

ON THE GEOMETRY OF RANDOM NETWORKS*

A. KRZYWICKI

Laboratoire de Physique Théorique, Bâtiment 210, Université Paris-Sud
91405 Orsay, France

(Received August 5, 2003)

The Krakow-Orsay collaboration has applied methods borrowed from equilibrium statistical mechanics and analytic combinatorics to study the geometry of random networks. Results contained in a series of recent publications and concerning networks that are either uncorrelated or causal are briefly overviewed.

PACS numbers: 02.50.Cw, 05.40.-a, 05.50.+q, 87.18.Sn

1. Introduction

The purpose of this communication is to overview the results published recently in the Physical Review E by the Kraków–Orsay collaboration [1–3] and devoted to the geometry of random networks¹. The discussion is sketchy and aimed to give the reader only a general idea of what has been achieved. All the useful details can be found in the original papers.

Network study is not our original field of research: we are trying to exploit the experience gained working on a different subject — quantum gravity, or, more precisely, statistical mechanics of random manifolds — in another context and to fill the gap between two communities, that are differently motivated but often confronted to manifestly similar problems. This is a status report, in the same vein as my talk at the Utrecht symposium in 2001, addressed to a similar audience [5].

There are two complementary approaches to random networks, and actually to numerous complex systems: the diachronic and the synchronic one. In the former one focuses on the time evolution of the system. It is particularly suitable if the aim is to uncover the evolution dynamics. In the latter one works at a fixed time, considering an ensemble of related systems, with

* Presented at the Workshop on Random Geometry, Kraków, Poland, May 15–17, 2003.

¹ Excellent general reviews on network physics can be found in Ref. [4].

the aim of finding common structural traits. Being primarily interested in the geometry of networks, we have adopted the synchronic approach.

The goal is to develop a statistical mechanics of random networks. In the statistical mechanics of gases one starts with an ideal gas. Analogously, working with uncorrelated nodes is a natural first step in our research. For definiteness, we consider undirected networks only.

2. Networks with uncorrelated nodes

2.1. Formulation of the model

The model is compactly defined by writing the partition function as a formal integral

$$Z \sim \int_{-\infty}^{+\infty} d\phi \exp \frac{1}{\kappa} \left[-\phi^2/2\lambda + \sum_{n>0} p_n \phi^n \right]. \quad (1)$$

The set of “coupling constants” p_n is eventually identified with the degree distribution, while κ and λ are control parameters. Of course, strictly speaking, the integral does not exist. However, the perturbative series in the “coupling constants” is well defined and the individual terms can be, as in field theory, represented by Feynman diagrams. The idea is to identify the labeled Feynman diagrams of the minifield theory defined by (1) with the graphs representing the network and to attach to every such graph a weight equal to the corresponding Feynman amplitude. All this is explained in detail in Refs. [1, 2].

Consider the ensemble where N and L , the number of nodes and links, respectively, are fixed. Using the Feynman rules one finds that the weight w of a labeled non-degenerate graph — *i.e.* one without tadpoles and multiple connections between nodes — is up to an irrelevant factor given by

$$w \sim \prod_{j=1}^N n_j! p_{n_j}, \quad (2)$$

where n_j is the degree of the j -th node. This graph is, in general, not connected.

It follows from the obvious identity

$$\sum_{j=1}^N n_j = 2L \quad (3)$$

that, at fixed N and L , all non-degenerate graphs are equiprobable when p_n has the Poisson form $p_n \sim c^n/n!$, with some constant c . Hence, in this case

graphs are those of the classical Erdős–Rényi theory [7]. In general, the statistical ensemble discussed in this section is a generalization of the classical ensemble of random graphs to allow for an arbitrary degree distribution (see later).

Notice, that, because of (3), the relative weights of microstates are invariant under the transformation $p_n \rightarrow c^n p_n$. We shall see later that the ambiguity is lifted when one fixes the ratio L/N .

2.2. A few words on trees

In the “quasi-classical” limit $\kappa \rightarrow 0$ only connected tree graphs contribute to $W = \kappa \ln Z$. The integral in (1) can be calculated using the saddle-point approximation. The saddle-point condition reads

$$\Phi = \lambda \sum_{n>0} n p_n \Phi^{n-1}, \quad (4)$$

and one easily checks that $\Phi = \partial W / \partial p_1$, which means that Φ generates tree graphs with one external node marked. Although Eq. (4) can be exactly solved by Lagrange inversion, it is sufficient to use a more direct approach [6]: Eq. (4) can only be satisfied when λ is smaller than some critical value λ_c . Hence Φ is a singular function of $\delta\lambda = \lambda_c - \lambda$. Furthermore, only the singular part of Φ is of real physical interest, since it determines the behavior of arbitrarily large trees. And this singular part is readily found directly from (4). The result is used to determine the distribution of the smallest distance between pairs of nodes, the Hausdorff and spectral dimensions, *etc.* All this is discussed at length in Ref. [1], where, among others, the results of Ref. [6] are extended to the interesting case of scale-free graphs. I shall not enter into more details here.

2.3. Algorithmic considerations

Equation (2) gives a weight to each microstate. For given L/N , this, in essence, defines the statistical ensemble. However, in order to make this definition being of any use, we supplement it with an explicit recipe enabling one to construct graphs, *e.g.* on a computer. To this end we define a local *move* transforming one graph into another. A succession of such moves is a Markov process. The initial state of the system is rapidly forgotten and graphs are sampled with relative frequency given by (2). The whole procedure is a specific application of the so-called Metropolis method, widely used in other branches of statistical physics [8].

One move consists of three steps ². First, we sample two distinct nodes, say j and k . Second, we pick one neighbor of j , say i . Third, we rewire $ij \rightarrow ik$ with probability

$$\text{Prob}(ij \rightarrow ik) = \min \left(1, \frac{p_{n_k+1} p_{n_j-1}}{p_{n_k} p_{n_j}} \right). \quad (5)$$

It is evident, that as far as the modifications of node degrees are concerned, the algorithm is identical to that defining the so called balls-in-boxes model [9], defined by the partition function

$$z \sim \sum_{\{n_i\}} \prod_{j=1}^N p_{n_j} \delta \left(M - \sum_{j=1}^N n_j \right) \quad (6)$$

and describing M balls distributed with probability $\sim p_n$ among N boxes (in our case $M = 2L$). The constraint represented by the Kronecker delta is in the limit $N \rightarrow \infty$ satisfied “for free” by virtue of the law of large numbers provided $M/N = \langle n \rangle \equiv \sum_n n p_n / \sum_n p_n$. When the last condition is met, the occupation number of one box $\rightarrow p_n$ when $N \rightarrow \infty$.

The isomorphy of the graph and balls-in-boxes model implies that the degree distribution in the graph model tends asymptotically to p_n provided

$$L = \frac{1}{2} N \langle n \rangle. \quad (7)$$

Although the relative weights of microstates are invariant under the transformation $p_n \rightarrow c^n p_n$, $\langle n \rangle$ is not. Hence, the ratio L/N is fixed, once one has decided that the degree distribution should be p_n .

There is a problem, however. Graphs generated by the above described algorithm are, in general, degenerate (as are the objects constructed in the well-known paper by Molloy and Reed [10]; these construction is often misused in the physics literature as a method of generating graphs, without due attention to the degeneracy problem).

Our algorithm is local. The creation of degeneracies is therefore easily forbidden: It suffices to check that i and k are neither identical nor connected. But this check introduces a bias. The point is discussed in the next subsection.

² An equivalent and simpler definition is given in Refs. [1,2]. The one given here makes the useful mapping on the balls-in-boxes model more evident (*cf.* footnote [15] in Ref. [2]).

2.4. Finite-size effects and the degeneracy problem

The use of the Metropolis method guarantees that the degree distribution approaches p_n for large N , provided one has enough statistics, *i.e.* when $Np_n \gg 1$, even if one forbids degeneracies. However, this condition is not satisfied when p_n has a fat tail. Then, there are large fluctuations in the tail and introducing a constraint can bias the sample.

Assume that $p_n \sim n^{-\beta}$ at large n . At finite N the tail of p_n cannot extend to infinity, because there exists some $n_{\max}$ such that the expected number of nodes with degrees $n > n_{\max}$ is less than unity. Neglecting all correlations one easily finds the scaling law

$$n_{\max} \sim N^{1/(\beta-1)}. \quad (8)$$

It is easily seen that np_n can actually be very small well below this natural cut-off.

The bias associated with rejecting degeneracies can be evaluated [2]. Consider the symmetric adjacency matrix C_{ij} : the elements of say the m -th row sum up to n , the degree of the m -th node. These elements equal either 0 or 1 when the graph is non-degenerate, they are just positive integers when the graph is degenerate. We wish to compare the number of ways the m -th node can be connected to n other nodes, when one accepts or rejects degeneracies. The problem reduces to counting the number of ways to place n balls in $N-1$ boxes, but is not altogether trivial, since one has to take into account the symmetry factors that appear in the weights of the degenerate graphs as well as the shape of the degree distribution. The result at large N is

$$\frac{\text{\#without degeneracy}}{\text{\#with possible degeneracies}} \sim \exp \left[-\text{const} \frac{n^2}{N} \right]. \quad (9)$$

Notice, that although n/N is always small, n^2/N may be large. We observe, that at fixed n the rejection of degeneracies does not introduce any bias at asymptotic N . However, at large n the rejection of degeneracies introduces a non-uniform deformation of the spectrum. Actually, there is a cut-off scaling like $\sqrt{N}$. This cut-off is smaller than the “natural” cut-off given by (8) when $2 < \beta < 3$. And this is not a marginal case. The β exponent is like that for most interesting networks! Apparently, forbidding degeneracies introduces a kind of “kinematic” correlation at finite N . It is important to stress that this is a property of the model, not a deficiency of the algorithm. Let us also mention, that the conclusions of the above heuristic argument are confirmed by numerical simulations.

There is a mathematical conclusion of the above discussion: the algorithm is fine. To my knowledge this is the only efficient algorithm generating non-degenerate graphs with a given degree distribution³.

There are also physical conclusions: Independently of any specific model, inter-node correlations are necessarily present in observed scale-free networks, where the tail of the degree distribution manifestly extends beyond a cut-off scaling like $\sqrt{N}$. Also, the thermodynamical limit can be rather tricky for scale-free networks.

2.5. Recent results by other people

I would like to mention a very nice result obtained by Fronczak *et al.* [11]. They have calculated analytically the average internode distance in graphs with uncorrelated nodes:

$$\langle \text{shortest path} \rangle \sim \frac{\ln N}{\ln (\langle n^2 \rangle / \langle n \rangle - 1)} . \quad (10)$$

This formula has been proposed earlier, by other groups, but the derivation has never been satisfactory, in my opinion. The problem is that the average shortest path has to grow like a power of N for a generic tree with uncorrelated nodes [6]. Thus, a derivation leading to the logarithmic behavior must use arguments that do not work for trees. This condition is satisfied in Ref. [11], but not in earlier publications claiming the same result. Notice that the coefficient in front of $\ln N$ diverges at the percolation threshold, *i.e.* when $\langle n(n-2) \rangle \rightarrow 0^+$ (*cf.* the celebrated reference [10]), at the transition to the regime dominated by trees.

Another set of related and interesting results is presented in Ref. [12]. These authors have calculated, among others, the distribution of connected components and found the size of the percolation cluster above the percolation threshold. They have also calculated the conditional degree distribution of nodes belonging to the percolation cluster.

There are many other results of the classical theory that could be extended to graphs with a given degree distribution. Indeed, a comprehensive discussion of the classical theory is a subject of a fat book [7]. But, we feel we have understood some of the most salient features of the model without correlations. Also, we have a numerical control of the model. Hence, we are eager to move to the next item on our agenda, *i.e.* the problem of correlations.

³ We are, of course, ready to share our numerical code with interested people.

3. Correlations

A comprehensive theory of correlation in networks does not exist. It is straightforward to generalize the model of the preceding section, introducing pairwise correlations between degrees of neighbor nodes. Specific proposals to this effect have been made, for example, in Refs. [13,14]. However, it seems to me, that correlations of a different nature are particularly important from the phenomenology point of view:

- Correlations induced by the growth dynamics.
- Clustering, *i.e.* the fact that neighbors of a randomly chosen node are directly linked to each other more frequently than by chance.

A work on clustering is in progress, but we do not have yet results significant enough to be presented here. Let me only mention that we are dealing with a very specific class of matrix models. On the other hand, we have developed a synchronic approach to growth processes, which is I believe worth mentioning:

We focus our attention on trees, actually on labeled rooted trees, in order to be able to proceed analytically. We consider a static ensemble, but assume that the networks are endowed with a causal structure. We say a tree is endowed with a causal structure when the labels always appear in growing numerical order as one moves along the tree from the root towards an arbitrary node. One can imagine that these labels refer to the time of node formation. The approach is complementary to the more standard diachronic one. It turns out that the presence of a causal structure generates internode correlations, once one has summed over all possible labelings. It is, therefore, of interest to consider models where these specific correlations do not interfere with correlations of a different origin. Hence, we assume that microstate weights factorize, as in Eq. (2). I have no place to enter into details, which can be found in Ref. [3] (see also the talk by P. Bialas [15]). Let me shortly summarize the most significant results:

- Some of the most popular growing network models, like Barabasi–Albert’s [16], can be reformulated in our static formalism. The original results are recovered in an elegant fashion. This shows that the widely accepted distinction between growing and equilibrium networks is not really correct. The opposition between diachrony and synchrony is to large extent an illusion, except when one is interested in specific phenomena, like aging, intrinsically reflecting the running of time.

- We derive a closed, general formula for the degree distribution.
- We also derive a closed formula for the correlation between the degree of an ancestor and that of its descendent, when they are separated by a geodesic distance r . Typically, the average descendent degree falls like $1/r$ [17]. Manifestly, this implies a long-range correlation.

— We further derive a general formula for the distribution of the shortest paths connecting nodes to the root. Using this formula we show that, generically, the length of an average such path grows at most like $\ln N$, in contrast to the uncorrelated trees where the growth is power-like [1, 6].

4. Concluding remarks

I am tempted to share with you a speculation, which does not rest on any solid basis, but may animate someone's imagination. Most present works on networks can be classified under the following headlines:

- Geometry of networks.
- Phenomenology of networks observed in nature.
- Matter on quenched random networks (this includes *e.g.* Ising spins living on networks, or the propagation of diseases).

What is manifestly missing, as far as I know, is a study of networks whose geometry is interacting with matter living on it (like in the models of quantum gravity, we have been working on). I am not sure that it would be relevant for the present day phenomenology, although some experts tell me that it might find applications in the theory of traffic and communication. Nevertheless, I believe it would be interesting to develop, at least, some models of that kind. I am pretty sure they would find applications in the future.

This work was partially supported by the EC IHP Grant No. HPRN-CT-1999-000161. Laboratoire de Physique Théorique is Unité Mixte du CNRS UMR 8627.

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