RANDOM MATRIX THEORY AND PARAMETRIC DYNAMICS OF SPECTRA*

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A unified treatment of the distribution of eigenvalues for different matrix ensembles is given with the help of parametric dynamics of levels and symplectic reduction of it.

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1. Introduction

Level dynamics is the parametric motion of eigenvalues of finite matrices X such as

$$X(\lambda) = X_0 + \lambda Y_0\,. \tag{1}$$

The real parameter λ may be looked upon as the weight of a perturbation Y_0 of an "initial" X_0 [1, 2]. If Hermitian, such matrices arise as Hamiltonians of quantum systems and level dynamics is then just a fancy variant of perturbation theory for energy spectra.

Non-Hermitian matrices of the form (1) are encountered, *e.g.* as generators of dissipative quantum dynamics, for instance in master equations $i\frac{d}{dt}\rho = X\rho$ for density operators ρ.

To derive differential equations governing the parametric level dynamics of Hermitian matrices (1) one differentiates with respect to λ the eigenvalue equation $X(\lambda)|\psi_n(\lambda)\rangle = q_n(\lambda)|\psi_n(\lambda)\rangle$ multiplies with the adjoint eigenvector $\langle\psi_m(\lambda)|$. The resulting set of differential equations can be revealed as a classical Hamiltonian flow for a fictitious one-dimensional gas of interacting particles (see below) [3]. The conventional strategy *cannot* be applied to arbitrary complex matrices $X(\lambda)$. First it can happen that such a matrix does not have N eigenvectors — in fact, the non-Hermitian generators of dissipative quantum dynamics are often of precisely this non-diagonalizable type.

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What is, however, even more important, is that the obtained flow can not be written as a Hamiltonian one with a *real* Hamilton function. Although this is the situation perfectly acceptable from the point of view of theory of Hamiltonian dynamical system, it is very inconvenient for making connections to the Random Matrix Theory — the main goal of the present paper. If the Hamilton function is complex the canonical equilibrium ensemble can not be constructed in a sensible way.

The fact that a real Hamilton function does not exist for the obtained in this way dynamical equations is connected to the impossibility of diagonalizing an arbitrary complex matrix by an unitary transformation — usually a general complex similarity transformation is needed to achieve the goal. On the other hand, even for non-diagonalizable matrices, by a unitary similarity transformation U we can reduce a matrix X to the triangular form. The eigenvalues of X then appear as the diagonal elements $(U^{-1}XU)_{nn}$; all elements $(U^{-1}XU)_{mn}$ below the diagonal (*i.e.* the ones with $m > n$) vanish, but the $(U^{-1}XU)_{mn}$ with $m < n$ do not.

2. Level dynamics of real symmetric matrices

2.1. Equations of motion

To present briefly the above outlined approach let us consider dynamical equations for the eigenvalues of a real symmetric $N \times N$ matrix X undergoing the parametric motion

$$\begin{aligned} X(\lambda) &= X_0 + \lambda Y_0\,, \\ X_0 &= X_0^{\mathrm{T}} = X_0^{\dagger}\,, \\ Y_0 &= Y_0^{\mathrm{T}} = Y_0^{\dagger}\,, \end{aligned} \tag{2}$$

and the corresponding eigenvalue equation X

$$X|\psi_n(\lambda)\rangle = q_n(\lambda)|\psi_n(\lambda)\rangle\,; \tag{3}$$

here $|\psi_n(\lambda)\rangle, n = 1, \ldots, N$, are the orthonormalized eigenvectors of X and q_n the corresponding eigenvalues (energies). By differentiating (3) with respect to λ and taking matrix elements between the eigenvectors $|\psi_n\rangle$ we obtain a closed system of differential equations for the quantities q_n, $p_n := \langle\psi_n|Y_0|\psi_n\rangle$, and $l_{mn} := \langle\psi_m|Y_0|\psi_n\rangle(q_n - q_m)$,

$$\begin{aligned} \dot{q}_n &= \frac{dq_n}{d\lambda} = p_n\,, \\ \dot{p}_n &= \frac{dp_n}{d\lambda} = -2\sum_{k\neq n}\frac{l_{nk}l_{kn}}{(q_n - q_k)^3}\,, \end{aligned}$$

$$\dot{l}_{mn} \;=\; \frac{dl_{mn}}{d\lambda} = -\sum_{k\neq m,n} l_{mk}l_{kn}\left(\frac{1}{(q_m-q_k)^2}-\frac{1}{(q_n-q_k)^2}\right). \qquad (4)$$

The system (4) fully describes the fate of the eigenvalues when the parameter λ , a fictitious time, is changed. The equations generalize the dynamics of the Calogero system [4–6]. As already advertised, the system can be treated as a Hamiltonian one (see [1–3]). To see that this is the case, we define the following Poisson brackets for the phase-space variables q_n, p_n, and l_{mn}:

$$\begin{aligned}
\{p_m, q_n\} &= \delta_{mn}\,, \quad \{p_m, p_n\} = 0 = \{q_m, q_n\}\,,\\
\{l_{mn}, p_i\} &= 0 = \{l_{mn}, q_i\}\,,\\
\{l_{mn}, l_{ij}\} &= \frac{1}{2}(\delta_{mj}l_{ni} + \delta_{ni}l_{mj} - \delta_{nj}l_{mi} - \delta_{mi}l_{nj})\,, \qquad (5)
\end{aligned}$$

and check that the dynamical equations (4) turn out to be the Hamilton ones

$$\dot{q}_n = \{\mathcal{H}, q_n\}\,, \quad \dot{p}_n = \{\mathcal{H}, p_n\}\,, \quad \dot{l}_{mn} = \{\mathcal{H}, l_{mn}\}\,, \qquad (6)$$

with the Hamilton function

$$\mathcal{H}(q,p,l) = \frac{1}{2}\sum_n p_n^2 + \frac{1}{2}\sum_{n\neq m}\frac{l_{mn}^2}{(q_m-q_n)^2}\,. \qquad (7)$$

One may think of the equations (4) or (6) as describing the time evolution of a one-dimensional gas of particles with positions q_n and canonically conjugate momenta p_n experiencing repulsive two-body interactions with the coupling strengths l_{mn} also undergoing temporal changes according to the third equation of (4). It will be noted that the gas in question must expand forever, due to the interparticle repulsion and the lack of confining forces. Such expansion is in fact obvious from the parametric motion (2) of the matrix $X(\lambda) = X_0 + \lambda Y_0$: for Y generic the eigenvalues of $X(\lambda)$ behave roughly like λ times the eigenvalues of Y_0 and thus fly apart indefinitely as $\lambda \to \infty$.

Not subject to such explosion are the levels of the real symmetric matrix

$$X(\lambda) = X_0\cos\lambda + Y_0\sin\lambda\,. \qquad (8)$$

In full analogy to the foregoing this yields level dynamics with unchanged Poisson brackets but the Hamilton function

$$\mathcal{H}(q,p,l) = \frac{1}{2}\sum_n p_n^2 + \frac{1}{2}\sum_n q_n^2 + \frac{1}{2}\sum_{n\neq m}\frac{l_{mn}^2}{(q_m-q_n)^2} \qquad (9)$$

which differs from (7) by an added harmonic confining potential. The change from the unbound parametric matrix motion (2) to the bounded one (8) can be understood as a non-canonical transformation of the fictitious-particle dynamics involving a λ-dependent rescaling of the coordinates as well as a non-linear reparametrization of the time λ [3, 7].

The fictitious gas with the binding Hamilton function (9) offers a convenient starting point for a statistical theory of spectra of real symmetric matrices X. The energy surface $\mathcal{H} = \text{const.}$ is compact and yields a normalizable canonical phase space density $\rho = \exp(-\mathcal{H})$. By integrating ρ over the variables p_n and l_{mn} we obtain a density of the fictitious-particle coordinates, *i.e.* a joint density of the eigenvalues of X,

$$P(q_1, q_2, \ldots, q_n) \sim \exp\left(-\left(\frac{1}{2}\sum_n q_n^2\right)\right) \prod_{i<j} |q_i - q_j| \,. \tag{10}$$

This is in fact the distribution known from the Gaussian Orthogonal Ensemble (GOE) of random-matrix theory [8], which appears here as a consequence of equilibrium statistical mechanics for an associated many-body system.

The case of general complex Hermitian matrix X can be analyzed without special alterations of the above outlined scheme and repeated for quaternion-real Hermitian matrices. In both cases the canonical equilibrium leads to distribution of energy levels complying with known from the Random Matrix Theory Gaussian Unitary (GUE) or Gaussian Symplectic (GSE) Ensemble.

Even the level dynamics of unitary Floquet matrices of periodically kicked quantum systems (see [3]) can be cast into the form of Hamiltonian matrix dynamics. Such Floquet matrices have the structure

$$F = \exp(-i\lambda V) F_0 \tag{11}$$

with V Hermitian and F_0 unitary, both of dimension N, and λ once more a real control parameter. Of interest is the fate of the eigenphases upon varying λ. The resulting equations read in this case

$$\begin{aligned}
\frac{dq_n}{d\lambda} &= p_n \,, \\
\frac{dp_n}{d\lambda} &= -\frac{1}{4}\sum_{k\neq n} l_{nk} l_{kn} \frac{\cos\left(\frac{q_n - q_k}{2}\right)}{\sin^3\left(\frac{q_n - q_k}{2}\right)} \,, \\
\frac{dl_{mn}}{d\lambda} &= -\frac{1}{4}\sum_{k\neq m,n} l_{mk} l_{kn} \left(\frac{1}{\sin^2\left(\frac{q_m - q_k}{2}\right)} - \frac{1}{\sin^2\left(\frac{q_n - q_k}{2}\right)}\right) ,
\end{aligned} \tag{12}$$

with q_n — eigenphases of F, *i.e.* $F(\lambda)|\psi_n(\lambda)\rangle = \mathrm{e}^{-iq_n(\lambda)}|\psi_n(\lambda)\rangle$, $p_n := \langle\psi_n|V|\psi_n\rangle$, and $l_{mn} := i(\mathrm{e}^{iq_m}\langle\psi_m|V|\psi_n\rangle\mathrm{e}^{-iq_m} - \langle\psi_m|V|\psi_n\rangle)$. The equations, generalizing the Sutherland system [9], are also Hamiltonian, with the Hamilton function

$$\mathcal{H}(q,p,l) = \frac{1}{2}\sum_n p_n^2 + \frac{1}{8}\sum_{n,m;n\neq m}\frac{l_{mn}^2}{\sin^2\left(\frac{q_m-q_n}{2}\right)} \tag{13}$$

and the same Poisson brackets (5) as in the Hermitian case. Upon assuming appropriate symmetries one recovers quasi-energy levels distributions of Dyson's Circular Orthogonal, Unitary and Symplectic Ensembles (COE, CUE, and CSE) as equilibrium distributions.

2.2. Hamiltonian formulation and reduction

To understand the origin of the universal scheme encountered, valid for all classical Gaussian and unitary ensembles I shall now rederive the Hamiltonian form of level dynamics for real symmetric matrices using a different method which lends itself more easily to generalizations. The method employs a certain symmetry inherent in the equation (2) or rather in its differential form

$$\dot{X} = Y\,, \quad \dot{Y} = 0\,. \tag{14}$$

The system of differential equations (14) describes a free motion in the space M of pairs of real symmetric matrices (X,Y). Forming the matrix elements of the matrices X and Y into canonically conjugate pairs (X_{ij},Y_{ij}) with $i\leq j$ of "positions" X_{ij} and "momenta" Y_{ij}, and defining the Poisson brackets

$$\begin{aligned}\{Y_{ii},X_{kk}\} &= \delta_{ik}\,, \quad \{Y_{ij},X_{kl}\} = \tfrac{1}{2}\delta_{ik}\delta_{jl} \text{ for } i<j,\ k<l\,,\\ \{Y_{ij},Y_{kl}\} &= 0 = \{X_{ij},X_{kl}\}\,,\end{aligned} \tag{15}$$

we reproduce the free motion (14) as the Hamilton equations

$$\dot{X}_{ij} = \{\mathcal{H},X_{ij}\}\,, \quad \dot{Y}_{ij} = \{\mathcal{H},Y_{ij}\}\,, \quad i,j = 1,\ldots,N \tag{16}$$

with the Hamilton function

$$\mathcal{H} = \frac{1}{2}\sum_i Y_{ii}^2 + \sum_{i<j} Y_{ij}^2 = \frac{1}{2}\mathrm{tr}Y^2. \tag{17}$$

We furnish the phase space M with a differential two-form ω which in our case reads

$$\omega = \sum_i dY_{ii}\wedge dX_{ii} + 2\sum_{i<j} dY_{ij}\wedge dX_{ij} = \mathrm{tr}(dY\wedge dX)\,. \tag{18}$$

As in the second member of the foregoing equation we shall usually omit the indices ij on X and Y and indicate summations as the trace operation; in employing that shorthand we must always keep in mind that only the X_{ij}, Y_{ij} with $i \leq j$ are our independent variables, and identify $X_{ij} = X_{ji}$.

The symplectic two-form ω is the exterior derivative, $\omega = d\alpha$, of a symplectic potential α; in the present example the potential reads

$$\alpha = \mathrm{tr}(Y dX)\,. \tag{19}$$

The symplectic form ω is used to define the Poisson brackets in a standard way. One introduces, namely, the Hamiltonian vector field $\mathcal{X}_\mathcal{F}$ of the smooth function $\mathcal{F}$ is defined by requiring that the action of ω on the pair of vector fields $\mathcal{X}_\mathcal{F}$ and $\mathcal{Z}$ (with an arbitrary $\mathcal{Z}$),

$$\mathcal{Z} = \mathrm{tr}\left(\mathcal{Z}_X \frac{\partial}{\partial X} + \mathcal{Z}_Y \frac{\partial}{\partial Y}\right), \tag{20}$$

gives the negative change of $\mathcal{F}$ along $\mathcal{Z}$,

$$\omega(\mathcal{X}_\mathcal{F}, \mathcal{Z}) = -d\mathcal{F}(\mathcal{Z}) = -\mathrm{tr}\left(\mathcal{Z}_X \frac{\partial \mathcal{F}}{\partial X} + \mathcal{Z}_Y \frac{\partial \mathcal{F}}{\partial Y}\right). \tag{21}$$

We find explicitly

$$\mathcal{X}_\mathcal{F} = \mathrm{tr}\left(\frac{\partial \mathcal{F}}{\partial Y}\frac{\partial}{\partial X} - \frac{\partial \mathcal{F}}{\partial X}\frac{\partial}{\partial Y}\right), \tag{22}$$

and define the Poisson bracket $\{\mathcal{F}, \mathcal{G}\}$ of two functions as

$$\{\mathcal{F}, \mathcal{G}\} = \omega(\mathcal{X}_\mathcal{F}, \mathcal{X}_\mathcal{G}). \tag{23}$$

It is easy to check that this definition gives back the original Poisson brackets (15) when we employ the particular symplectic two-form (18).

Observe now that the symplectic form ω (18) (and hence also the Poisson brackets (15)) as well as the Hamilton function $\mathcal{H}$ (17) are invariant under orthogonal transformations,

$$(X, Y) \longrightarrow (OXO^{-1}, OYO^{-1}), \quad OO^{\mathrm{T}} = I\,. \tag{24}$$

This symmetry under the group $\mathrm{O}(N)$ comes with a constant of the motion, (the momentum map)

$$\mu(X, Y) = [Y, X]\,. \tag{25}$$

The $\mathrm{O}(N)$-invariance can be used to change coordinates on M in order to obtain the eigenvalues of the matrix X among the new coordinates. To

that end we choose the particular orthogonal transformation (24) which diagonalizes X,

$$O^{-1}XO = \mathrm{diag}(q_1, q_2, \ldots, q_N) =: Q\,. \tag{26}$$

Obviously the matrix O is not uniquely determined by the matrix X. Nevertheless we can always choose one such a matrix for each X that the below outlined construction of coordinates is correct.

We subject the matrices Y and μ to that very same transformation and call the resulting matrices

$$P := O^{-1}YO\,, \quad l := O^{-1}\mu O\,. \tag{27}$$

To rewrite our Hamiltonian dynamics (14) alias (16) in new coordinates we consider the differential of X. With the help of the one-form

$$W := O^{-1}dO = -W^{\mathrm{T}} \tag{28}$$

we get

$$dX = d(OQO^{-1}) = O\,(dQ + [W, Q])\,O^{-1}. \tag{29}$$

We shall also need the differential dW

$$dW = d(O^{-1}dO) = -O^{-1}dOO^{-1} \wedge dO = -W \wedge W. \tag{30}$$

We can now express our symplectic potential α in the new coordinates,

$$\alpha = \mathrm{tr}(Y\,dX) = \mathrm{tr}(P(dQ + [W, Q])) = \mathrm{tr}(P dQ) - \mathrm{tr}(lW). \tag{31}$$

As its differential we have the symplectic two-form

$$\begin{aligned}\omega \;&=\; d\alpha = \mathrm{tr}(dP \wedge dQ) - \mathrm{tr}(dl \wedge W) + \mathrm{tr}(lW \wedge W)\\ &=\; \sum_i dp_i \wedge dq_i + 2\sum_{i<j} dl_{ij} \wedge W_{ij}\\ &\quad -2\sum_{i<j<k} \left(l_{ij}W_{jk} \wedge W_{ik} + l_{ik}W_{ij} \wedge W_{jk} + l_{jk}W_{ik} \wedge W_{ij}\right),\end{aligned} \tag{32}$$

where p_i with $i = 1, \ldots, N$ are the diagonal elements of the matrix P and l_{ij} the elements of the antisymmetric matrix l.

As new coordinates on M we choose now N functions q_i, N functions p_i, $N(N-1)/2$ elements of the antisymmetric matrix l and $N(N-1)/2$ independent coordinates parameterizing the orthogonal matrix O. A short calculation expresses the Hamilton function

$$\mathcal{H} = \tfrac{1}{2}\mathrm{tr}Y^2 = \tfrac{1}{2}\mathrm{tr}O^{-1}Y^2O \tag{33}$$

in terms of the new coordinates. As a result we find exactly the Hamiltonian function of level dynamics (7). Moreover, upon using (21), (23), and (32) we derive the Poisson brackets in the new variables and recover the ones previously given in (5). The considerations carry over immediately to real symmetric matrices of the form (8).

Adaptation to the unitary case (11) demands only minor changes. The phase space is now parameterized by pairs of matrices (X, Y), defined as

$$X = F\,, \quad Y = iF^{-1}V\,. \tag{34}$$

The symplectic structure given by the same form ω (18) together with the Hamilton function

$$\mathcal{H} = -\tfrac{1}{2}\mathrm{tr}(XY)^2 \tag{35}$$

leads to (34) as the solution of the Hamilton equations. The procedure quite analogous to the one presented above for Hermitian matrices gives the desired equations (12). To this end one diagonalizes F with the help of an unitary U:

$$U^\dagger F U = \exp(-iQ)\,, \quad Q = \mathrm{diag}(q_1, q_2 \ldots, q_N) \tag{36}$$

and, as previously, defines l as the image of the constant of the motion (the momentum mapping) $\mu = [X, Y]$ under the rotation by U

$$l := U^\dagger \mu U\,, \tag{37}$$

which, after straightforward calculations, gives

$$l = i(\exp(iQ)v\exp(-iQ) - v)\,, \tag{38}$$

with $v := U^\dagger V U$. The equations of motion expressed in the new variables q_n, $p_n := v_{nn}$, and l_{nm} are now given by (12), and the Hamilton function (35) reduces to (13). Other symmetry classes are treated in the same way, the only differences concern the diagonalizing matrix U, which reduces to a real orthogonal one in for COE and a unitary, symplectic one for CSE.

3. Level dynamics of complex matrices

The parametric motion (1) for arbitrary complex matrices can be written as a real one, instead of pairs of complex matrices (X, Y) we may choose quadruples of real ones through $X = X^{(1)} + iX^{(2)}, Y = Y^{(1)} + iY^{(2)}$ with $X^{(a)}, Y^{(a)}, a = 1, 2$; the phase space is then a $4N^2$ dimensional real manifold. The dynamics becomes Hamiltonian if we introduce the following symplectic structure

$$\alpha = \mathrm{Re}\,\mathrm{tr}(Y dX^\dagger)\,, \quad \omega = d\alpha = \mathrm{Re}\,\mathrm{tr}(dY \wedge dX^\dagger)\,, \tag{39}$$

and a *real, positive* Hamilton function

$$\mathcal{H} = \frac{1}{2}\mathrm{tr}YY^\dagger = \frac{1}{2}\sum_{i,j}^{N}\left(Y_{ij}^{(1)}\right)^2 + \sum_{i,j}^{N}\left(Y_{ij}^{(2)}\right)^2 . \tag{40}$$

As already mentioned one cannot mimic exactly the procedure applied in the Hermitian and unitary cases. Nevertheless the symplectic structure (39), as well as the Hamilton function (40) just introduced are invariant under the action of the unitary group $U(N)$

$$(X,Y) \longrightarrow (UXU^{-1}, UYU^{-1}) , \quad UU^\dagger = I . \tag{41}$$

As in the previous section, we find the (matrix) constant of the motion

$$\mu(X,Y) := \frac{1}{2}\left(\left[Y, X^\dagger\right] + \left[Y^\dagger, X\right]\right) . \tag{42}$$

Now a unitary transformation (41) putting X to the triangular form

$$Z := U^\dagger XU , \quad Z_{ij} = 0 \text{ for } i > j ; \tag{43}$$

can be always found (see *e.g.* [10]) with the diagonal elements Z_{ii} being the eigenvalues of X. The same transformation performed on the matrices Y and μ,

$$P := U^\dagger YU , \quad l := U^\dagger \mu(X,Y)U = \frac{1}{2}\left([P, Z^\dagger] + [P^\dagger, Z]\right) , \tag{44}$$

yields (in general full) complex matrices P and l. Now, one can reexpress the symplectic forms (39) in terms of P, l and the 1-form

$$W := U^\dagger dU = -W^\dagger \tag{45}$$

as

$$\begin{aligned} \alpha &= \mathrm{Re}\,\mathrm{tr}(YdX^\dagger) = \mathrm{Re}\,\mathrm{tr}(P(dZ^\dagger + [Z^\dagger, W^\dagger])) = \mathrm{Re}\,\mathrm{tr}(PdZ^\dagger) - \mathrm{tr}(lW) , \\ \omega &= d\alpha = \mathrm{Re}\,\mathrm{tr}(dP \wedge dZ^\dagger) - \mathrm{tr}(dl \wedge W) + \mathrm{tr}(lW \wedge W) . \end{aligned} \tag{46}$$

Mere inspection reveals the Poisson bracket between Z and P to be canonical,

$$\{P_{ij}, Z_{kl}^*\} = 2\delta_{ik}\delta_{jl} , \quad \{P_{ij}, Z_{kl}\} = \{Z_{ij}, Z_{kl}^*\} = \{P_{ij}, P_{kl}^*\} = 0 \tag{47}$$

and the Poisson brackets of P_{ij} and Z_{ij} with l_{kl} and U_{kl} to vanish. Moreover, the Poisson brackets for the l_{ij} do not depend on the U_{kl}. Indeed, let $\mathcal{F}$,

$\mathcal{G}$ be two functions depending on the components of l only and $\mathcal{X}_\mathcal{F}$, $\mathcal{X}_\mathcal{G}$ the corresponding Hamiltonian vector fields calculated according to (21). A straightforward calculation shows that

$$\{\mathcal{F},\mathcal{G}\} = \omega(\mathcal{X}_\mathcal{F},\mathcal{X}_\mathcal{G}) = -\mathrm{tr}\left(l\left[\left(\frac{\partial\mathcal{F}}{\partial l}\right)^{\mathrm{T}},\left(\frac{\partial\mathcal{G}}{\partial l}\right)^{\mathrm{T}}\right]\right), \tag{48}$$

where $\frac{\partial\mathcal{F}}{\partial l}$ is the matrix with the elements $\frac{\partial\mathcal{F}}{\partial l_{ij}}$ etc. In particular, for the l_{ij} themselves we obtain the brackets related to the Lie algebra $u(N)$,

$$\{l_{pq}, l_{mn}\} = l_{pn}\delta_{mq} - l_{mq}\delta_{pn}. \tag{49}$$

As in the case of Hermitian matrices, we can compactify the motion by considering (8) instead of (1) and look for the canonical equilibrium distribution. After integrating the irrelevant variables we recover for the distribution of the eigenvalues

$$P(\{Z_{ii}\}) \sim \exp\left(-\sum_i^N |Z_{ii}|^2\right)\prod_{i<j}|Z_{ii}-Z_{jj}|^2 \tag{50}$$

which is well known as the joint density of eigenvalues of Ginibre's ensemble [11] of random complex matrices.

4. Summary and conclusions

In the previous section I have shown how, with the help of parametric level dynamics, obtain the known distributions of eigenvalues for various ensembles of unitary matrices (Gaussian and circular orthogonal, unitary and symplectic, as well as Ginibre ensembles) using a unifying approach of reducing the dynamics via symmetry of the underlying symplectic structures. It is to stress that there are still some conceptual problems connected with the presented approach. For example all considered dynamical systems posses many independent, additional integrals of motion ([5, 6]), which should be taken into account when calculating equilibrium distributions. First step in this direction were taken in [12,13], where it was shown that corrections due to other constants of motion are of the order $1/N$ where N is the dimension of the considered matrices.

More detailed treatment of the presented results are given in [14, 15] on which the present paper was entirely based.

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