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CONTINUUM BRANCHED POLYMERS*

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We discuss geometric paths and review the theory of continuous trees which has been developed in the last 12 years. We explain the relation of continuum trees to the extensively studied discrete trees.

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1. Introduction

Random geometry (or quantum geometry) can be viewed as classical statistical mechanics of random geometrical objects: paths, trees (or branched polymers), surfaces, higher dimensional manifolds. Quantum geometry arises naturally in particle physics and quantum gravity, where the functional integrals used for quantization are often most naturally viewed as being taken over geometrical objects. Random geometry arises also in many branches of condensed matter physics, *e.g.*, in the study of phase boundaries, vesicles, branched polymers, foams *etc.* For a review of this theory up to 1997, written from the point of view of quantum field theory, see [1].

The first step in the analysis of random geometry is usually to define a class of discrete approximations to the geometrical objects under investigation. These discrete approximations are described by a finite number of parameters so that the functional integrals become finite dimensional integrals or a convergent sum of finite dimensional integrals. The second step is the construction of a scaling limit of the theory at an appropriate critical point. The scaling limit of the Green functions in the discrete theory are generally believed to describe the underlying continuum theory. The ultimate goal in the study of models in quantum geometry is to understand in detail how to integrate over the space of continuous geometrical objects. This requires the construction of a measure on the appropriate space of continuum paths, surfaces, trees, *etc.*

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It is fair to say that in the case of surfaces and higher dimensional manifolds the last step has not been achieved. In the case of higher dimensional manifolds even the first step is problematic since in general bounds which ensure the convergence of discrete approximations are not known, see [1]. In [2] it is shown how to intgrate over continuous geometric paths and the measure on such paths is constructed. Roughly speaking this is achieved by dividing the Wiener measure by the action of the diffeomorphism group on parametrized paths. We review this theory in the next section. In the third section we describe Aldous's theory of random continuum trees, see [3] for a detailed review for probabilists and [4] for a different perspective aimed at physicists.

Trees and continuous trees are interesting in their own right as an example of a relatively simple random geometry. More importantly, discrete surfaces can be represented as certain labelled trees [5] so it is possible that one can define integration over continuous random surfaces by integrating over random labelled continuous trees.

2. Geometric paths

It is well known that the Euclidean free field propagator can be expressed formally as

$$(-\Delta + m^2)^{-1}(x,y) \equiv G(x,y) = \int\limits_{\omega: x \to y} \mathrm{e}^{-m|\omega|} D\omega, \tag{1}$$

where the functional integral is over all *geometric paths* ω from x to y and $|\omega|$ denotes the length of ω. A geometric path is an equivalence class of parametrized paths $\Omega : [0,1] \mapsto \boldsymbol{R}^d$ under the equivalence relation $\sim$ where $\Omega \sim \Omega'$ if and only if there is an increasing diffeomorphism φ of the unit interval such that $\Omega' = \Omega \circ \varphi$.

Let us denote the set of all geometric paths from x to y by $P(x,y)$. In [2] a probability measure μ is constructed on $P(x,y)$ such that

$$\langle F(\omega) \rangle \equiv \int F(\omega) d\mu = \frac{1}{G(x,y)} \int\limits_{\omega: x \to y} \mathrm{e}^{-m|\omega|} F(\omega) D\omega, \tag{2}$$

where the equality above means that natural discrete approximations to the functional integral, *e.g.*, the approximation using lattice paths with lattice spacing a, converge to the expectation value with respect to μ as $a \to 0$. It comes as no surprise that the above equality holds because the measure μ is in fact constructed as a limit of measures on discretized paths. In [2] two different schemes are introduced, one using lattice paths and the other using picewise linear paths with an intrinsic metric. The ensuing continuum limits are equivalent. Similarly, using the same techniques, one can construct a probability measure on the space of geometric paths with an arbitrary endpoint.

The measure μ can also be described directly in terms of the Wiener measure. Let W_t be the Wiener measure with variance t on paths from x to y parametrized by the unit interval. Then the measure of a set of geometric paths $A \subset P(x, y)$ is given by

$$\mu(A) = \int_0^\infty \mathrm{e}^{-m^2 t/2} W_t(\pi^{-1}(A))\, dt\,, \tag{3}$$

where π is the canonical mapping of parametrized paths onto their equivalence classes in $P(x, y)$. This result is established in [2] together with the corresponding result for paths with a free endpoint.

Geometric paths have no intrinsic structure and they cannot be parametrized by arclength since their local behaviour is the same as that of Wiener paths which have infinte arclength. However, the points on the geometric path are linearly ordered since such ordering on a parametrized paths is preserved by increasing diffeomorphisms. It therefore makes sense to ask whether a geometric path in $\boldsymbol{R}^d$ hits a certain set A or whether it hits a number of mutually disjoint sets $A_1, \ldots, A_n$ in that order. The typical functions one could imagine integrating with respect to the measure μ are therefore of the type

$$\chi(\omega) = \begin{cases} 1, & \text{if } \omega \text{ first hits } A \text{ and then } B \\ 0, & \text{otherwise} \end{cases}\,, \tag{4}$$

where A and B are two disjoint subsets of $\boldsymbol{R}^d$. In [2] a convenient family of sets of geometric paths to integrate over is defined. These sets are called cylinder sets by analogy with the cylinder sets of parametrized paths and it is shown that the measures of of cylinder sets define μ uniquely. The measures of cylinder sets can be evaluated in terms of Green functions for Dirichlet propagators. The simplest example is

$$\mu(Z(A)) = \frac{G_A^{\mathrm{D}}(x, y)}{G(x, y)}, \tag{5}$$

where A is a sufficiently nice subset of $\boldsymbol{R}^d$ which contains x and y, $G_A^{\mathrm{D}}(x, y)$ is the Dirichlet Green function for the Helmholtz operator with data on the boundary of A and $Z(A)$ is the set of all geometric paths from x to y which are contained in the set A.

3. Continuous random trees

We begin by discussing the theory of discrete random trees. By discrete trees we understand rooted planar trees, *i.e.*, one of the vertices of each tree is singled out and called the root and the trees should be thought of as embedded in the plane so that the links are cyclically ordered around each vertex. The standard theory of discrete trees is defined by the partition function

$$Z(x) = \sum_t x^{|t|} w(t), \tag{6}$$

where the sum runs over all rooted planar trees t, $|t|$ denotes the number of links in t and $w(t)$ is the weight factor of the tree t. Typically the weight factor $w(t)$ is chosen to be a product of weight factors associated with vertices of the tree, *i.e.*,

$$W(t) = \prod_{v \in t} \nu_k \,, \tag{7}$$

where ν_k is the weight factor for a vertex v in t of order $\sigma(v) = k$. With these definitions the partition function satisfies the equation

$$Z(x) = x \sum_{k=1}^{\infty} \nu_k Z^{k-1}(x). \tag{8}$$

This equation can be analysed in order to determine the critical behaviour of the discrete theory, see [6, 7]. The critical behaviour is universal provided the ν_k go to zero sufficiently fast with k.

The weight factors defined above give rise to a probability measure on trees with a fixed number n of links:

$$P(\mathcal{T} = t) = C \prod_{v \in t} \nu_{\sigma(v)}, \tag{9}$$

where C is a normalization factor. Alternatively one can write

$$P(\mathcal{T} = t) = C \prod_{k \geq 1} \nu_k^{D_k(t)}, \tag{10}$$

where $D_k(t)$ is the number of vertices of order k in t. If

$$\nu_k = \left(\frac{1}{2}\right)^k, \tag{11}$$

then all trees with n vertices are equally probable and

$$P(\mathcal{T} = t) = \frac{1}{\sharp\{t : |t| = n\}} = \frac{n!(n-1)!}{(2n-2)!}. \tag{12}$$

For the following discussion it is most convenient to focus on trees with a fixed number n of vertices and use the fundamental fact [8] that there is a one to one correspondence between trees with $n+1$ vertices and "discrete excursions" of duration $2n$, *i.e.*, random walks on the positive integers which begin and end at 0. This correspondence is best explained by a picture, see Fig. 1. We draw the tree in the plane. The excursion $X(i)$, where $X(i)$ is the distance of the walker from the root after i steps, is defined by starting at the root and walking counterclockwise around the tree, visiting the vertices

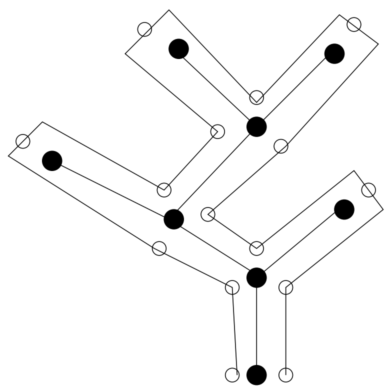

Fig. 1. A rooted planar tree with 8 vertices and the associated contour walk.

sucessively and traversing each link twice as indicated in the figure. Fig. 2 shows $X(i)$ as a function of i. A measure on discrete trees obviously defines a measure on discrete excursions via the correspondence. *A priori* we do not know how to define continuous trees but the correspondence between discrete excursions and discrete trees allows us to define continuous trees as the objects associated with continuous excursions in the same way as discrete trees are associated with discrete excursions.

Let $f : \boldsymbol{R}^+ \mapsto \boldsymbol{R}^+$ be a continuous function such that $f(0) = 0$, $f(s) > 0$ for $0 < s < \gamma$ and $f(s) = 0$ for $s > \gamma$. Such a function is said to be a continuous excursion of duration γ. We define a pseudometric on the closed interval $[0, \gamma]$ by putting

$$d(s, s') = f(s) + f(s') - 2 \inf_{s \le r \le s'} f(r) \tag{13}$$

for $s, s' \in [0, \gamma]$ and $s < s'$. This is not a metric because we can have $d(s, s') = 0$ with $s \neq s'$ but d is easily seen to satisfy the triangle inequality. Now define an equivalence relation $\sim$ on $[0, \gamma]$ by $s \sim s'$ if $d(s, s') = 0$. Then the *tree coded by the excursion* f is defined to be the metric space $[0, \gamma]/\sim$.

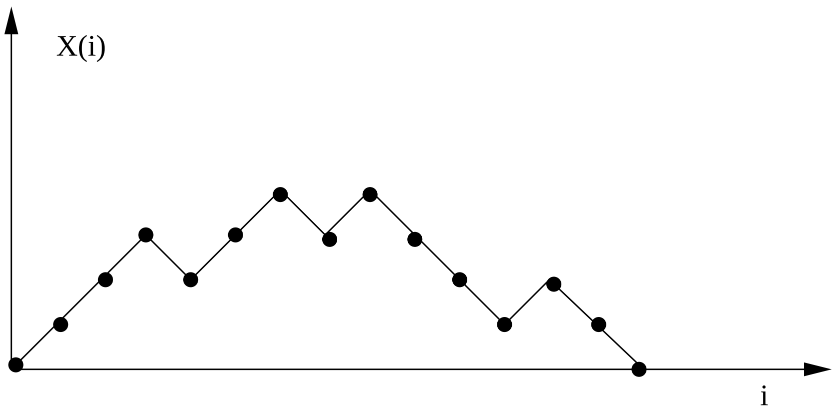

Fig. 2. The excursion $X(i)$ corresponding to the tree on Fig. 1.

It is perhaps most convenient to think of the continuous tree using the genealogical picture. Then $f(s)$ should be thought of as the generation of the individual s. If $s < s'$ then s is an ancestor of s' if and only if

$$f(s) = \inf_{s \leq r \leq s'} f(r) \tag{14}$$

and

$$\inf_{s<r<s'} f(r) = \text{generation of the last common ancestor of } s \text{ and } s'. \tag{15}$$

If we now have a measure on the space of continuous excursions it gives rise to a measure on the continuous trees via the correspondence we have described above. We would of course like to have a measure on the continuous trees that is in a natural sense a limit of the measure on discrete trees. This is taken care of by a theorem whose most general form is due to Aldous [9,10]. Let $X_t^n \equiv n^{-1/2} X([2nt])$ with $0 \leq t \leq 1$ be a stochastic process where $X(i)$ is a discrete excursion and $[x]$ denotes the integer part of x. Let B_t be a Brownian excursion of duration 1 (which can be thought of as a Brownian motion which starts at 0 at time 0 and is constrained to be positive until it returns to 0 at time $t = 1$, see [11,12] for a more precise discussion). Let σ be the variance of the offspring probability distribution associated with the weight factors ν_k. This is the distribution defined by

$$\text{Prob.}(q \text{ children}) \propto \nu_{q+1}. \tag{16}$$

With the above definitions, Aldous's theorem states that the process X_t^n converges weakly to $2\sigma^{-1} B_t$ as $n \to \infty$.

The Brownian excursion can be characterized by the density

$$q_s(x) = \left(\frac{2}{\pi}\right)^{1/2} s^{-3/2}(1-s)^{-3/2}x^2\mathrm{e}^{-x^2/2s(1-s)} \tag{17}$$

which is the probability density for an excursion of duration 1 to be located at $x > 0$ at time $s \in (0,1)$. The above formula does however not yield much insight into the structure of the associated random continuous trees but it can be used to calculate some of their properties. For example, the occupation density ℓ_h of the tree at distance h from the root which for a given excursion W_t is defined by

$$\ell_h = \frac{d}{ds}\int \chi_{\{W_t \le s\}}dt, \tag{18}$$

where χ_A is the characteristic function of the set A, has the average

$$\langle \ell_h \rangle = 4h\mathrm{e}^{-2h^2}. \tag{19}$$

More insight into the structure is obtained by a direct construction of the random continuous tree due to Aldous. We will now outline this direct construction which is based on taking a half line, cutting it up in a particular random way and gluing the pieces together in another particular random way.

Let $0 = T_0 < T_1 < T_2 < T_3 < \ldots$ be the times of a Poisson process with rate t, *i.e.*, the rate increases linearly with time so we can think of the times T_i as the clicks in a Geiger counter measuring the radioactivity of a substance whose decay rate increases linaerly with time. Construct the line segments $[T_i, T_{i+1})$ for $i = 0, 1, 2, \ldots$. We now define an infinte rooted tree inductively. Let t_1 be the interval $[0, T_1)$ with 0 the root. Inductively, let t_{k+1} be the tree obtained by gluing the interval $[T_k, T_{k+1})$ to t_k at a random point chosen with the uniform probability distribution on t_k. The limiting tree as $k \to \infty$ is the random continuous tree and it can be shown to be the same as the one associated with Brownian excursion. This random tree can be realized rigorously as a random subset of a suitable Banach space [10] and not surprisingly this random subset has Hausdorff dimension 2.

Another picture of the random continuous tree can be obtained by looking at the subtree spanned by a number of points randomly chosen. Let μ be the measure on the random continuous tree induced by the Lebesgue measure on the unit interval. Now we are viewing the continuous tree as being generated by a Brownian excursion. Choose n points on the tree according to the measure μ. These n points together with the root span a subtree all of whose vertices have order one or three (the probability of finding a higher order vertex is zero) and $2n-1$ egdes with lengths $\ell_1, \ell_2, \ldots, \ell_{2n-1}$. Of course the tree is not completely specified by the edgelengths, it has some particular topology τ. Let

$$L = \sum_{i=1}^{2n-1} \ell_i. \tag{20}$$

Then [10, 11]

$$\text{Prob.}(\tau, \ell_1, \ell_2, \ldots, \ell_{2n-1}) = cL\mathrm{e}^{-L^2/2}, \tag{21}$$

where c is a normalization constant. It is quite interesting that this probability is independent of the topology τ and it can be shown to characterize the random continuum tree uniquely. Several other properties of the continuum tree are discussed in [3].

4. Conclusion

Aldous' theory of the random continuous tree provides a complete quantum geometry of trees. We have above outlined his approach and stated some of the more important results. In principle one can use this theory to answer any question about random continuous trees. Some properties of continuous trees are not completely understood, *e.g.*, it is not known how to define Brownian motion on a continuous tree in a precise way even though some suggestions are provided in [3]. One can therefore not calculate directly the spectral dimension of continuous trees while the spectral dimension of their discrete siblings are known to be 4/3 [13].

The most interesting potential application of Aldous' theory of continuous trees to quantum geometry would be to construct a measure on continuous random surfaces as suggested in [5]. There are some technical obstacles to carrying this programme through but using the correspondence between trees and surfaces seems to be the most promising way of getting a hold on integration theory for random surfaces.

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