QUANTUM MECHANICS, RANDOM MATRICES AND BMN GAUGE THEORY*

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We review how the identification of gauge theory operators representing string states in the *pp*-wave/BMN correspondence and their associated anomalous dimension reduces to the determination of the eigenvectors and the eigenvalues of a simple quantum mechanical Hamiltonian and analyze the properties of this Hamiltonian. Furthermore, we discuss the role of random matrices as a tool for performing explicit evaluation of correlation functions.

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1. Introduction

BMN gauge theory can be characterized as the theory which appears in the lower right corner of the diagram in figure 1. The upper line of the diagram symbolizes the celebrated AdS/CFT conjecture which relates $\mathcal{N}=4$ Super Yang Mills theory with gauge group $SU(N)$ to type IIB string theory on the ten-dimensional geometry $AdS_5 \times S_5$ [1]. Unfortunately, on this geometry the IIB string theory has so far resisted quantization. It was understood in [2], however, that imposing a Penrose limit (left vertical arrow) on $AdS_5 \times S_5$ one can obtain a simpler ten-dimensional geometry, known as a *pp*-wave, on which quantization of the IIB string is possible using light cone gauge [3, 4]. After this discovery an intriguing question was of course what would be the corresponding procedure on the gauge theory side. The answer was provided by Berenstein, Maldacena and Nastase who showed that the appropriate operation consisted in considering a certain sub-sector of the gauge theory and imposing a certain limit [5] (right vertical arrow). The theory which results from this operation is denoted as BMN gauge theory

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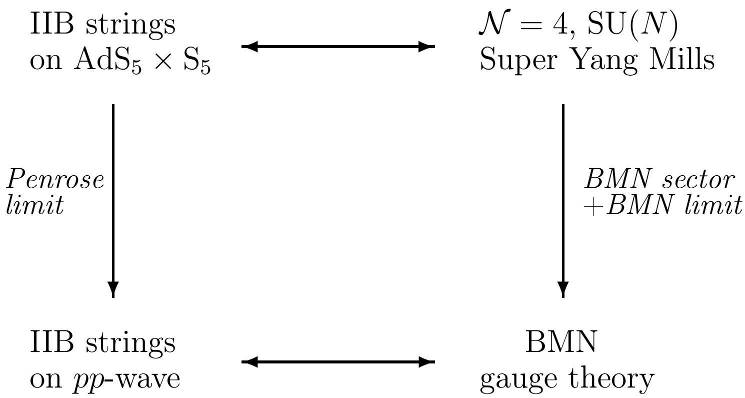

Fig. 1. The emergence of BMN gauge theory.

and is via the original AdS/CFT correspondence conjectured to be the gauge theory dual of the IIB *pp*-wave string.

As most gauge theories $\mathcal{N} = 4$ Super Yang Mills is described by two parameters the number of colours, N, and the 't Hooft coupling constant, $\lambda = g^2_{\rm YM} N$. The 't Hooft coupling constant governs the field theoretical loop expansion and N governs the topological expansion [6]. As just mentioned, to define BMN gauge theory one has to consider a certain sub-sector of $\mathcal{N} = 4$ Super Yang Mills. Part of this procedure consists in introducing a new parameter, J, the SO(2) R-charge. The SO(2) R-charge is a charge associated with a particular SO(2) sub-group of the bosonic SO(6) symmetry group of the $\mathcal{N} = 4$ Super Yang Mills theory. A BMN sector of $\mathcal{N} = 4$ Super Yang Mills can then be described as the set of operators for which the SO(2) R-charge takes some particular value, J, and BMN gauge theory is obtained by, in such a sector, considering the limit

$$J \to \infty, \qquad N \to \infty, \qquad g^2_{\rm YM} \text{ finite}, \qquad \frac{J^2}{N} \text{ finite}. \tag{1}$$

BMN-gauge theory is again described by two parameters

$$\lambda' = \frac{g^2_{\rm YM} N}{J^2}, \qquad \text{and} \qquad g_2 = \frac{J^2}{N}. \tag{2}$$

The parameter λ' is an effective gauge coupling constant which governs the loop expansion [5] and g_2 is an effective genus counting parameter which governs the topological expansion [7, 8].

Berenstein, Maldacena and Nastase argued that in the *pp*-wave/BMN correspondence string states should map to gauge invariant operators with the following identification of the quantum numbers

$$E_{\rm l.c.} = \Delta - J. \tag{3}$$

Here $E_{\rm l.c.}$ is the light cone energy of the string state and Δ and J are respectively the conformal dimension and the SO(2) R-charge of the gauge theory operator. For a detailed discussion of the string theory side of the correspondence, see for instance the reviews [9, 10].

Operators which have definite conformal dimensions are characterized by their two-point functions taking the form

$$\left\langle \tilde{\mathcal{O}}_A(x)\bar{\tilde{\mathcal{O}}}_B(0)\right\rangle = \frac{\delta_{AB}}{|x|^{2\Delta_A}}C_A, \tag{4}$$

with Δ_A the conformal dimension and C_A some normalization constant. In reference [5] an explicit map between the states of the quantized *free* IIB *pp*-wave string and certain operators in the dual gauge theory was suggested. In particular, the mapping gave rise to a prediction for the conformal dimension of these operators. This prediction, known as the BMN square root formula, re-sums the entire loop expansion, *i.e.* it can be expanded as an infinite power series in λ'. However, it is limited to the case $g_2 = 0$, *i.e.* to the planar gauge theory. The BMN prediction has been confirmed to all orders in λ' [11, 12]. What we shall discuss is what happens when one includes *non-planar* corrections in the gauge theory. More precisely, we shall work at first order in λ' and to all orders in g_2. Accordingly, we can write Eq. (4) as

$$\left\langle \tilde{\mathcal{O}}_A(x)\bar{\tilde{\mathcal{O}}}_B(0)\right\rangle = C_A\frac{\delta_{AB}}{|x|^{2\Delta_A^0}}\left(1+\lambda'(\delta\Delta)_A\log|x\Lambda|^{-2}\right). \tag{5}$$

Here Δ_A^0 is the tree-level conformal dimension, $(\delta\Delta)_A$ is the one loop correction and Λ is some (divergent) renormalization scale. One expects that the one-loop correction can be expanded in genus as follows[1]

$$(\delta\Delta)_A = (\delta\Delta)_A^{(0)} + g_2^2(\delta\Delta)_A^{(1)} + g_2^4(\delta\Delta)_A^{(2)} + \dots\,. \tag{6}$$

In our analysis we will consider only operators built from scalar fields. The $\mathcal{N} = 4$ Super Yang Mills theory has three complex scalars, ψ, ϕ and Z

[1] In the full $\mathcal{N} = 4$ Super Yang Mills theory one expects an expansion similar to Eqs. (5) and (6) with λ and $\frac{1}{N}$ replacing λ' and g_2. However, some one-loop anomalous dimensions have a large-N expansion involving odd powers of $\frac{1}{N}$ and some do not even have a well-defined double expansion in λ and $\frac{1}{N}$ [13]. Similar complications have not been encountered in the BMN limit so far.

all in the adjoint representation of the gauge group. By the choice of a particular SO(2) sub-group of the SO(6) symmetry group of $\mathcal{N} = 4$ Super Yang Mills one singles out one of these three fields, say Z, and the SO(2) R-charge is then given as the quantum number conjugate to the phase of Z. For scalar operators the SO(2) R-charge thus simply counts the number of Z-fields. Furthermore, for such operators the tree-level conformal dimension just equals the number of fields. A particular set of scalar operators are the following with tree level conformal dimension $J+2$

$$\Omega_n^{J_0,J_1,\ldots,J_k}(x) = \sum_{p=0}^{J_0} \mathrm{e}^{2\pi i p n/J_0}\,\mathrm{Tr}(\phi Z^p \psi Z^{J_0-p})\,\mathrm{Tr}\,Z^{J_1}\ldots \mathrm{Tr}\,Z^{J_k}(x), \qquad \sum_{i=0}^{k} J_i = J. \quad (7)$$

These operators have well-defined conformal dimensions in the *planar* BMN limit and correspond in that limit to direct products of $(k+1)$ string states, k of which are vacuum states and one is a state with two oscillators excited [5][2]. Once one takes into account non-planar contributions in the gauge theory, however, the operators in Eq. (7) start to mix and do no longer have well-defined conformal dimensions. To find the gauge theory operators which correctly represent the string states we have to carry out a (re-)diagonalization process. In Section 2 we will show that this process can be described as the process of finding the eigenvectors and the eigenvalues of a simple quantum mechanical Hamiltonian. Subsequently, in Section 3 we will explain how techniques from the field of random matrices can be used to perform explicit evaluation of correlation functions. The reason why random matrices appear at all is that the propagators of the scalar fields in $\mathcal{N} = 4$ Super Yang Mills take the form[3] (with $\bar{Z}$ the Hermitian conjugate of Z)

$$\left\langle Z_{ij}(x)\bar{Z}_{kl}(0)\right\rangle_{free} = \frac{g^2_{\mathrm{YM}}}{8\pi^2x^2}\,\delta_{il}\delta_{jk} \equiv \frac{g^2_{\mathrm{YM}}}{8\pi^2x^2}\,\left\langle Z_{ij}\bar{Z}_{kl}\right\rangle, \quad (8)$$

$$\langle Z_{ij}(x)Z_{kl}(0)\rangle = \left\langle \bar{Z}_{ij}(x)\bar{Z}_{kl}(0)\right\rangle = 0, \quad (9)$$

and similarly for ψ and ϕ. Here $\left\langle Z_{ij}\bar{Z}_{kl}\right\rangle$ is easily recognized as a propagator of a zero-dimensional complex matrix model (*cf.* Section 3). Thus whenever the space-time dependence of a correlator can be factored out matrix model techniques come in handy and we sketch how such techniques make it possible to derive exact, all genera results for certain correlation functions [7,8,15–17]. In the last section we briefly discuss some new insights on the integrability of $\mathcal{N} = 4$ Super Yang Mills which have been obtained on the basis of the quantum mechanical formalism [13].

[2] Operators of a similar type having well-defined planar conformal dimensions in the *full* gauge theory have been constructed in reference [14].

[3] We use the notation and normalization of reference [15].

2. The quantum mechanics

Gauge theory operators which are to represent string states must be operators with well-defined conformal dimensions. Such operators are characterized by being eigenstates of the dilatation operator, $\hat{D}$, with eigenvalue equal to the conformal dimension. Equivalently, their two-point functions take the canonical form in Eq. (4) which at one-loop level in BMN gauge theory looks as in Eq. (5). Considering a basis of operators $\{\mathcal{O}_\alpha\}$ with identical tree-level conformal dimension, Δ, a two-point function will generically read

$$\left\langle \mathcal{O}_\alpha(x)\bar{\mathcal{O}}_\beta(0)\right\rangle = \frac{1}{|x|^{2\Delta}}\left(S_{\alpha\beta} + T_{\alpha\beta}\log|x\Lambda|^{-2}\right), \tag{10}$$

with $S_{\alpha\beta}$ and $T_{\alpha\beta}$ respectively a tree-level and a one-loop mixing matrix. As pointed out in reference [18] (see also [19]) it is possible to read off the matrix elements of the dilatation operator in the basis $\{\mathcal{O}_\alpha\}$ from these mixing matrices. Let us split the dilatation operator in a tree level part $\hat{D}_0$ and a one-loop correction $\delta\hat{D}$, *i.e.*

$$\hat{D} = \hat{D}_0 + \delta\hat{D}. \tag{11}$$

Our aim is to find linear combinations of the states $\{\mathcal{O}_\alpha\}$ which are eigenstates of the dilatation operator when the one-loop correction is taken into account. Denoting the sought for eigenstates as $\tilde{\mathcal{O}}_A$ we have

$$\mathcal{O}_\alpha = V_{\alpha A}\tilde{\mathcal{O}}_A, \tag{12}$$

and

$$(\delta\hat{D})\tilde{\mathcal{O}}_A = (\delta\Delta)_A\tilde{\mathcal{O}}_A, \tag{13}$$

as well as the (equivalent) relation (4). In particular in the basis $\{\mathcal{O}_\alpha\}$ it holds that

$$(\delta\hat{D})\mathcal{O}_\alpha = V_{\alpha A}(\delta\Delta)_A V^{-1}_{A\beta}\mathcal{O}_\beta \equiv (\delta D)_{\alpha\beta}\mathcal{O}_\beta. \tag{14}$$

Inserting Eq. (12) into $\left\langle \mathcal{O}_\alpha(x)\bar{\mathcal{O}}_\beta(0)\right\rangle$ and making use of Eq. (4) one finds

$$\left\langle \mathcal{O}_\alpha(x)\bar{\mathcal{O}}_\beta(0)\right\rangle = \frac{1}{|x|^{2\Delta}}\left(V_{\alpha A}C_A V^\dagger_{A\beta} + V_{\alpha A}C_A(\delta\Delta)_A V^\dagger_{A\beta}\log|x\Lambda|^{-2}\right). \tag{15}$$

Comparison to Eq. (10) then gives

$$T_{\alpha\gamma}S^{-1}_{\gamma\beta} = (\delta D)_{\alpha\beta}. \tag{16}$$

Let us now specialize to the following set of operators

$$\mathcal{O}_p^{J_0,J_1,\ldots,J_k}(x) = \mathrm{Tr}(\phi Z^p\psi Z^{J_0-p})\ \mathrm{Tr}\, Z^{J_1}\ldots\mathrm{Tr}\, Z^{J_k}(x), \qquad \sum_{i=0}^{k} J_i = J. \tag{17}$$

These operators all have tree level conformal dimension $J+2$. To find the linear combinations with definite conformal dimensions at one-loop level we must diagonalize the matrix $(\delta D)_{\alpha\beta} = T_{\alpha\gamma} S^{-1}_{\gamma\beta}$. Taking into account only planar contributions we should reproduce the BMN operators in Eq. (7) and the associated BMN prediction. Including non-planar corrections we go beyond BMN and enter into the domain of *interacting* IIB *pp*-wave strings. Due to the simple form of the scalar propagators in $\mathcal{N}=4$ Super Yang Mills theory (*cf.* Eqs. (8) and (9)) one can express the elements of the tree-level mixing matrix $S_{\alpha\beta}$ as expectation values in a zero-dimensional field theory *i.e.*

$$S_{\alpha\beta} = \left\langle \mathcal{O}_\alpha \bar{\mathcal{O}}_\beta \right\rangle , \tag{18}$$

where $\mathcal{O}_\alpha$ and $\bar{\mathcal{O}}_\beta$ now consist of space-time independent fields and contractions are carried out using the Feynman rules

$$\left\langle Z_{ij} \bar{Z}_{kl} \right\rangle = \delta_{il}\delta_{jk}, \qquad \langle ZZ \rangle = \left\langle \bar{Z}\bar{Z} \right\rangle = 0, \tag{19}$$

and similarly for ψ and ϕ. The relation (18) defines an inner product on the space of states that we shall denote as the gauge theory inner product. Moreover, as shown in reference [15, 20], the matrix elements $T_{\alpha\beta}$ for operators involving only scalar fields can be expressed in an analogous manner by means of an effective Hamiltonian, $\hat{H}$. More precisely, one has

$$T_{\alpha\beta} = \left\langle \mathcal{O}_\alpha \, \hat{H} \, \bar{\mathcal{O}}_\beta \right\rangle , \tag{20}$$

where the notation and the contraction rules are as above and where $\hat{H}$ encodes the combinatorial structure of the $\mathcal{N}=4$ Super Yang Mills interaction

$$\hat{H} = -\frac{g^2_{\rm YM}}{8\pi^2} \; : \left(\mathrm{Tr}[\bar{Z},\bar{\phi}][Z,\phi] + \mathrm{Tr}[\bar{Z},\bar{\psi}][Z,\psi] + \mathrm{Tr}[\bar{\phi},\bar{\psi}][\phi,\psi] \right) : . \tag{21}$$

Here the normal ordering means that contractions between two fields of $\hat{H}$ are forbidden. Notice that the operator $\hat{H}$ is Hermitian with respect to the gauge theory inner product. Unlike believed until recently, determining the matrix elements $(\delta D)_{\alpha\beta} = T_{\alpha\gamma} S^{-1}_{\gamma\beta}$ does not require determining neither S nor T as we shall now explain. In evaluating (20) one can perform the contractions in any convenient order. In particular, one may start by contracting $\hat{H}$ with $\mathcal{O}_\alpha$. In doing so for an operator of the type (17) one observes that the contraction produces a linear combination of operators of the same type, *i.e.*

$$\hat{H}\mathcal{O}_\alpha = H_{\alpha\gamma} \, \mathcal{O}_\gamma . \tag{22}$$

Thus, we have

$$T_{\alpha\beta} = \left\langle (\hat{H}\mathcal{O}_\alpha) \, \bar{\mathcal{O}}_\beta \right\rangle = H_{\alpha\gamma} \left\langle \mathcal{O}_\gamma \, \bar{\mathcal{O}}_\beta \right\rangle = H_{\alpha\gamma} S_{\gamma\beta}, \tag{23}$$

or comparing to (16)

$$H_{\alpha\beta} = T_{\alpha\gamma} S^{-1}_{\gamma\beta} = (\delta D)_{\alpha\beta}. \tag{24}$$

Hence, to determine the matrix elements of $\delta\hat{D}$ we only need to determine the expansion coefficients $H_{\alpha\gamma}$ in Eq. (22). The matrix $H_{\alpha\gamma}$ is obviously not Hermitian. However, it is related to its Hermitian conjugate by a similarity transformation, *i.e.*

$$H^{\dagger}_{\alpha\beta} \equiv H^{*}_{\beta\alpha} = S^{-1}_{\alpha\gamma} H_{\gamma\delta} S_{\delta\beta}, \tag{25}$$

which can be seen by first contracting $\hat{H}$ with $\bar{\mathcal{O}}_\beta$ in Eq. (23). This in particular implies that $\delta\hat{D}$ has real eigenvalues as we expect. Furthermore, the eigenvectors of $H_{\alpha\beta}$ corresponding to different eigenvalues are automatically orthogonal with respect to the gauge theory inner product.

Applying the operator $\hat{H}$ in equation (21) to a state of the type (17) using the contraction rules (19) one finds that $\hat{H}$ can conveniently be split into three parts, a trace-conserving one $\hat{H}_0$, a trace-increasing one $\hat{H}_+$ and a trace-decreasing one $\hat{H}_-$, *i.e.*

$$\hat{H} = -\frac{g^2_{\mathrm{YM}} N}{4\pi^2}\left(\hat{H}_0 + \frac{1}{N}\hat{H}_+ + \frac{1}{N}\hat{H}_-\right), \tag{26}$$

where

$$\hat{H}_0 \mathcal{O}_p^{J_0,J_1,\ldots,J_k} = \mathcal{O}_{p+1}^{J_0,J_1,\ldots,J_k} - 2\mathcal{O}_p^{J_0,J_1,\ldots,J_k} + \mathcal{O}_{p-1}^{J_0,J_1,\ldots,J_k}, \tag{27}$$

takes the form of a discrete second derivative and where the expressions for $\hat{H}_+\mathcal{O}$ and $\hat{H}_-\mathcal{O}$ can be found in reference [20][4]. Starting from the discrete Hamiltonian we can derive a continuum, *i.e.* a BMN version. To do so we must consider $J \to \infty$, $N \to \infty$ while keeping fixed $\lambda' = \frac{g^2_{\mathrm{YM}} N}{J^2}$ and $g_2 = \frac{J^2}{N}$ (*cf.* Eqs. (1) and (2)). Preparing for this we introduce the quantities

$$r_i = \frac{J_i}{J}, \quad i \in \{0, 1, \ldots, k\}, \quad \sum_{i=0}^{k} r_i = 1, \quad x = \frac{p}{J_0} \le r_0. \tag{28}$$

Then we imagine that all the J_i become very large so that we can view the r_i and x as continuous variables. Accordingly, we replace our discrete set of operators in Eq. (17) with a continuum set of states, *i.e.*

$$\mathcal{O}_p^{J_0,J_1,\ldots,J_k} \longrightarrow |x; r_1, \ldots, r_k\rangle, \tag{29}$$

[4] The trace-conserving part $\hat{H}_0$,when acting on a general single trace operator of scalar fields has been identified as a Hamiltonian of an integrable SO(6) spin chain [21].

where the order of the r-quantum numbers is unimportant. Finally, imposing the BMN limit in the explicit expression for $\hat{H}\mathcal{O}$ one finds

$$\hat{H} \longrightarrow \frac{\lambda'}{4\pi^2} h, \qquad h = h_0 + g_2(h_+ + h_-), \tag{30}$$

where obviously the discrete second derivative in (27) turns into a continuous one, *i.e.*

$$h_0 = -\partial_x^2, \tag{31}$$

and where h_+ and h_- become more involved integro-differential operators (*cf.* reference [20]). It is important to notice that the expression (30) is exact at one-loop order. In the planar limit ($g_2 = 0$) we have $h = h_0$. All information about higher genera contributions is encoded in a single trace-splitting term $g_2 h_+$ and a single trace-joining term $g_2 h_-$. In particular, there are no terms of higher order in g_2. The continuum Hamiltonian not allowing for an exact diagonalization it becomes natural to split h into a free part h_0 and a perturbation with g_2 playing the role of a perturbation parameter

$$h = h_0 + g_2 V, \qquad V = h_+ + h_-. \tag{32}$$

The free Hamiltonian h_0 is easily diagonalized. Due to the cyclicity of the trace the variable x is effectively periodic and the eigenstates of h_0 are simply the Fourier modes

$$|n; r_1, \ldots, r_k\rangle = \frac{1}{\sqrt{r_0}} \int_0^{r_0} dx\, \mathrm{e}^{2\pi i n x / r_0}\, |x; r_1, \ldots, r_k\rangle, \qquad n \in \mathcal{Z}. \tag{33}$$

The corresponding eigenvalues read

$$E^{(0)}_{|n;r_1,\ldots,r_k\rangle} = 4\pi^2 \frac{n^2}{r_0^2}. \tag{34}$$

We notice that in the planar limit we, as promised, reproduce the BMN operators in Eq. (7) up to normalization. In the basis given by (33) the perturbation acts as follows [20]

$$\begin{aligned}
&h_+ |n; r_1, \ldots, r_k\rangle = \\
&\quad \int_0^{r_0} dr_{k+1} \sum_{m=-\infty}^{\infty} \frac{4m \sin^2\left(\pi n \frac{r_{k+1}}{r_0}\right)}{\sqrt{r_0}\sqrt{r_0 - r_{k+1}}\left(m - n\frac{r_0 - r_{k+1}}{r_0}\right)} |m; r_1, \ldots, r_{k+1}\rangle, \\
&h_- |n; r_1, \ldots, r_k\rangle = \\
&\quad \sum_{i=1}^{k} \sum_{m=-\infty}^{\infty} \frac{4\, r_i\, m \sin^2\left(\pi m \frac{r_i}{r_0 + r_i}\right)}{\sqrt{r_0}\sqrt{r_0 + r_i}\left(m - n\frac{r_0 + r_i}{r_0}\right)} |m; r_1, \ldots, \not{r}_i, \ldots, r_k\rangle,
\end{aligned} \tag{35}$$

where the quantity r_0 refers to the state on the left hand side of the equations. We can now proceed by quantum mechanical perturbation theory to evaluate order by order in g_2 the non-planar corrections to our eigenstates and eigenvalues (*i.e.* to the gauge theory operators dual to string states and their associated conformal dimensions). For that purpose it is convenient to to introduce an inner product on the space of states given in (33)

$$\langle n; s_1, \dots s_l | m; r_1, \dots, r_k \rangle = \delta_{kl}\, \delta_{mn} \sum_{\pi \in S_k} \prod_{i=1}^{k} \delta(s_i - r_{\pi(i)}), \tag{36}$$

where the sum runs over permutations of k elements. This inner product is only a computational tool which makes it possible to represent calculations in the usual language of quantum mechanics.

As is well-known, the first order energy shift in quantum mechanical perturbation theory is given by the diagonal elements of the perturbation — provided there are no degeneracies or matrix elements between degenerate states vanish. Our perturbation is entirely off-diagonal but we have huge degeneracies. For simplicity, let us consider the case of a single trace state $|n\rangle$. Such a state is degenerate with a multi-trace state $|m; r_1, \dots, r_k\rangle$ if $n = \pm \frac{m}{1-(r_1+\dots+r_k)}$ [22]. The perturbation $(h_+ + h_-)$ can at worst have non-vanishing matrix elements between states for which the number of traces differs by one. However, from the explicit form of the matrix elements in Eq. (35) we see that such matrix elements vanish for degenerate states.[5] Thus, we *can* actually use the formulas from first order non-degenerate perturbation theory and conclude that there is no energy shift for the state $|n\rangle$ at this order but that the state itself gets corrected through mixing with double trace states (that are not degenerate with $|n\rangle$). Since the degeneracies are not lifted at leading order in perturbation theory we also have to worry about these at next to leading order. Defining

$$\mathcal{P}_{|\alpha\rangle} = \frac{1 - |\alpha\rangle\langle\alpha|}{E^{(0)}_{|\alpha\rangle} - h_0}, \tag{37}$$

the familiar formulas for the second order correction to energies and states in *non-degenerate* perturbation theory read

$$E^{(2)}_{|\alpha\rangle} = \langle\alpha| V\, \mathcal{P}_{|\alpha\rangle} V |\alpha\rangle = \sum_{\beta\neq\alpha} \frac{|\langle\alpha|V|\beta\rangle|^2}{E^{(0)}_{|\alpha\rangle} - E^{(0)}_{|\beta\rangle}}, \tag{38}$$

[5] Notice that this statement is only true in the BMN limit and not in the full $\mathcal{N} = 4$ Super Yang Mills theory (*cf.* reference [13]).

and

$$|\alpha\rangle^{(2)} = \mathcal{P}_{|\alpha\rangle} V \mathcal{P}_{|\alpha\rangle} V |\alpha\rangle = \sum_{\beta\neq\alpha} |\beta\rangle \frac{\langle\beta|V\mathcal{P}_{|\alpha\rangle}V|\alpha\rangle}{E^{(0)}_{|\alpha\rangle} - E^{(0)}_{|\beta\rangle}}, \tag{39}$$

where the last expression holds for an off-diagonal perturbation. Naively applying the formula (38) to our state $|n\rangle$ the only non-vanishing contributions come from intermediate double trace states of the type $|m;r\rangle$ which are not degenerate with $|n\rangle$, the sum is finite and gives the value presented in references [15,22]. However, applying the formula (39) to our state $|n\rangle$ we encounter a divergence if the matrix element $\langle m; r_1, r_2|V\mathcal{P}_{|n\rangle}V|n\rangle$ is non-vanishing for $n = \pm\frac{m}{1-r_1-r_2}$ or if $\langle -n|V\mathcal{P}_{|n\rangle}V|n\rangle \neq 0$. The possibility of such a divergence was first discussed in reference [22]. In the present formalism it is simple to evaluate the problematic matrix elements and what one finds is that the latter vanishes whereas the former is *non-vanishing* for $n = +\frac{m}{1-r_1-r_2}$ [20]. Thus non-degenerate perturbation theory can in general not be applied to the state $|n\rangle$. The only case for which it remains valid is the case $n = 1$ where degeneracy with multi-trace states is excluded [22]. To correctly find the second order energy shift to the state $|n\rangle$ we have to diagonalize the operator

$$\hat{M} = V\mathcal{P}_{|n\rangle}V \tag{40}$$

in the space of states degenerate with $|n\rangle$, see for instance [23]. There will be non-vanishing matrix elements between $(2k+1)$ and $(2k+3)$ states for all k as well as non-vanishing matrix elements connecting k-trace states with k-trace states. So far it has not been possible to carry out the required diagonalization. The breakdown of non-degenerate perturbation theory was interpreted in [24] on the string theory side as an instability causing a single string state to decay into degenerate triple string states.

3. Random matrices

The role matrix models play in with BMN gauge theory is similar to the role they play in the study of 2D quantum gravity, see for instance [25], (and in the Dijkgraaf–Vafa approach to supersymmetric gauge theories [26]). They constitute a convenient tool for handling the combinatorics of Feynman diagrams with trivial space-time dependence. (Another point of view is presented in references [27, 28].)

Consider a two-point function of operators built from scalar fields. Due to the simple form of the propagators, *cf.* Eqs. (8) and (9), (or due to conformal invariance), at tree level one can immediately factor out the space-time dependence and one is left with a correlation function in a zero-dimensional field theory. The same is true in the case of three-point functions. Furthermore, as a consequence of supersymmetry, some operators have two- and three-point functions which are protected, *i.e.* which do not get any quantum corrections. Examples of such operators are BMN operators with zero or one impurity, *i.e.*

$$\Omega^{J_1,\ldots,J_k}(x) \;=\; \mathrm{Tr}\, Z^{J_1} \ldots \mathrm{Tr}\, Z^{J_k}(x), \qquad \sum_{i=1}^{k} J_i = J, \tag{41}$$

$$\Omega_0^{J_0,J_1,\ldots,J_k}(x) \;=\; \mathrm{Tr}\, \phi Z^{J_0}\, \mathrm{Tr}\, Z^{J_1} \ldots \mathrm{Tr}\, Z^{J_k}(x), \qquad \sum_{i=0}^{k} J_i = J, \tag{42}$$

or two-impurity operators with mode-number $n = 0$, *cf.* Eq. (7). For these operators, which correspond to supergravity states on the string theory side, tree-level two- and three-point functions are thus exact and can conveniently be obtained using techniques from the field of random matrices. As an example, let us consider the following two-point function

$$\left\langle \Omega^J(x) \bar{\Omega}^{J_1,\ldots,J_k}(0) \right\rangle = \left(\frac{g_{\mathrm{YM}}^2}{8\pi^2 x^2} \right)^J \left\langle \mathrm{Tr}\, Z^J \prod_{i=1}^{k} \mathrm{Tr}\, \bar{Z}^{J_i} \right\rangle. \tag{43}$$

Here we have factored out the trivial space-time dependence and the remaining expectation value is to be evaluated using the contraction rules (19). As we are instructed to take $J \to \infty$ in the BMN limit the combinatorics of these contractions would be very involved were it not for the existence of matrix model techniques. We can represent the expectation value above as the following matrix integral

$$\left\langle \mathrm{Tr}\, Z^J \prod_{i=1}^{k} \mathrm{Tr}\, \bar{Z}^{J_i} \right\rangle = \int dZ d\bar{Z} \exp\left(- \mathrm{Tr}\, \bar{Z} Z \right) \mathrm{Tr}\, Z^J \prod_{i=1}^{k} \mathrm{Tr}\, \bar{Z}^{J_i}, \tag{44}$$

with measure

$$dZ d\bar{Z} = \prod_{i,j=1}^{N} \frac{d\mathrm{Re} Z_{ij}\, d\mathrm{Im} Z_{ij}}{\pi}, \tag{45}$$

as the Gaussian term produces exactly the contraction rule in Eq. (19). A matrix integral like the one in Eq. (44) can be evaluated using an old method due to Ginibre [29], see also [30]. Diagonalizing Z by a similarity transformation *i.e.* writing

$$Z = X D X^{-1}, \tag{46}$$

where X as well as D are complex matrices and D is diagonal it becomes possible to integrate out the non-diagonal degrees of freedom [29]. Doing so leaves one with an integral over only diagonal degrees of freedom which can be evaluated *exactly* [15, 17]. The result reads

$$\left\langle \mathrm{Tr}\, Z^J \prod_{i=1}^{k} \mathrm{Tr}\, \bar{Z}^{J_i} \right\rangle \tag{47}$$
$$= \frac{1}{J+1} \left\{ \frac{\Gamma(N+J+1)}{\Gamma(N)} - \sum_{i=1}^{k} \frac{\Gamma(N+J-J_i+1)}{\Gamma(N-J_i)} + \sum_{1\le i_1<i_2\le k} \frac{\Gamma(N+J-J_{i_1}-J_{i_2}+1)}{\Gamma(N-J_{i_1}-J_{i_2})} - \cdots + (-)^k \frac{\Gamma(N+1)}{\Gamma(N-J)} \right\}.$$

We stress that since the correlation function (43) is known to be protected we have hereby determined its value to all orders in the loop expansion and to all genera. A similar statement of course holds in the BMN limit where we get

$$\frac{1}{J\,N^J} \left\langle \mathrm{Tr}\, Z^J \prod_{i=1}^{k} \mathrm{Tr}\, \bar{Z}^{J_i} \right\rangle \longrightarrow g_2^{k-1} \prod_{i=1}^{k} \frac{\sinh\left(\frac{g_2\, r_i}{2}\right)}{\frac{g_2}{2}}. \tag{48}$$

The method of Ginibre allows one to evaluate (in principle) any expectation value involving a product composed of factors of the type $\mathrm{Tr}\, Z^{J_i}$ and $\mathrm{Tr}\, \bar{Z}^{K_i}$. This type of expectation value is also accessible by character expansion [31, 32]. Using either approach one can obtain exact expressions for a large number of protected two- and three-point functions, for instance

$$\left\langle \Omega_0^J(x) \bar{\Omega}_0^{J_0, J_1, \ldots, J_k}(0) \right\rangle = \left(\frac{g_{\mathrm{YM}}^2}{8\pi^2 x^2} \right) \frac{1}{J+1} \left\langle \Omega^{J+1}(x) \bar{\Omega}^{J_0+1, J_1, \ldots, J_k}(0) \right\rangle, \tag{49}$$

where to arrive at (49) we have contracted by hand the two impurities.

In general correlation functions of scalar BMN operators are not protected. Of course, at *tree level* any two- or three-point function of such operators can be reduced to an expectation value in a zero-dimensional Gaussian one-matrix model, the strategy being the same as above: One first factors out the trivial space-time dependence and next contracts by hand the finite number of impurities. However, not all the resulting matrix model expectation values can be evaluated by the method of Ginibre or by character expansion. As an example, let us consider a tree-level two-point function of operators of the type appearing in Eq. (7)

$$\left\langle \Omega_n^J(x) \bar{\Omega}_m^J(0) \right\rangle = \left(\frac{g_{\mathrm{YM}}^2}{8\pi^2 x^2} \right)^{J+2} \sum_{p,q=0}^{J} \mathrm{e}^{2\pi i(np-mq)/J} \left\langle \mathrm{Tr}(Z^p \bar{Z}^q)\, \mathrm{Tr}(Z^{J-p} \bar{Z}^{J-q}) \right\rangle. \tag{50}$$

Clearly the product of traces in Eq. (50) is not diagonalized by the similarity transformation (46)[6]. However, there exists a strategy by means of which one can evaluate any expectation value of traces of words of Z's and $\bar{Z}$'s order by order in the genus expansion, namely a strategy based on loop equations [16]. Loop equations express the invariance of the matrix model partition function under certain analytical re-definitions of the integration variables. These equations are most conveniently expressed in terms of generating functions. For the correlator in Eq. (50) the relevant generating function is

$$W(x_1,y_1;x_2,y_2) = \left\langle \mathrm{Tr}\left(\frac{1}{x_1 - Z}\frac{1}{\bar{y}_1 - \bar{Z}}\right)\mathrm{Tr}\left(\frac{1}{x_2 - Z}\frac{1}{\bar{y}_2 - \bar{Z}}\right)\right\rangle . \tag{51}$$

This function fulfills

$$\begin{aligned} &W(X\mathrm{e}^{\frac{-i\pi n}{J}}, X\mathrm{e}^{\frac{-i\pi m}{J}}; X\mathrm{e}^{\frac{i\pi n}{J}}, X\mathrm{e}^{\frac{i\pi m}{J}}) = \\ &\quad \mathrm{e}^{i\pi(m-n)}\sum_{J=0}^{\infty}(X\bar{X})^{-J-2}\sum_{p,q=0}^{J}\left\langle \mathrm{Tr}(Z^{J-p}\bar{Z}^{J-q})\,\mathrm{Tr}(Z^p\bar{Z}^q)\right\rangle \mathrm{e}^{2i\pi(np-mq)/J}, \end{aligned} \tag{52}$$

and allows us to easily recover the sum in Eq. (50) by a contour integration (over $X\bar{X}$). The result of the contour integration depends on the analyticity structure of $W(x_1,y_1;x_2,y_2)$. It can be shown that $W(x_1,y_1;x_2,y_2)$ only has singularities in the form of poles [16]. For the choice of arguments of W in Eq. (52) the position and the order of the poles, not surprisingly, depend on n and m. Taking this into account one finds in the BMN limit at tree level and to genus one [7, 8, 16]

$$\left\langle \Omega_n^J(x)\bar{\Omega}_m^J(0)\right\rangle \longrightarrow \left(\frac{g_{\mathrm{YM}}^2}{8\pi^2x^2}\right)^{J+2} S_{nm}, \qquad S_{nm} = \delta_{nm} + g_2^2 M_{nm} + \mathcal{O}(g_2^4),$$

where S_{nm} was defined (more generally) in Eq. (10) and where

$$M_{nm} = M_{mn} = \begin{cases} \frac{1}{24} & n = m = 0 \\ 0 & n \neq 0, m = 0 \\ \frac{1}{60} - \frac{1}{24\pi^2n^2} + \frac{7}{16\pi^4n^4} & n = m \text{ and } n \neq 0 \\ \frac{1}{48\pi^2n^2} + \frac{35}{128\pi^4n^4} & n = -m \text{ and } n \neq 0 \\ \frac{2\pi^2(n-m)^2-3}{24(n-m)^4\pi^4} + \frac{2n^2-3nm+2m^2}{8n^2m^2(n-m)^2\pi^4} & |n| \neq |m| \text{ and } n \neq 0 \neq m \end{cases}$$

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[6] There exists another possibility for diagonalizing a complex matrix, namely writing $Z = V\mathcal{D}W^\dagger$ with V and W unitary and $\mathcal{D}$ diagonal. Exploiting this on can obtain expectation values of products of traces of the form $\mathrm{Tr}(\bar{Z}Z)^k$ and that even for a general $\mathrm{U}(N)\times\mathrm{U}(N)$ invariant potential [33, 34], but also not this method applies to a correlation function like the one in Eq. (50).

The genus two result can be found in references [8, 16]. One can of course also, by means of the effective vertex H express the one-loop correction to non-protected two-point functions, *i.e.* the quantity $T_{\alpha\beta}$, as a matrix model expectation value. As stressed earlier, when aiming at determining conformal dimensions one has no need of knowing neither $S_{\alpha\beta}$ nor $T_{\alpha\beta}$. The quantity $S_{\alpha\beta}$ nevertheless has an interesting interpretation on the string theory side as it is conjectured to provide the transition between gauge theory operators and string states encompassing the effects of string interactions [35–37].

4. Conclusion

Matrix model techniques played an important role in the early investigations of BMN gauge theory leading to the discovery of the genus counting parameter g_2 and allowing, via the use of effective vertices, for the first calculations of higher genus corrections to the one-loop anomalous dimension of BMN operators [7, 8, 15, 22]. Later, it was understood that by focusing on the dilatation operator of the $\mathcal{N} = 4$ Super Yang Mills theory these calculations could be considerably simplified, being equivalent to the diagonalization of a simple quantum mechanical Hamiltonian [20]. The quantum mechanical picture applies also to the full $\mathcal{N} = 4$ Super Yang Mills theory and this has revealed a very promising and yet to be explored underlying integrability structure [13]. Integrability at the planar one-loop level was established in [21] for scalar operators and was recently generalized to all operators in [38, 39]. The study of [13] provided evidence for two-loop integrability and lead to the conjecture that integrability would hold at all loop orders.

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REFERENCES

[1] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998).

[2] M. Blau, J. Figueroa-O'Farrill, C. Hull, G. Papadopoulos, *Class. Quant. Grav.* **19**, L187 (2002).

[3] R.R. Metsaev, *Nucl. Phys.* **B625**, 70 (2002).

[4] R.R. Metsaev, A.A. Tseytlin, *Phys. Rev.* **D65**, 126004 (2002).

[5] D. Berenstein, J.M. Maldacena, H. Nastase, *J. High Energy Phys.* **0204**, 013 (2002).

[6] G. 't Hooft, *Nucl. Phys.* **B72**, 461 (1974).

[7] C. Kristjansen, J. Plefka, G. W. Semenoff, M. Staudacher, *Nucl. Phys.* **B643**, 3 (2002).

[8] N.R. Constable, D.Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov, W. Skiba, *J. High Energy Phys.* **0207**, 017 (2002).

[9] A. Pankiewicz, `hep-th/0307027`.

[10] J. Plefka, `hep-th/0307101`.

[11] D.J. Gross, A. Mikhailov, R. Roiban, *Ann. Phys.* **301**, 31 (2002).

[12] A. Santambrogio, D. Zanon, *Phys. Lett.* **B545**, 425 (2002).

[13] N. Beisert, C. Kristjansen, M. Staudacher, *Nucl. Phys.* **B564**, 131 (2003).

[14] N. Beisert, *Nucl. Phys.* **B659**, 79 (2003).

[15] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff, M. Staudacher, *Nucl. Phys.* **B650**, 125 (2003).

[16] B. Eynard, C. Kristjansen, *J. High Energy Phys.* **0210**, 027 (2002).

[17] K. Okuyama, *J. High Energy Phys.* **0211**, 043 (2002).

[18] R. A. Janik, Phys. Lett. B **549**, 237 (2002).

[19] D.J. Gross, A. Mikhailov, R. Roiban, *J. High Energy Phys.* **0305**, 025 (2003).

[20] N. Beisert, C. Kristjansen, J. Plefka, M. Staudacher, *Phys. Lett.* **B558**, 229 (2003).

[21] J.A. Minahan, K. Zarembo, *J. High Energy Phys.* **0303**, 013 (2003).

[22] N.R. Constable, D.Z. Freedman, M. Headrick, S. Minwalla, *J. High Energy Phys.* **0210**, 068 (2002).

[23] A. Messiah, *Quantum Mechanics*, North-Holland Publishing Company, Amsterdam 1962.

[24] D.Z. Freedman, U. Gursoy, `hep-th/0305016`.

[25] J. Ambjørn, B. Durhuus, T. Jonsson, *Quantum Geometry*, Cambridge University Press, 1997.

[26] R. Dijkgraaf, M. T. Grisaru, C.S. Lam, C. Vafa, D. Zanon, `hep-th/0211017`.

[27] R. de Mello Koch, A. Jevicki, J.P. Rodrigues, `hep-th/0209155`.

[28] R. de Mello Koch, A. Donos, A. Jevicki, J.P. Rodrigues, `hep-th/0305042`.

[29] J. Ginibre, *J. Math. Phys.* **6**, 440 (1965).

[30] M.L. Mehta, *Random Matrices*, 2nd ed., Academic Press, 1990.

[31] I.K. Kostov, M. Staudacher, *Phys. Lett.* **B394**, 75 (1997).

[32] S. Corley, A. Jevicki, S. Ramgoolam, *Adv. Theor. Math. Phys.* **5**, 809 (2002).

[33] T.R. Morris, *Nucl. Phys.* **B356**, 703 (1991).

[34] J. Ambjørn, C.F. Kristjansen, Y.M. Makeenko, *Mod. Phys. Lett.* **A7**, 3187 (1992).

[35] D. Vaman, H. Verlinde, `hep-th/0209215`.

[36] J. Pearson, M. Spradlin, D. Vaman, H. Verlinde, A. Volovich, *J. High Energy Phys.* **0305**, 022 (2003).

[37] M. Spradlin, A. Volovich, *Phys. Lett.* **B565**, 253 (2003).

[38] N. Beisert, hep-th/0307015.

[39] N. Beisert, M. Staudacher, hep-th/0307042.