as a tensor under observer transformations

$$\delta^{\mathrm{obs}}_f\theta^{\mu\nu} \;=\; \tfrac{1}{2}\{f^\sigma,\partial_\sigma\theta^{\mu\nu}\}_\star - \tfrac{1}{2}\{\partial_\sigma f^\mu,\theta^{\sigma\nu}\}_\star - \tfrac{1}{2}\{\partial_\sigma f^\nu,\theta^{\mu\sigma}\}_\star\,. \tag{3.6}$$

Since $\theta^{\mu\nu}$ is constant and should stay constant we must leave out special conformal transformations. This means that f^α is at most linear in x^μ and (3.6) reduces to

$$\delta^{\mathrm{obs}}_f\theta^{\mu\nu} \;=\; -\partial_\sigma f^\mu\theta^{\sigma\nu} - \partial_\sigma f^\nu\theta^{\mu\sigma}\,. \tag{3.7}$$

The gauge and conformal transformations constructed so-far fulfill the algebra

$$\begin{aligned}
\left[\delta_f^{\mathrm{obs}}, \delta_g^{\mathrm{obs}}\right] &= \delta_h^{\mathrm{obs}}\,, & h^\alpha &= \{g^\beta, \partial_\beta f^\alpha\}_\star - \{f^\beta, \partial_\beta g^\alpha\}_\star\,, && (3.8)\\
\left[\delta_{\lambda_1}^g, \delta_{\lambda_2}^g\right] &= \delta_{\lambda_3}^g\,, & \lambda_3 &= -\mathrm{i}[\lambda_1, \lambda_2]_\star + \delta_{\lambda_1}^g \lambda_2 - \delta_{\lambda_2}^g \lambda_1\,, && (3.9)\\
\left[\delta_\lambda^g, \delta_f^{\mathrm{obs}}\right] &= \delta_{\lambda^{\mathrm{obs}}(\lambda)}^g\,, & \lambda^{\mathrm{obs}}(\lambda) &= \tfrac{1}{2}\{f^\alpha, \partial_\alpha \lambda\}_\star - \delta_f^{\mathrm{obs}} \lambda\,. && (3.10)
\end{aligned}$$

Comment: In the commutative case the conformal transformations of the gauge field (obtained by setting $\theta^{\mu\nu}$ to zero in (3.5)) lead to an energy-momentum tensor which is not gauge invariant and thus unphysical. This deficit is corrected by adding a field-dependent gauge transformation to the conformal transformation [23]. The algebra of conformal transformations closes hereafter only up to gauge transformations. The important observation is that the commutative conformal transformations of the gauge field may be written as a gauge covariant part and a part which forms a gauge transformation. Subtracting this gauge transformation renders gauge covariant transformations. In the noncommutative case this does not work. Due to the noncommutative product, (3.5) cannot be rewritten as a covariant part and a gauge transformation. It turns out that the missing part can be interpreted as the θ-dependency of the gauge field which is given by the Seiberg–Witten differential equation (see also [12]).

4. Particle Lorentz transformations

We first attempt to define particle transformations by the (naive) dispartment

$$\delta_f^{\mathrm{par}} A_\mu := \delta_f^{\mathrm{obs}} A_\mu\,, \tag{4.1}$$

$$\delta_f^{\mathrm{par}} \theta^{\mu\nu} := 0\,. \tag{4.2}$$

Inverse particle transformations are then given by

$$\delta_f^{-\mathrm{par}} A_\mu := 0\,, \tag{4.3}$$

$$\delta_f^{-\mathrm{par}} \theta^{\mu\nu} := \delta_f^{\mathrm{obs}} \theta^{\mu\nu}\,, \tag{4.4}$$

in order to fulfill (3.4). However, it turns out that the transformations (4.1)–(4.5) do not fulfill the algebra (3.10):

$$\begin{aligned}
\left[\delta^g_\lambda, \delta^{\rm par}_f\right]\theta^{\alpha\beta} &= 0\,,\\
\left[\delta^g_\lambda, \delta^{\rm par}_f\right]A_\mu &= \delta^g_{\lambda'}A_\mu + (\delta^{-{\rm par}}_f\theta^{\alpha\beta})\tfrac{1}{2}\{\partial_\alpha A_\mu, \partial_\beta\lambda\}_\star\,,\\
\lambda' &= \tfrac{1}{2}\{f^\alpha, \partial_\alpha\lambda\}_\star - \delta^{\rm par}_f\lambda\,,\\
\left[\delta^g_\lambda, \delta^{-{\rm par}}_f\right]\theta^{\alpha\beta} &= 0\,,\\
\left[\delta^g_\lambda, \delta^{-{\rm par}}_f\right]A_\mu &= \delta^g_{\lambda''}A_\mu - (\delta^{-{\rm par}}_f\theta^{\alpha\beta})\tfrac{1}{2}\{\partial_\alpha A_\mu, \partial_\beta\lambda\}_\star\,,\\
\lambda'' &= -\delta^{-{\rm par}}_f\lambda\,,
\end{aligned}\tag{4.5}$$

where we used

$$\delta(A\star B) = (\delta A)\star B + A\star(\delta B) + \frac{\rm i}{2}(\delta\theta^{\alpha\beta})\partial_\alpha A\star\partial_\beta B\,,\tag{4.6}$$

and

$$[f^\alpha, *]_\star = {\rm i}\theta^{\mu\nu}\partial_\mu f^\alpha\partial_\nu *\ .\tag{4.7}$$

The non-closure of the algebra (4.5) represents a serious problem: Given an observable $\mathcal{O}$, $\delta^g_\lambda\mathcal{O} = 0$, consider the transformed observable, $\mathcal{O}' = \delta^{\rm par}_f\mathcal{O}$. Since the background field $\theta^{\mu\nu}$ breaks Lorentz invariance $\mathcal{O}'$ does not equal $\mathcal{O}$. It should, however, still be an observable, $\delta^g_\lambda\mathcal{O}' = 0$. This implies that the commutator $\left[\delta^g_\lambda, \delta^{\rm par}_f\right]$ must close up to another symmetry transformation of the theory. Thus, due to (4.5) we are lead to conclude that the transformations (4.1)–(4.4) do not display the correct (inverse) particle transformations.

Physically acceptable particle transformations, which we are about to construct, are required to

- involve observer transformations of the gauge field, since the limit $\theta^{\mu\nu}\to 0$ should cast ordinary conformal transformations
- *not* to involve observer transformations of the background field, since this would be a trivial dispartment
- fulfill the algebra (3.8) and (3.10).

To proceed we write down particle transformation in the general form

$$\delta^{\rm par}_f A_\mu := \delta^{\rm obs}_f A_\mu - \Xi_{f,\mu}\,,\tag{4.8}$$

$$\delta^{\rm par}_f\theta^{\mu\nu} := \Upsilon^{\mu\nu}_f\,,\tag{4.9}$$

followed by

$$\delta_f^{-\mathrm{par}} A_\mu := \Xi_{f,\mu}\,, \tag{4.10}$$

$$\delta_f^{-\mathrm{par}} \theta^{\mu\nu} := \delta_f^{\mathrm{obs}} \theta^{\mu\nu} - \Upsilon_f^{\mu\nu}\,, \tag{4.11}$$

where $\Upsilon_f^{\mu\nu} \neq \delta_f^{\mathrm{obs}} \theta^{\mu\nu}$ and $\Xi_{f,\mu} \neq \delta_f^{\mathrm{obs}} A_\mu$. Further, both $\Upsilon_f^{\mu\nu}$ and $\Xi_{f,\mu}$ must be directly proportional to $\theta^{\mu\nu}$

$$\Upsilon_f^{\mu\nu} = \theta^{\alpha\beta} \Upsilon_{f,\alpha\beta}^{\mu\nu}\,, \qquad \Xi_{f,\mu} = \theta^{\alpha\beta} \Xi_{f,\alpha\beta\mu}\,. \tag{4.12}$$

Next we make the ansatz[1]

$$\left[\delta_\lambda^g, \delta_f^{\mathrm{par}}\right] = \delta^g_{\lambda_f^{\mathrm{par}}(\lambda)}\,, \qquad \left[\delta_\lambda^g, \delta_f^{-\mathrm{par}}\right] = \delta^g_{\lambda_f^{-\mathrm{par}}(\lambda)}\,, \tag{4.13}$$

where $\lambda_f^{\mathrm{par}}(\lambda)$ and $\lambda_f^{-\mathrm{par}}(\lambda)$ are field dependent gauge parameters yet to be determined. In (4.13) the second condition follows from the first via (3.4) and (3.10). Finally, we also impose the condition that particle transformations must fulfill the conformal algebra

$$\left[\delta_f^{\mathrm{par}}, \delta_g^{\mathrm{par}}\right] = \delta_{h'}^{\mathrm{par}}\,. \tag{4.14}$$

However, we shall in the following not make use of this final condition[2]. If we apply the first commutator in (4.13) to $\theta^{\mu\nu}$ we immediately find that $\Upsilon_f^{\mu\nu}$ cannot transform under gauge transformations and thus cannot involve the gauge field (which would also violate our initial assumption that θ should remain constant). Thus

$$\Upsilon_f^{\mu\nu} = 0\,. \tag{4.15}$$

Next, we apply the commutator in (4.13) involving particle transformations to the noncommutative gauge field and attempt to solve for $\Xi_{f,\mu}$. The full solution consist of a covariant term $\Omega_{f,\mu}$ and a gauge transformation $D_\mu \lambda'_f$. In total we have

$$\Omega_{f,\mu} + D_\mu \lambda'_f = \tfrac{1}{2}\{f^\alpha, \partial_\alpha A_\mu\}_\star + \tfrac{1}{2}\{\partial_\mu f^\alpha, A_\alpha\}_\star + \Xi_{f,\mu}\,. \tag{4.16}$$

Let us first consider the term $\frac{1}{2}\{f^\alpha, \partial_\alpha A_\mu\}_\star$. This term is not proportional to $\theta^{\mu\nu}$ and cannot be reorganized to form (part of) a gauge transformation. The only option is to embed it in the covariant field polynomial $\Omega_{f,\mu}(A,\theta)$. We thus introduce the covariant quantity $\hat{f}^\alpha$

$$\hat{f}^\alpha = f^\alpha + \theta^{\sigma\beta} A_\beta \partial_\sigma f^\alpha\,, \tag{4.17}$$

[1] We assume that no further symmetries characterize the theory.

[2] It is important when dealing with the ambiguities in the Seiberg–Witten map.

and write

$$\Omega_{f,\mu}(A,\theta) \;=\; \tfrac{1}{2}\{\hat{f}^{\alpha}, F_{\alpha\mu}\}_\star + \Omega'_{f,\mu}(A,\theta)\,, \tag{4.18}$$

where $\Omega'_{f,\mu}(A,\theta)$ is a general covariant quantity and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu]_\star$ is the noncommutative field strength tensor. Using (4.7) as well as a graded Jacobi identity, we find

$$\begin{aligned}\Omega_{f,\mu}(A,\theta) \;=\;& \frac{1}{2}\{f^{\alpha}, \partial_\alpha A_\mu\}_\star - \frac{1}{2}\{f^{\alpha}, \partial_\mu A_\alpha\}_\star - \frac{1}{2}\theta^{\sigma\beta}\partial_\sigma f^{\alpha}\{A_\alpha, \partial_\beta A_\mu\}_\star \\ &+\frac{1}{2}\theta^{\sigma\beta}\partial_\sigma f^{\alpha}\{A_\beta, F_{\alpha\mu}\}_\star + \frac{\mathrm{i}}{2}[A_\mu, \{f^{\alpha}, A_\alpha\}_\star]_\star + \Omega'_{f,\mu}(A,\theta)\,.\end{aligned} \tag{4.19}$$

The first term in (4.19) is what we wanted. The second and fifth terms are not proportional to $\theta^{\mu\nu}$ and should, due to (4.16), be transformed into a gauge transformation:

$$-\frac{1}{2}\{f^{\alpha}, \partial_\mu A_\alpha\}_\star + \frac{\mathrm{i}}{2}[A_\mu, \{f^{\alpha}, A_\alpha\}_\star]_\star = D_\mu\left(-\frac{1}{2}\{f^{\alpha}, A_\alpha\}_\star\right) + \frac{1}{2}\{\partial_\mu f^{\alpha}, A_\alpha\}_\star. \tag{4.20}$$

Finally, the last term in (4.19) must be proportional to θ according to

$$\Omega'_{f,\mu}(A,\theta) = \theta^{\alpha\beta}\Omega'_{f,\mu\alpha\beta}(A,\theta)\,. \tag{4.21}$$

In total we find

$$\begin{aligned}&D_\mu\big(\lambda'_f - \tfrac{1}{2}\{f^{\alpha}, A_\alpha\}_\star\big) + \Omega'_{f,\mu}(A,\theta) \\ &+\tfrac{1}{2}\theta^{\sigma\beta}\partial_\sigma f^{\alpha}\big(-\{A_\alpha, \partial_\beta A_\mu\}_\star + \{A_\beta, F_{\alpha\mu}\}_\star\big) = \Xi_{f,\mu}\,,\end{aligned} \tag{4.22}$$

and we read off

$$\lambda'_f \;=\; \tfrac{1}{2}\{f^{\alpha}, A_\alpha\}_\star\,, \tag{4.23}$$

$$\Xi_{f,\mu} \;=\; -2\theta^{\sigma\beta}\partial_\sigma f^{\alpha}\,\Xi_{\mu\alpha\beta} - \theta^{\alpha\beta}\Omega'_{f,\mu\alpha\beta} \tag{4.24}$$

with

$$\Xi_{\mu\alpha\beta} = \tfrac{1}{4}\{A_\alpha, \partial_\beta A_\mu\}_\star - \tfrac{1}{4}\{A_\beta, F_{\alpha\mu}\}_\star\,. \tag{4.25}$$

In the following we shall set $\Omega'_{f,\mu\alpha\beta}$ and simply bear in mind that all results are valid up to a covariant term. Notice that $\Xi_{f,\mu}$ is only determined up to θ-dependent gauge transformations. If we add and subtract the gauge transformation

$$\begin{aligned}\kappa D_\mu\big(\theta^{\sigma\beta}\partial_\sigma f^{\alpha}\{A_\beta, A_\alpha\}_\star\big) \;=\;& \kappa\theta^{\sigma\beta}\partial_\sigma f^{\alpha}\big(-\{A_\alpha, F_{\beta\mu}\}_\star + \{A_\alpha, \partial_\beta A_\mu\}_\star \\ &-\{A_\beta, F_{\alpha\mu}\}_\star + \{A_\beta, \partial_\alpha A_\mu\}_\star\big)\end{aligned} \tag{4.26}$$

to (4.22), we obtain for $\kappa = \frac{1}{2}$ the transformation $\alpha \leftrightarrow \beta$ in $\Xi_{\mu\alpha\beta}$ and for $\kappa = \frac{1}{4}$ the anti-symmetric expression

$$\Xi_{\mu\alpha\beta} \;=\; \tfrac{1}{8}\{A_\alpha, \partial_\beta A_\mu + F_{\beta\mu}\}_\star - \tfrac{1}{8}\{A_\beta, \partial_\alpha A_\mu + F_{\alpha\mu}\}_\star\,, \tag{4.27}$$

which means that we may write (4.24) as

$$\Xi_{f,\mu} \;=\; \left(\delta_f^{-\mathrm{par}}\theta^{\alpha\beta}\right)\Xi_{\mu\alpha\beta}\,. \tag{4.28}$$

The gauge parameter λ'_f now reads

$$\lambda'_f \;=\; \tfrac{1}{2}\{f^\alpha + \tfrac{1}{2}\theta^{\sigma\beta}\partial_\sigma f^\alpha A_\beta, A_\alpha\}_\star\,. \tag{4.29}$$

In total, we find

$$\delta_f^{-\mathrm{par}} A_\mu \;:=\; \left(\delta_f^{-\mathrm{par}}\theta^{\alpha\beta}\right)\Xi_{\mu\alpha\beta}\,. \tag{4.30}$$

If we write this in the suggestive form

$$\delta_f^{-\mathrm{par}} A_\mu \;:=\; \left(\delta_f^{-\mathrm{par}}\theta^{\alpha\beta}\right)\frac{dA_\mu}{d\theta^{\alpha\beta}}\,, \tag{4.31}$$

we are lead to the Seiberg–Witten differential equation

$$\frac{dA_\mu}{d\theta^{\alpha\beta}} = \tfrac{1}{8}\{A_\alpha, \partial_\beta A_\mu + F_{\beta\mu}\}_\star - \tfrac{1}{8}\{A_\beta, \partial_\alpha A_\mu + F_{\alpha\mu}\}_\star\,. \tag{4.32}$$

Using (3.9) we find the field dependent gauge parameter $\lambda_f^{\mathrm{par}}(\lambda)$

$$\lambda_f^{\mathrm{par}}(\lambda) = -\tfrac{1}{4}\theta^{\sigma\beta}\partial_\sigma f^\alpha\left(\{A_\alpha, \partial_\beta\lambda\}_\star - \{A_\beta, \partial_\alpha\lambda\}_\star\right) + \tfrac{1}{2}\{f^\alpha, \partial_\alpha\lambda\}_\star - \delta_f^{\mathrm{par}}\lambda\,, \tag{4.33}$$

and since

$$\lambda_f^{\mathrm{obs}}(\lambda) = \lambda_f^{\mathrm{par}}(\lambda) + \lambda_f^{-\mathrm{par}}(\lambda)\,, \tag{4.34}$$

we may write down the gauge parameter $\lambda_f^{-\mathrm{par}}(\lambda)$ as well:

$$\lambda_f^{-\mathrm{par}}(\lambda) = +\tfrac{1}{4}\theta^{\sigma\beta}\partial_\sigma f^\alpha\left(\{A_\alpha, \partial_\beta\lambda\}_\star - \{A_\beta, \partial_\alpha\lambda\}_\star\right) - \delta_f^{-\mathrm{par}}\lambda\,. \tag{4.35}$$

Let us consider under which circumstances $\lambda_f^{-\mathrm{par}}(\lambda)$ vanishes, *i.e.* gauge parameters $\tilde\lambda$ for which

$$\left[\delta^g_{\tilde\lambda}, \delta_f^{-\mathrm{par}}\right] = 0\,. \tag{4.36}$$

Comparing (4.35) we find this to be the case whenever

$$\delta_f^{-\mathrm{par}}\tilde{\lambda} = \frac{1}{4}\theta^{\sigma\beta}\partial_\sigma f^\alpha\big(\{A_\alpha, \partial_\beta\tilde{\lambda}\}_\star - \{A_\beta, \partial_\alpha\tilde{\lambda}\}_\star\big)\,, \tag{4.37}$$

which we again may write in the form of a differential equation

$$\delta_f^{-\mathrm{par}}\tilde{\lambda} = \big(\delta_f^{\mathrm{obs}}\theta^{\alpha\beta}\big)\frac{d\tilde{\lambda}}{d\theta^{\alpha\beta}}\,, \tag{4.38}$$

which thus gives us a differential equation for the gauge parameter

$$\frac{d\tilde{\lambda}}{d\theta^{\alpha\beta}} = \tfrac{1}{8}\{A_\alpha, \partial_\beta\tilde{\lambda}\}_\star - \tfrac{1}{8}\{A_\beta, \partial_\alpha\tilde{\lambda}\}_\star\,. \tag{4.39}$$

This is exactly the condition on the gauge parameter one uses in the usual derivation of the Seiberg–Witten map.

5. Gauge equivalence

Let us in the following take a closer look at (4.13) perturbed around the point $\theta = 0$. If we set $f^\alpha = \varepsilon x^\alpha$ (dilatation) we find that $\delta_f^{-\mathrm{par}} = \varepsilon\theta_{ij}\frac{d}{d\theta_{ij}}$ and further

$$\left[\frac{d}{d\theta_{ij}}, \delta_\lambda^g\right] = \delta_{\lambda_{ij}}^g\,, \tag{5.1}$$

where $\lambda_{ij} = \varepsilon^{-1}\theta_{ij}^{-1}\lambda^{-\mathrm{par}}(\lambda)$. We first calculate

$$\begin{aligned}\frac{d^2\delta_\lambda^g A_i}{d\theta_{j_2k_2}d\theta_{j_1k_1}} = &\Bigg(\delta^g_{\lambda_{j_2k_2}(\lambda_{j_1k_1}(\lambda))} + \delta^g_{\lambda_{j_1k_1}(\lambda)}\frac{d}{d\theta_{j_2k_2}}\\ &+\delta_\lambda^g\frac{d}{d\theta_{j_2k_2}}\frac{d}{d\theta_{j_1k_1}} + \delta^g_{\lambda_{j_2k_2}(\lambda)}\frac{d}{d\theta_{j_1k_1}}\Bigg)A_i\,,\end{aligned} \tag{5.2}$$

setting $\theta = 0$ and generalizing it to any order n and inserting the results into a Taylor expansion of the fields and gauge parameters around $\theta = 0$ we find

$$\Big(\delta_\lambda^g A_i\Big)[a,\theta] = \delta^g_{l[\lambda,a,\theta]}\Big(A_i[a,\theta]\Big)\,, \tag{5.3}$$

where the commutative gauge parameter $l[\lambda, a, \theta]$ is given by

$$l[\lambda,a,\theta] = \big(\lambda\big)_{\theta=0} + \theta_{j_1k_1}(\lambda_{j_1k_1}(\lambda))_{\theta=0} + \tfrac{1}{2}\theta_{j_2k_2}\theta_{j_1k_1}(\lambda_{j_1k_1}(\lambda_{j_2k_2}(\lambda)))_{\theta=0} + \ldots\,. \tag{5.4}$$

Using (4.35) we find

$$\lambda_{jk} = \frac{1}{4}\big(\{A_j, \partial_k \lambda\}_\star - \{A_k, \partial_j \lambda\}_\star\big) - 2\frac{d\lambda}{d\theta_{jk}}\,. \tag{5.5}$$

Equation (5.3) equals the original gauge equivalence condition given by Seiberg and Witten in [1] whenever $\big(\lambda_{jk}(\lambda)\big)_{\theta=0} = 0$ which is the case when the gauge parameter fulfills (4.39). Without further conditions on the gauge parameter equation (5.3) reassembles the condition originally found in [5]. Further, if we define the vector $a_\star = \frac{1}{2}\theta_{jk} a_j \partial_k$ and scale the noncommutative parameter with a scaling factor t, $\theta_{jk} \to t\theta_{jk}$, we find

$$l[\lambda, a, \theta] = \sum_{n=0}^{\infty} \frac{(a_\star + \partial_t)^n(\lambda)}{(n+1)!}\bigg|_{\theta=0}\,, \tag{5.6}$$

which is the same expression for the gauge parameter l found in [5].

6. Conclusion

We have demonstrated that the derivation of covariant coordinate transformations naturally lead to the construction of the Seiberg–Witten differential equation. Further, we have shown that the derivation may be carried through without using additional information about the action. Finally, we find that our approach is closely connected to the analysis carried out in [5] within the Kontsevich star-product formalism. However, the derivation presented here differs on a few important points: First of all, in the above analysis we do not only specify any details on the gauge group, *i.e.* Abelian or non-Abelian. Thus, the derivation works equally well for matrix-valued gauge fields. Further, as demonstrated in [11] for spin-half fields and in [24] for scalar fields presents in a supersymmetric model, the method works for any field transforming under gauge transformations.

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