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STRINGS AND PARTICLES WITH EXTRINSIC CURVATURE*

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We discuss some of the similarities between strings and particles with extrinsic curvature. We shall highlight the appearance of extra classical symmetries that appear in particular actions.

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1. Introduction

The standard Nambu–Goto string has an action proportional to the area of the worldsheet

$$S = T \int d^2\xi \sqrt{g}\,, \tag{1}$$

where $g = \det g_{ab}$. The induced world-sheet metric is given by $g_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$. We shall consider for simplicity the case in which the target space is flat $g_{\mu\nu} = \eta_{\mu\nu}$.

The action (1) has an obvious two-dimensional diffeomorphism invariance. We can also introduce tangent vectors e^μ_a and then the metric becomes: $g_{ab} = e^\mu_a e^\mu_b$. In terms of the tangent vectors the diffeomorphism symmetry corresponds to a local SO(2) rotation

$$e^\mu_a \to U_{ab}(\xi) e^\mu_b\,. \tag{2}$$

The Nambu–Goto action only involves the metric on the two dimensional worldsheet which is an *intrinsic* quantity. Other possible intrinsic quantities are the Riemann tensor and its derivatives. In two dimensions the Ricci

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scalar determines the full Riemann tensor: $R_{abcd} = \frac{1}{2}R\,(g_{ac}g_{bd} - g_{ad}g_{bc})$. The addition of the simplest term only affects the action globally

$$\int d^2\xi R\sqrt{g} = 2\pi\chi\,, \tag{3}$$

where χ is the Euler characteristic of the two-dimensional surface. Therefore, one must consider more complicated actions involving R and its derivatives.

However, one may also consider string actions based on *extrinsic* quantities. This was first suggested by Polyakov and Kleinert. One introduces, in addition to the tangential vectors e_a^μ on the world-sheet, a set of normal vectors n_i^μ (see Fig. 1).
These satisfy

$$\begin{aligned} e_a^\mu e_b^\nu &= g_{ab}\,, \\ e_a^\mu n_i^\mu &= 0\,, \\ n_i^\mu n_j^\mu &= \delta_{ij}\,. \end{aligned} \tag{4}$$

There is some obvious freedom in choosing this system and in a string action this corresponds to the local symmetries

- SO(2): $e_a^\mu \to U_{ab}(\xi)e_b^\mu$,
- SO$(D-2)$: $n_i^\mu \to U_{ij}(\xi)n_i^\mu$.

The similarity between these expressions follows from the fact that one can think of the presence of the string breaking the SO(D) space–time symmetry down to SO(2)×SO$(D-2)$.

By taking covariant derivatives of the equations (4) we find the Gauss–Codazzi equations (see for example [1])

$$\begin{aligned} \nabla_a e_b^\mu &= \nabla_a\nabla_b X^\mu = K_{ab}^i n_i^\mu\,, \\ \nabla_a n_i^\mu &= -K_i{}^b{}_a e_b^\mu\,, \end{aligned} \tag{5}$$

where the quantity K_{ab}^i is known as the extrinsic curvature tensor.

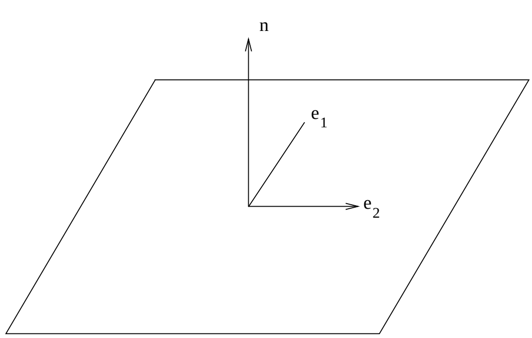

Fig. 1. Normal and tangential vectors.

1.1. Polyakov–Kleinert model

A string theory based on extrinsic curvature was proposed independently by Polyakov [2] and Kleinert [3]. The action they considered is given by

$$S = \frac{1}{2\alpha_0} \int d^2\xi \sqrt{g} \left(g^{ab} K^i_{ab} \right)^2 , \tag{6}$$

where the induced metric is the same as before. Using the Gauss–Codazzi equations we find the action (6) can be written in the equivalent forms

$$\begin{aligned} S_{\mathrm{PK}} &= \frac{1}{2\alpha_0} \int d^2\xi \sqrt{g} \, (\nabla_a n^\mu_i)^2 \\ &= \frac{1}{2\alpha_0} \int d^2\xi \sqrt{g} \left(\nabla^2 X^\mu \right)^2 . \end{aligned} \tag{7}$$

There are many problems with the quantization of this action due to the presence of the higher derivative terms. The theory has even more ghosts than the standard string. However, it has been found that the Euclidean model has a number of extremely attractive features. It is asymptotically free at high energies and exhibits confinement in the infra-red making it much closer to QCD than the standard string. There is also a spontaneous generation of string tension [4]. It is mainly these features, and the search for a stringy description of QCD, that have led to considerable interest in this model.

2. Discrete gonihedric string

We shall now momentarily leave the world of continuum physics to discuss the gonihedric string model [5–7]. This is based on an action which measures the length of a surface

$$S = m \sum_{<ij>} |X_i - X_j| \, |\pi - \alpha| , \tag{8}$$

where the quantities X_i and the angle α are indicated in Fig. 2.

To motivate the claim that this measures the length of a surface let us consider the action of a rectangular cylinder of length L (see Fig. 3). At each corner there is an angle of $\frac{\pi}{2}$ and, therefore, it is clear that for large L the action is given by $S = 2\pi m L$ and hence we see it is proportional to the length. This property differs markedly from that of the standard Nambu–Goto string which is proportional to the area. The fact that an action proportional to the length of a surface leads to a less divergent path-integral has been a primary motivation for this type of action.

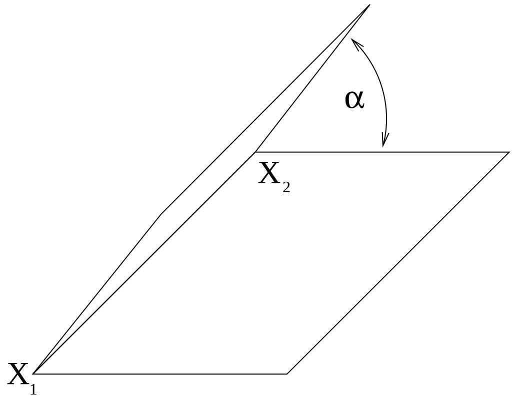

Fig. 2. The angle between two surfaces.

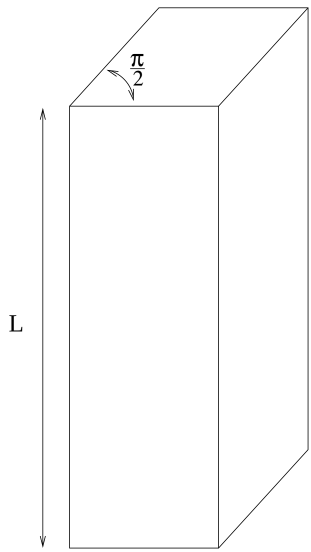

Fig. 3. "Length" of a surface

In the continuum the gonihedric action was shown to become

$$\begin{aligned} S &= m \int d^2\xi \sqrt{g} \sqrt{\left(g^{ab} K^i_{ab}\right)^2} \\ &= m \int d^2\xi \sqrt{g} \sqrt{(\nabla^2 X^\mu)^2} \,. \end{aligned} \tag{9}$$

In the work of Savvidy (see [8] and references therein) two separate continuum theories were associated to the gonihedric action

$$S = m \int d^2\xi \sqrt{g} \sqrt{(\nabla^2 X^\mu)^2}\,. \tag{10}$$

These are

- Model A (continuum gonihedric string):

$$\int \mathcal{D}X^\mu \mathrm{e}^{-S[X^\mu]}\,, \qquad g_{ab} = \partial_a X^\mu \partial_b X^\mu\,. \tag{11}$$

- Model B:

$$\int \mathcal{D}X^\mu \mathcal{D}g_{ab} \mathrm{e}^{-S[X^\mu, g_{ab}]}\,. \tag{12}$$

In the first case the metric is the induced one whereas in the second it is an independent degree of freedom.

3. Continuum gonihedric string (model A)

The perturbative calculations in this model (see [12, 13] and talk presented by A.R. Fazio [14]) showed the existence of $D-3$ physical degrees of freedom. Here one would generically expect $D-2$ physical degrees of freedom with two unphysical modes due to the two-dimensional diffeomorphism symmetry. The fact that there is another unphysical degree of freedom suggests that there may be some extra symmetry.

The canonical quantization of this action seems to be even harder than the Polyakov–Kleinert one due to the presence of the extra square-root. We shall therefore proceed, as we always do in physics, to consider a simpler example which we hope captures much of the essential details. In particular we shall see that this model possesses an extra, rather subtle and unexpected, symmetry.

3.1. Point particle analogy

We shall consider the direct analogue of the action (9) in just one-dimension parameterised by time, t. This action has been previously considered in [9] and is given by

$$S = m \int dt \sqrt{g} \sqrt{(\nabla^2 X^\mu)^2}\,. \tag{13}$$

Now the "metric" is given by

$$g_{tt} = \frac{dX^\mu}{dt}\frac{dX^\mu}{dt} = \dot{X}^2\,. \tag{14}$$

The Laplacian is given by

$$\nabla^2 X^\mu = \frac{1}{\sqrt{g}}\partial_t\left(\sqrt{g}g^{tt}\partial_t X^\mu\right) = \frac{1}{\dot{X}^2}\left(\ddot{X}^\mu - \frac{\dot{X}\ddot{X}}{\dot{X}^2}\dot{X}^\mu\right)\,. \tag{15}$$

Rather than discuss separately the quantization of such systems with higher derivatives it is convenient introduce an extra coordinate

$$q^\mu = \dot{X}^\mu \tag{16}$$

and re-write our system as a first order one in X^μ and q^μ. Using the above expressions we find the action (13) becomes

$$S = m\int dt\sqrt{q^2}\sqrt{\left(\frac{\dot{q}^\mu - \frac{q\dot{q}}{q^2}q^\mu}{q^2}\right)^2} + \lambda^\mu\left(q^\mu - \dot{X}^\mu\right)\,, \tag{17}$$

where the Lagrange multiplier λ^μ is used to enforce the condition (16). In the canonical formulation X^μ, q^μ and λ^μ are treated as independent variables. They have the canonically conjugate momenta

$$\begin{aligned} P^\mu &= \frac{\partial L}{\partial \dot{X}^\mu} = -\lambda^\mu\,,\\ p^\mu &= \frac{\partial L}{\partial \dot{q}^\mu} = \frac{1}{\sqrt{q^2}}\frac{\dot{q}^\mu - \frac{q\dot{q}}{q^2}q^\mu}{\sqrt{\left(\dot{q}^\mu - \frac{q\dot{q}}{q^2}q^\mu\right)^2}}\,,\\ \pi^\mu &= \frac{\partial L}{\partial \dot{\lambda}^\mu} = 0\,. \end{aligned} \tag{18}$$

In the Hamiltonian formalism we impose the canonical Poisson brackets

$$\begin{aligned} \{P^\mu, X^\nu\}_{\text{P.B.}} &= \eta^{\mu\nu}\,,\\ \{p^\mu, q^\nu\}_{\text{P.B.}} &= \eta^{\mu\nu}\,,\\ \{\pi^\mu, \lambda^\nu\}_{\text{P.B.}} &= \eta^{\mu\nu}\,. \end{aligned} \tag{19}$$

From the forms of these momenta we have immediately the so-called primary constraints

$$\begin{aligned} P^\mu + \lambda^\mu &= 0\,,\\ p^2q^2 - 1 &= 0\,,\\ \pi^\mu &= 0\,. \end{aligned} \tag{20}$$

The Poisson bracket between the first and third of these is given by

$$\{P^\mu + \lambda^\mu, \pi^\nu\}_{\mathrm{P.B.}} = -\eta^{\mu\nu} \tag{21}$$

and, therefore, it forms what is called a second-class constraint. There is a standard procedure for eliminating these by redefining the Poisson bracket [11]. Then we can set $\lambda^\mu = -P^\mu$ and $\pi^\mu = 0$ in all further expressions. The canonical Hamiltonian is given by

$$\begin{aligned} H &= P\dot{X} + p\dot{q} - L \\ &= Pq\,. \end{aligned} \tag{22}$$

Now we must compute the Poisson bracket of the constraint $p^2q^2 - 1 = 0$ with H which generates further constraints. We must then compute the algebra of these with themselves and H. In this way we get the closed set of constraints

$$\begin{aligned} p^2q^2 - 1 &= 0\,, \\ Pp &= 0\,, \\ Pq &= 0\,, \\ P^2 &= 0\,. \end{aligned} \tag{23}$$

The quantity P^μ, conjugate to the space–time coordinate X^μ, represents the target–space momenta. Hence, using the final constraint, we see that the spectrum is *massless*.

The fact that we found two first class constraints means that the action possesses *two* gauge symmetries. One is obvious: it is simply one-dimensional diffeomorphism invariance. Indeed as is usual in such systems the Hamiltonian (22) appears as one of the constraints. The other symmetry, however, is much more subtle. It was found in [10] to be a non-local (in X^μ space) and non-linear W_3 symmetry. One particularly interesting consequence of this symmetry is that the coordinate X^μ is *not* actually a gauge invariant quantity! This fact was realised long before the full symmetry had been discovered [9]. One can think of a helical motion of a particle with overall velocity exceeding that of light. The physical degrees of freedom correspond to the projection onto a plane where all quantities move at $v = c$. There is, therefore, no violation of causality.

We should emphasize that this extra symmetry and one less degree of freedom *only* occurs for the action (13). For instance a similar analysis of the point-particle analogue of the Polyakov–Kleinert model reveals that it only has the obvious diffeomorphism symmetry. It would be interesting to extend this canonical analysis to the continuum gonihedric string (9) to understand if the factors of $D - 3$ found in the perturbative analysis were also due to extra symmetry.

4. Model B

It is not clear if any geometrical interpretation can be given to the second model. However, it possesses the interesting property [8] that it has local Weyl symmetry *i.e.* invariance under: $g_{ab} \to e^{\rho(\xi)}$. Therefore, one may go to the conformal gauge in which the action takes the form

$$S = m \int d^2\xi \sqrt{(\partial^2 X^\mu)^2}\,, \tag{24}$$

and all derivatives are flat space ones. This action can be simply linearised with the introduction of a Lagrange multiplier

$$S = \int d^2\xi \partial X^\mu \partial \pi^\mu + \lambda(\pi^2 - m^2)\,. \tag{25}$$

The equations of motion are given by

$$\begin{aligned} \partial^2 \pi^\mu &= 0\,, \\ \partial^2 X^\mu + 2\lambda \pi^\mu &= 0\,, \\ \pi^2 - m^2 &= 0\,. \end{aligned} \tag{26}$$

The last two equations can be solved for π^μ to give

$$\pi^\mu = \frac{m \partial^2 X^\mu}{\sqrt{(\partial^2 X^\mu)^2}}\,. \tag{27}$$

The physical quantisation of this system on the constrained Hilbert space seems extremely difficult due to the non-linear constraint on π^μ. It seems much simpler to perform a covariant quantisation in which one takes the action

$$S = \int d^2\xi \partial X^\mu \partial \pi^\mu \tag{28}$$

and imposes the constraints as physical state conditions. The equations of motion are now linear and are easily solved

$$\begin{aligned} \partial^2 \pi^\mu = 0\,, \qquad \pi^\mu &= m\varepsilon^\mu + \tau p^\mu + \text{oscillators}\,, \\ \partial^2 X^\mu = 0\,, \qquad X^\mu &= x^\mu + \tau y^\mu + \text{oscillators}\,. \end{aligned} \tag{29}$$

The canonical commutators are given by

$$[x^\mu, P^\nu] = i\eta^{\mu\nu}\,, \qquad [m\varepsilon^\mu, y^\nu] = i\eta^{\mu\nu}\,. \tag{30}$$

The momenta conjugate to the real coordinate X^μ is $P^\mu = \partial_\tau \pi^\mu$. The physical state condition is given by

$$\left(\pi^2 - m^2\right) |\ \text{phys.}\rangle = 0\,. \tag{31}$$

On the lowest modes this gives

$$\begin{aligned} \varepsilon^2 &= 1\,, \\ \varepsilon P &= 0\,, \\ P^2 &= 0\,. \end{aligned} \tag{32}$$

The first of these implies that ε must be space-like and the last implies that the spectrum is massless. It is unclear what the physical spectrum of the theory, with all the oscillators included, is and whether it is unitary.

We briefly mention that the linearised form (25) admits a simple supersymmetric generalisation [15]

$$S = m \int d^2 z d^2 \theta \left[D_\theta \mathbf{\Pi}^\mu D_{\bar\theta} \boldsymbol{X}^\mu + \mathbf{\Lambda} \left(\mathbf{\Pi}^2 - m^2 \right) \right]\,, \tag{33}$$

where $\mathbf{\Pi}^\mu, \boldsymbol{X}^\mu, \mathbf{\Lambda}^\mu$ are $\mathcal{N} = 1$ super-fields. It is possible that this could result from the gauge fixing of an $\mathcal{N} = 1$ extrinsic super-gravity.

4.1. Point-particle analogue

We shall now discuss the point particle analogue of model B. The action is given by

$$S = m \int dt \sqrt{\ddot{X}^2}\,. \tag{34}$$

One should note that this is *not* the same as model A (13) because we have no diffeomorphism symmetry. Once again we can easily linearise this

$$S = \int dt \dot{\pi} \dot{X} + \lambda(\pi^2 - m^2)\,. \tag{35}$$

The equations of motion can be solved for π^μ giving

$$\pi^\mu = \frac{m \ddot{X}^\mu}{\sqrt{\ddot{X}^2}}\,. \tag{36}$$

Solving the unconstrained equations (with $\lambda = 0$) we find

$$\begin{aligned} \ddot{\pi}^\mu &= 0\,, \qquad \pi^\mu = m\varepsilon^\mu + p^\mu t\,, \\ \ddot{X}^\mu &= 0\,, \qquad X^\mu = x^\mu + y^\mu t\,. \end{aligned} \tag{37}$$

The constraints are again $\left(\pi^2 - m^2\right) \mid \text{phys.}\rangle = 0$ giving the conditions

$$\begin{aligned} \varepsilon^2 &= 1\,, \\ \varepsilon p &= 0\,, \\ p^2 &= 0\,. \end{aligned} \tag{38}$$

These are exactly the same as the constraints resulting from the zero modes of the string. This is expected due to the fact that the centre of mass of a string behaves as a point particle.

We shall see that, although there is no diffeomorphism symmetry, the action (34) actually possesses a non-local symmetry. As before we write the Lagrangian in a first order formalism

$$S = \int dt \sqrt{\dot{q}^2} + \lambda^\mu \left(\dot{X}^\mu - q^\mu\right). \tag{39}$$

The canonical momenta are given by

$$\begin{aligned} P^\mu &= \frac{\partial L}{\partial \dot{X}^\mu} = -\lambda^\mu\,, \\ p^\mu &= \frac{\partial L}{\partial \dot{q}^\mu} = \frac{\dot{q}^\mu}{\sqrt{\dot{q}^2}}\,, \\ \pi^\mu &= \frac{\partial L}{\partial \dot{\lambda}^\mu} = 0\,. \end{aligned} \tag{40}$$

Again we find the primary constraints

$$\begin{aligned} P^\mu + \lambda^\mu &= 0\,, \\ p^2 - 1 &= 0\,, \\ \pi^\mu &= 0 \end{aligned} \tag{41}$$

and as before we remove λ^μ and π^μ by a redefinition of the Poisson bracket. We are then left with the single primary constraint $p^2 - 1 = 0$.

The canonical Hamiltonian is given by

$$\begin{aligned} H &= P\dot{X} + p\dot{q} - L \\ &= Pq\,, \end{aligned} \tag{42}$$

which is exactly as before. The secondary constraints are given by

$$\begin{aligned} \{H, p^2\}_{\text{P.B.}} &= Pp\,, \\ \{H, Pp\}_{\text{P.B.}} &= P^2\,. \end{aligned} \tag{43}$$

The spectrum of the theory is again massless. We have just one first class constraint $p^2 - 1 = 0$. This is the generator of the symmetry

$$\{p^2 - 1, q^\mu\}_{\text{P.B.}} = 2p^\mu \,. \tag{44}$$

Therefore, the symmetry is given by

$$\delta q^\mu = p^\mu = \frac{\dot{q}^\mu}{\sqrt{\dot{q}^2}} \varepsilon(t) \,. \tag{45}$$

This implies

$$\delta \dot{q}^\mu = \frac{\ddot{q}^\mu}{\sqrt{\dot{q}^2}} \varepsilon(t) - \frac{\dot{q}\ddot{q}\dot{q}^\mu}{\sqrt{\dot{q}^2}^3} \varepsilon(t) + \frac{\dot{q}^\mu}{\sqrt{\dot{q}^2}} \dot{\varepsilon}(t) \tag{46}$$

and now we can check the invariance of the action

$$\begin{aligned} S = \int dt \sqrt{\dot{q}^2} \;\; &\rightarrow \;\; \int dt \sqrt{(\dot{q} + \delta\dot{q})^2} \\ &= \;\; \int dt \sqrt{\dot{q}^2} + \frac{\dot{q}\delta\dot{q}}{\sqrt{\dot{q}^2}} \\ &= \;\; S + \int dt \dot{\varepsilon}(t) \\ &= \;\; S \,. \end{aligned} \tag{47}$$

However, in X^μ space this is *non-local*

$$\delta \dot{X}^\mu = \frac{\ddot{X}^\mu}{\sqrt{\ddot{X}^2}} \varepsilon(t) \,. \tag{48}$$

Therefore, we also have a non-local and non-linear symmetry in this model.

5. Conclusion

We have seen that there are some interesting extra classical symmetries in certain extrinsic curvature actions for point particles. One would expect that much of this would extend to the string case but the canonical quantisation of that system appears much more involved. As the gonihedric model was first considered as a lattice model it would be interesting to investigate if these extra symmetries are also present there.

One of the most important questions is whether these theories give sensible quantum field theories. Most of the understanding of two-dimensional models is based on symmetries rather than explicit Lagrangians. Therefore, one feels that a proper understanding of the symmetries of these theories is of central importance.

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