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GAUGE FIELDS-STRINGS DUALITY AND TENSIONLESS SUPERSTRINGS*

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The duality map between gauge theories and strings suggests that when the gauge theory is in the weak coupling regime the dual string tension effectively tends to zero, $\alpha' \to \infty$. This observation of Sundborg and Witten initiates a fresh interest to the old problem of tensionless limit of standard string theory and to the description of its genuine symmetries. We approach this problem formulating tensionless string theory by means of geometrical concept of surface perimeter. The perimeter action uniquely leads to a tensionless theory.

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1. Introduction

It is a longstanding problem to describe low energy behavior of QCD in terms of string-like extended objects. Naive identification of standard string theory spectrum with hadronic spectrum encounters a number of conceptual problems connected with the appearance of massless states containing graviton in the string spectrum, soft behavior of high energy scattering amplitudes, opposite to what one should expect in parton-like picture of asymptotically free gauge theories and, not the least, higher dimensional space-time.

Essential progress was achieved in [1–4] where the AdS/CFT correspondence was proposed relating the classical supergravity approximation $R^2_{\mathrm{AdS}}/\alpha' \gg 1$ of closed IIB strings moving on ten-dimensional curved space-time background $\mathrm{AdS}_5 \times S^5$ with large 't Hooft coupling regime of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory $R^4_{\mathrm{AdS}}/\alpha'^2 = \lambda \equiv g^2_{\mathrm{YM}} N$. In this duality map one side is weakly coupled, the other is strongly coupled and there is a

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natural prescription relating physical quantities in string and gauge theories, in particular, identifying stringy states corresponding to the leading Regge trajectory with highly excited gauge theory operators [5–7]. The conjecture was applied to calculate Wilson loops, anomalous dimensions, *etc.* on the gauge theory side at strong coupling regime.

If one adopts the strong form of the Maldacena conjecture stating that two theories are exactly the same for all values of coupling constants [1], then it is important to understand what string theory is like in the opposite limit when the gauge theory is in the weak coupling regime [8–10, 21] . In that regime the string tension $T = 1/2\pi\alpha' = \sqrt{\lambda}/R^2_{\rm AdS}$ effectively tends to zero and it is natural to assume that free gauge theory, $\lambda \ll 1$, corresponds to zero tension string theory, that is to the string theory at extreme energies [11–13]. The dual description of weakly interacting gauge theory states on the boundary, in particular, operators with minimal twist, consisting of bilinear high spin tensors, on the AdS_5 side are supposed to be expressed in terms of locally interacting massless gauge fields of arbitrarily high spin [8–10,14,17,19,20,22]. The gauge theory correlation functions on the boundary define a high spin field theory in the bulk with nontrivial interaction vertices [15, 16, 18], and the resulting holographically dual classical gauge field theory would be the effective description of the desirable string theory in the bulk, celebrated symmetric phase of string theory [11].

This development initiates a fresh interest to the old problem of tensionless limit of the standard string theory and to the description of its genuine symmetries [8, 9, 11, 13, 29–31]. In recent publications we approached this problem formulating tensionless strings by means of geometrical concept of surface perimeter, or its length [23,29]. The perimeter action uniquely leads to tensionless theory. It was suggested that nonlinear world-sheet sigma model which describes tensionless limit is defined by the following action:

$$S = m\,L = m\int d^2\zeta\sqrt{h}\sqrt{K^{ia}_{a}K^{ib}_{b}}, \tag{1}$$

here m has dimension of mass, h_{ab} is the induced metric and K^i_{ab} is the second fundamental form (extrinsic curvature) [1]. Instead of being proportional to the area of the surfaces, as it is the case in the standard string theory

$$A \simeq \frac{1}{2\pi\alpha'}\int\sqrt{1-V^2_{\perp}}\;dsdt\,,$$

the perimeter action (1) is proportional to the length of the surface. Due to the last property the model has two desirable features. First of all, when

[1] This action is essentially different in its geometrical meaning from the action considered in previous studies [28] where it is proportional to the spherical angle and has dimensionless coupling constant.

the surface degenerates into a single world line, the perimeter action (1) becomes proportional to the length of the world line [23, 29]

$$S \simeq m \int \sqrt{1-V_{\perp}^2}\, K(s,t)\; dsdt \rightarrow \; m \int \sqrt{1-V_{\perp}^2}\; 2\pi \; dt\,, \tag{2}$$

where $K(s,t) = d\varphi/ds$ is a string curvature. Thanks to this property the functional integral over surfaces simply transforms into the Feynman path integral for a point-like relativistic particle, naturally extending it to relativistic strings. Secondly, the action is equal to the perimeter of the flat Wilson loop $S = m(R+T)$, where R is space distance between quarks, therefore at the *classical level string tension is equal to zero*. The action (1) can be written in the equivalent form [23, 29]

$$S = \frac{m}{\pi} \int d^2\zeta \sqrt{h} \sqrt{(\Delta(h) X_\mu)^2}, \tag{3}$$

where $h_{ab} = \partial_a X_\mu \partial_b X_\mu$ is the induced metric, $\Delta(h) = 1/\sqrt{h}\; \partial_a \sqrt{h} h^{ab} \partial_b$ is Laplace operator[2], $a, b = 1, 2;\quad \mu = 0, 1, 2, \ldots, D-1$.

In the present work I shall consider so called model B [29], in which two fields h_{ab} — the world-sheet metric and X^μ — the embedding field are considered as independent, that is we abandon the relation $h_{ab} = \partial_a X_\mu \partial_b X_\mu$ between them. At this stage there is no direct relation of the model with embedding into space-time and the model can be considered as two-dimensional quantum gravity interacting with scalar fields X^μ. We shall refer to the original theory, where fields are not independent, as to model A. The interrelation between them is considered in [29].

We shall fix the conformal gauge $h_{ab} = \rho \eta_{ab}$ using the reparametrization invariance of the action (3) and represent it in two equivalent forms [29]

$$S = \frac{m}{\pi} \int d^2\zeta \sqrt{(\partial^2 X)^2} \;\; \Leftrightarrow \;\; \acute{S} = \frac{1}{\pi} \int d^2\zeta \{\, \varPi^\mu \,\partial^2 X^\mu - \varOmega\, (\varPi^2 - m^2)\,\}, \tag{4}$$

where we have introduced the independent field $\varPi^\mu$ and the Lagrange multiplier $\varOmega$. The system of equations which follows from $\acute{S}$

$$(\mathrm{I}) \qquad \partial^2 \varPi^\mu = 0, \qquad \partial^2 X^\mu - 2\varOmega \varPi^\mu = 0, \qquad \varPi^\mu \varPi_\mu = m^2 \tag{5}$$

is equivalent to the original equation for X^μ which follows from S

$$(\mathrm{I}') \qquad \partial^2 \left(m \frac{\partial^2 X_\mu}{\sqrt{(\partial^2 X)^2}} \right) = 0. \tag{6}$$

[2] The equivalence follows from the relations: $K_a^{ia} n_\mu^i = \Delta(h) X_\mu$, where n_i^μ are the normals and $K_a^{ia} K_b^{ib} = (\Delta(h) X_\mu)^2\,, \quad i, j = 1, 2, \ldots, D-2$

The Π^μ field has the form $\Pi^\mu = m\partial^2 X^\mu/\sqrt{(\partial^2 X)^2}$. In addition to the reparametrization invariance the system exhibits a new gauge symmetry. For a given parametrization the additional invariance has the form [29]

$$\partial^2 X^\mu \to \partial^2 X^\mu + 2\omega \Pi^\mu\,, \quad \Pi^\mu \to \Pi^\mu\,, \quad \Omega \to \Omega + \omega\,,$$

where $\omega = \partial_a \omega^a$ and ω^a is arbitrary vector field on the world-sheet. The above gauge transformation

$$\partial^2 X_\mu^{'} = \left(1 + \frac{2\omega}{\sqrt{(\partial^2 X)^2}}\right)\, \partial^2 X_\mu \tag{7}$$

defines a set of fields X_μ describing the same physics and can be seen as the gauge orbit of this extra symmetry. Notice that fields on a given gauge orbit are not related by reparametrization. This gauge symmetry renders the string space-time coordinate X^μ "less" physical, not gauge invariant observable. Instead, the string momentum $P^\mu = \partial_0 \Pi^\mu$ is a gauge invariant quantity because $\Pi_\mu^{'} = \Pi_\mu$.

Quantization of the bosonic string B and its massless spectrum has been derived in [29]. The absence of conformal anomaly requires the space-time to be 13-dimensional $D_c = 13$. In this string theory *all particles, with arbitrary large integer spin, are massless.* This pure massless spectrum is consistent with the tensionless character of the model and it was conjectured in [29] that it may describe unbroken phase of standard string theory when $\alpha^{'} \to \infty$ and all masses tend to zero $M_n^2 = \frac{1}{\alpha^{'}}(n-1) \to 0$ [11] .

Supersymmetric extension of the model B with $\mathcal{N} = 1$ world-sheet supersymmetry was constructed in [35]. Here I shall demonstrate that actually it possesses enhanced fermionic symmetry which elevates $\mathcal{N} = 1$ world-sheet supersymmetry to $\mathcal{N} = 2$ world-sheet supersymmetry. Indeed quantization of the supersymmetric model shows that its gauge algebra is identical with the well known $\mathcal{N} = 2$ world-sheet superalgebra [36]. This new field theory realization of the $\mathcal{N} = 2$ algebra is free from old problem [37–41] connected with the introduction of second space-time coordinate Y^μ, which was introduced in addition to the coordinates X^μ [36]. Instead, in this model we have naturally two left-movers q_1^μ and q_2^μ and two right-movers $\tilde{q}_1{}^\mu$ and $\tilde{q}_1{}^\mu$ of the X^μ field [29]

$$X_{\mathrm{L}}^\mu = x^\mu + \frac{1}{m}\pi^\mu \zeta^+ + \sum_{n=1}^{\infty} \sqrt{\frac{2}{nm^2}}\ \left\{ q_{1n}^\mu \sin(n\zeta^+) + q_{2n}^\mu \cos(n\zeta^+) \right\},$$

$$X_{\mathrm{R}}^\mu = x^\mu + \frac{1}{m}\pi^\mu \zeta^- + \sum_{n=1}^{\infty} \sqrt{\frac{2}{nm^2}}\ \left\{ \tilde{q}_{1n}^{\ \mu} \sin(n\zeta^-) + \tilde{q}_{2n}^{\ \mu} \cos(n\zeta^-) \right\}.$$

The conjugate field is described by a separate field Π^μ,

$$\Pi^\mu_{\mathrm{L}} = me^\mu + k^\mu\zeta^+ + \sum_{n=1}^{\infty}\sqrt{\frac{2m^2}{n}}\left\{-p^\mu_{1n}\cos(n\zeta^+) + p^\mu_{2n}\sin(n\zeta^+)\right\},$$

$$\Pi^\mu_{\mathrm{R}} = me^\mu + k^\mu\zeta^- + \sum_{n=1}^{\infty}\sqrt{\frac{2m^2}{n}}\left\{-\tilde{p}^\mu_{1n}\cos(n\zeta^-) + \tilde{p}^\mu_{2n}\sin(n\zeta^-)\right\},$$

therefore, we have two times more degrees of freedom than in the standard bosonic string theory. This result can be qualitatively understood if one takes into account the fact that the field equations here are of the fourth order (6). Notice that there is also doubling of zero modes, the new zero mode coordinates are e^μ and their conjugate variables are π^μ, they describe transversal polarizations [29].

In the first part of this article I shall review the quantization of the bosonic tensionless string and shall describe its symmetries. In the second part I shall present oscillator representation of the supersymmetric extension of the model and its quantization. In the last section the twisted topological string model will be constructed in analogy with the standard prescription for $\mathcal{N} = 2$ superconformal field theories [24–27, 42].

2. Closed bosonic strings

In this section I shall review some facts concerning solution and quantization of the closed bosonic string which was defined in the previous section and shall discuss algebraic structure of the corresponding gauge symmetries (13), (14) and (15). As we shall see they naturally contain Virasoro algebra as its subalgebra and additional new generators Θ_{nk} associated with new gauge symmetry (7) forming an Abelian subalgebra. The conformal algebra has here its classical form with twice larger central charge $2 \times \frac{D}{12} = \frac{D}{6}$. This result can be qualitatively understood if one takes into account the fact that the field equations here are of the fourth order and therefore we have two left and two right movers of X^μ field, two times more degrees of freedom than in the standard bosonic string theory. Therefore it is not surprising that the absence of conformal anomaly requires the space-time to be 13-dimensional: $D_c = 13$.

For the closed bosonic strings the mode expansion of X field (8) can be written in the form [29]:

$$X^\mu_{\mathrm{L}} = x^\mu + \frac{1}{m}\pi^\mu\zeta^+ + i\sum_{n\neq 0}\frac{1}{n}\,\beta^\mu_n \mathrm{e}^{-in\zeta^+},$$

$$X^\mu_{\mathrm{R}} = x^\mu + \frac{1}{m}\pi^\mu\zeta^- + i\sum_{n\neq 0}\frac{1}{n}\,\tilde{\beta}^\mu_n \mathrm{e}^{-in\zeta^-},$$

where $X^\mu = \frac{1}{2}(X^\mu_{\rm L}(\zeta^+) + X^\mu_{\rm R}(\zeta^-))$, and in similar manner $\Pi^\mu = \frac{1}{2}(\Pi^\mu_{\rm L}(\zeta^+) + \Pi^\mu_{\rm R}(\zeta^-))$

$$\Pi^\mu_{\rm L} = me^\mu + k^\mu \zeta^+ + i\sum_{n\neq 0} \frac{1}{n}\ \alpha^\mu_n {\rm e}^{-in\zeta^+},$$
$$\Pi^\mu_{\rm R} = me^\mu + k^\mu \zeta^- + i\sum_{n\neq 0} \frac{1}{n}\ \tilde{\alpha}^\mu_n {\rm e}^{-in\zeta^-}. \tag{8}$$

The nonzero commutator is $[\partial_\pm X^\mu_{\rm L,R}(\zeta), \partial_\pm \Pi^\nu_{\rm L,R}(\zeta')] = 2\pi i\ \eta^{\mu\nu}\delta\,'(\zeta - \zeta')$, with all others equal to zero. The momentum density operator is $2P^\mu = \partial_+\Pi^\mu_{\rm L} + \partial_-\Pi^\mu_{\rm R} = P^\mu_{\rm L} + P^\mu_{\rm R}$. This canonical commutation among the fields imply also the following commutation relations among the coefficients of the Fourier expansion (8):

$$[e^\mu, \pi^\nu] = [x^\mu, k^\nu] = i\eta^{\mu\nu}, \qquad [\alpha^\mu_n, \beta^\nu_k] = n\ \eta^{\mu\nu}\delta_{n+k,0} \tag{9}$$

and similar ones for $\tilde{\alpha}^\mu_n$ and $\tilde{\beta}^\mu_n$. All other commutators are equal to zero. They are connected with the creation and annihilation operators in the following way

$$\begin{aligned}
\alpha^\mu_0 &= k^\mu\,, & \beta^\mu_0 &= \frac{\pi^\mu}{m}\,,\\
\alpha^\mu_n &= m\sqrt{n}\ a^\mu_n\,, \quad n>0\,, & \beta^\mu_n &= \frac{1}{m}\sqrt{n}\ b^\mu_n\,, \quad n>0\,,\\
\alpha^\mu_{-n} &= m\sqrt{n}\ a^{+\mu}_n\,, \quad n>0\,, & \beta^\mu_{-n} &= \frac{1}{m}\sqrt{n}\ b^{+\mu}_n\,, \quad n>0\,,
\end{aligned} \tag{10}$$

with nonzero commutator $[a^\mu_n, b^{+\mu}_m] = \eta^{\mu\nu}\delta_{nm}$.

The Virasoro operators L_n and new operators $\Theta_{n,k}$ are defined as

$$L_n = \langle {\rm e}^{in\zeta^+} : P^\mu_{\rm L}\ \partial_+ X^\mu_{\rm L} :\rangle, \qquad \Theta_{n,l} = \langle {\rm e}^{in\zeta^+ + il\zeta^-} : \Pi^\mu\ \Pi^\mu - m^2 :\rangle \tag{11}$$

and have the form

$$\begin{aligned}
L_n &= \sum_l : \alpha_{n-l}\cdot\beta_l :\,, \qquad \tilde{L}_n = \sum_l : \tilde{\alpha}_{n-l}\cdot\tilde{\beta}_l :\,,\\
\Theta_{0,0} &= m^2(e^2-1) + \sum_{n\neq 0}\frac{1}{4n^2} : (\alpha_{-n}\ \alpha_n + \tilde{\alpha}_{-n}\ \tilde{\alpha}_n) :\,,\\
\Theta_{n,0} &= \frac{im}{n}\ e\cdot\alpha_n - \frac{1}{4}\sum_{l\neq 0,n}\frac{1}{(n-l)l} : \alpha_{n-l}\cdot\alpha_l :\,, \qquad n = \pm 1, \pm 2, \dots\,,\\
\Theta_{0,n} &= \frac{im}{n}\ e\cdot\tilde{\alpha}_n - \frac{1}{4}\sum_{l\neq 0,n}\frac{1}{(n-l)l} : \tilde{\alpha}_{n-l}\cdot\tilde{\alpha}_l :\,, \qquad n = \pm 1, \pm 2, \dots\,,\\
\Theta_{n,k} &= -\frac{1}{2nk} : \alpha_n\cdot\tilde{\alpha}_k :\,, \qquad n,k = \pm 1, \pm 2, \dots\,.
\end{aligned} \tag{12}$$

The conformal algebra has here its classical form but with twice larger central charge

$$[L_n\ ,\ L_k] = (n-k)L_{n+k} + \frac{D}{6}(n^3-n)\delta_{n+k,0} \tag{13}$$

and with the similar expression for right movers $\tilde{L}_n$. The reason that the central charge is twice bigger than in the standard bosonic string theory $2 \times \frac{D}{12} = \frac{D}{6}$ is simply because we have two left and two right movers of X_μ field. Such doubling of modes is reminiscent to the bosonic part of the $\mathcal{N}=2$ superstring [36]. In the last model there was an essential problem in identifying the Y^μ coordinates which are introduced in addition to the normal coordinates X^μ [36]. In our model the coordinate field X has simply two sets of commuting oscillators and the conjugate fields are described by the separate field Π .

The full extended gauge symmetry algebra of constraints (11) takes the form

$$\begin{aligned}
[L_n\ ,\Theta_{0,0}] &= -2n\,\Theta_{n,0}\,, & [\tilde{L}_n\ ,\Theta_{0,0}] &= -2n\,\Theta_{0,n}\,,\\
[L_n,\Theta_{k,0}] &= -(n+k)\,\Theta_{n+k,0}\,, & [\tilde{L}_n,\Theta_{k,0}] &= -2n\,\Theta_{k,n}\,,\\
[L_n,\Theta_{0,k}] &= -2n\,\Theta_{n,k}\,, & [\tilde{L}_n,\Theta_{0,k}] &= -(n+k)\,\Theta_{0,n+k}\,,\\
[L_n,\Theta_{k,l}] &= -(n+k)\,\Theta_{n+k,l}\,, & [\tilde{L}_n,\Theta_{k,l}] &= -(n+l)\,\Theta_{k,n+l}\,,
\end{aligned} \tag{14}$$

and one should stress that it is an essentially Abelian extension

$$[\Theta_{n,k},\Theta_{l,p}] = 0, \qquad n,k,l,p = 0,\pm 1,\pm 2,\ldots\ . \tag{15}$$

One can easily check that Jacobi identities between all these operators are satisfied, therefore the relations (13), (14) and (15) define Abelian extension of Virasoro algebra. The equations (12) suggest its oscillator representation.

To define the physical Hilbert space we should first impose the Virasoro constraints

$$\begin{aligned}
(L_0+a)\,\Psi_{\rm phys} &= 0\\
L_n\,\Psi_{\rm phys} &= 0, \qquad n=1,2\ldots
\end{aligned} \tag{16}$$

and then our new constraint Θ. The last operator has a linear and quadratic τ dependence which in fact uniquely define the spectrum of this string theory

$$(\Pi^2-m^2) = k^2\,\tau^2 + 2\{me\cdot k + k\cdot\Pi_{\rm oscil}\}\tau + \Pi^2_{\rm oscil} + 2me\cdot\Pi_{\rm oscil} + m^2(e^2-1)\,.$$

Indeed the first operator diverges quadratically with τ and the second one linearly. Therefore, in order to have normalizable states in physical Hilbert–Fock space one should impose corresponding constraints. We are enforced to define the physical Hilbert space as

$$k^2\,\Psi_{\rm phys} = 0\,, \quad e{\cdot}k\;\Psi_{\rm phys} = 0\,, \quad k{\cdot}\alpha_n\,\Psi_{\rm phys} = 0\,, \quad k{\cdot}\tilde{\alpha}_n\,\Psi_{\rm phys} = 0\,, \qquad n > 0\,. \tag{17}$$

The first equation states that *all physical states with different spins are massless*. This is consistent with the tensionless character of the theory. The rest of the constraints take the form

$$\Theta_{n,k}\,\Psi_{\rm phys} \;=\; 0\,, \qquad\qquad n,k = 0,1,2,\ldots\,. \tag{18}$$

Thus the physical Hilbert space is defined by the equations (16), (17) and (18). In the next section we shall consider $\mathcal{N}=1$ world-sheet supersymmetric extension of the above model [35] and shall demonstrate that it actually exhibits the $\mathcal{N}=2$ world-sheet supersymmetry.

3. $\mathcal{N}=1$ world-sheet supersymmetry

In the recent article [35] the authors constructed the $\mathcal{N}=1$ supersymmetric extension of the above model using world-sheet superfields [32–34, 43,44]. Both forms of the action (4) can be extended to the supersymmetric case as follows. For the basic fields (X,Π,Ω) in (4) one should introduce the corresponding superfields

$$\begin{aligned}
\hat{X}^\mu &= X^\mu + \bar{\vartheta}\,\Psi^\mu + \frac{1}{2}\bar{\vartheta}\vartheta F^\mu\,,\\
\hat{\Pi}^\mu &= \Pi^\mu + \bar{\vartheta}\eta^\mu + \frac{1}{2}\bar{\vartheta}\vartheta\,\Phi^\mu\,,\\
\hat{\Omega} &= \omega + \bar{\vartheta}\xi + \frac{1}{2}\bar{\vartheta}\vartheta\,\Omega\,,
\end{aligned} \tag{19}$$

where ϑ is an anti-commuting variable and shall define the supersymmetric action simply exchanging basic fields (X,Π,Ω) in (4) by corresponding superfields as follows

$$S = \frac{-i}{2\pi}\int d^2\zeta d^2\theta\left\{\hat{\Pi}^\mu\bar{D}D\hat{X}^\mu - 2\hat{\Omega}(\hat{\Pi}^2 - m^2)\right\}, \tag{20}$$

where

$$D_A \;=\; \frac{\partial}{\partial\bar{\vartheta}^A} - i(\rho^a\vartheta)_A\partial_a\,, \qquad \Psi^\mu_A(\zeta) \equiv \left(\;\Psi^\mu_-(\zeta)\,\Psi^\mu_+(\zeta)\;\right), \tag{21}$$

$$\eta^\mu_A(\zeta) \;\equiv\; \begin{pmatrix}\eta^\mu_-(\zeta)\\ \eta^\mu_+(\zeta)\end{pmatrix}, \qquad\qquad \xi_A(\zeta) \equiv \begin{pmatrix}\xi_-(\zeta)\\ \xi_+(\zeta)\end{pmatrix}, \tag{22}$$

μ is a space-time vector index, $A = 1, 2$ is a two-dimensional spinor index. $\bar{\Psi}^\mu = \Psi^{+\mu}\rho^0 = \Psi^{T\mu}\rho^0$ and ρ^α are two-dimensional Dirac matrices

$$\{\rho^a, \rho^b\} = -2\eta^{ab}. \tag{23}$$

In Majorana basis the ρs are given by

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{24}$$

and $i\rho^\alpha\partial_\alpha$ is a real operator. The two-dimensional chiral fields are defined as $\rho^3\Psi^\mu_\pm = \mp\Psi^\mu_\pm$, where $\rho^3 = \rho^0\rho^1$. Substituting superfields one can get the following expression for the action [35]

$$\begin{aligned} S \ = \ & \frac{1}{\pi}\int d^2\zeta \ \{ \ \Pi^\mu\partial^2 X^\mu + i\bar{\eta^\mu}\rho^a\partial_a\Psi^\mu - F^\mu\Phi^\mu \\ & -\Omega(\Pi^2 - m^2) - \omega(2\Pi^\mu\Phi^\mu + \bar{\eta^\mu}\eta^\mu) - 2\Pi^\mu \ \bar{\eta^\mu}\xi \ \} \, . \end{aligned} \tag{25}$$

The equations of motion are:

$$\text{(I)} \qquad \begin{aligned} \Phi^\mu \ &= \ 0 \, , \\ \partial^2\Pi^\mu \ &= \ 0 \, , \\ i\rho^a\partial_a\eta^\mu \ &= \ 0 \, , \\ 2\omega\Pi^\mu + F^\mu \ &= \ 0 \, , \\ \partial^2 X^\mu - 2\Omega\Pi^\mu - 2\omega\Phi^\mu - 2\bar{\eta^\mu}\xi \ &= \ 0 \, , \\ i\rho^a\partial_a\Psi^\mu - 2\Pi^\mu\xi - 2\omega\eta^\mu \ &= \ 0 \, , \end{aligned} \tag{26}$$

and the variation over Lagrange multipliers gives constraints

$$\text{(II)} \qquad \begin{aligned} \Pi^2 - m^2 \ &= \ 0 \, , \\ 2\Pi^\mu\Phi^\mu + \bar{\eta^\mu}\eta^\mu \ &= \ 0 \, , \\ 2\Pi^\mu \ \eta^\mu \ &= \ 0 \, . \end{aligned} \tag{27}$$

The first equation represents the constrain which appears in bosonic tensionless string theory and the last equation represents its fermionic partners. As we shall see the first one is the analog of Klein–Gordon equation and the second one is the analog of the Dirac equation. In some sense they are more important relations than the motion equations (I). The SUSY transformation is:

$$
\begin{aligned}
\delta X^\mu &= \bar\varepsilon \Psi^\mu\,,\\
\delta \Psi^\mu &= -i\rho^a \partial_a X^\mu \varepsilon + F^\mu \varepsilon,\\
\delta F^\mu &= -i\bar\varepsilon \rho^a \partial_a \Psi^\mu\,,\\
\delta \Pi^\mu &= \bar\varepsilon \eta^\mu\,,\\
\delta \eta^\mu &= -i\rho^a \partial_a \Pi^\mu \varepsilon + \Phi^\mu\ \varepsilon\,,\\
\delta \Phi^\mu &= -i\bar\varepsilon \rho^a \partial_a \eta^\mu\,,\\
\delta \omega &= \bar\varepsilon \xi\,,\\
\delta \xi^\mu &= -i\rho^a \partial_a \omega \varepsilon + \Omega \varepsilon\,,\\
\delta \Omega &= -i\bar\varepsilon \rho^a \partial_a \xi\,,
\end{aligned}
\tag{28}
$$

where the anti-commuting parameter ε is a two-dimensional spinor

$$
\varepsilon \equiv \begin{pmatrix} \varepsilon_- \\ \varepsilon_+ \end{pmatrix}.
$$

The action (25), equations (26) and the constraints (27) completely define the system which exhibits the supersymmetry (28).

As we shall see bellow the action (25) possesses enhanced fermionic symmetry which elevates $\mathcal{N} = 1$ world-sheet supersymmetry up to $\mathcal{N} = 2$ world-sheet supersymmetry. It is convenient to work in light-cone coordinates. In the light-cone coordinates the action takes the form

$$
\begin{aligned}
S \ = \ \frac{2}{\pi} \int d^2\zeta \{ &-2\Pi^\mu \partial_+ \partial_- X^\mu + i\eta^\mu_+ \partial_- \psi^\mu_+ + i\eta^\mu_- \partial_+ \psi^\mu_- - \tfrac{1}{2} F^\mu \Phi^\mu \\
&- \tfrac{1}{2} \Omega (\Pi^2 - m^2) - \omega (\Pi^\mu \Phi^\mu + i\eta^\mu_+ \eta^\mu_-) - i\Pi^\mu \eta^\mu_+ \xi_- + i\Pi^\mu \eta^\mu_- \xi_+ \}
\end{aligned}
\tag{29}
$$

and equations of motion can be solved. As one can see the SUSY solution of equations (26) is:

$$
i)\, \Omega = \omega = \xi = 0
$$

and the rest of the equations (I) reduce to the following form:

$$
\text{(I)} \qquad
\begin{aligned}
\partial^2 \Pi^\mu &= 0\,, \quad i\rho^a \partial_a \eta^\mu = 0\,, \quad \partial^2 X^\mu = 0\,,\\
i\rho^a \partial_a \Psi^\mu &= 0\,, \quad F^\mu = \Phi^\mu = 0
\end{aligned}
\tag{30}
$$

and should be accompanied by the constraints

$$
\text{(II)} \qquad \Pi^2 - m^2 = 0\,, \quad \bar{\eta^\mu} \eta^\mu = 0\,, \quad \Pi^\mu\ \eta^\mu = 0\,.
\tag{31}
$$

In the light-cone coordinates these equations are easy to solve since they take the from

$$\begin{aligned}
\partial_+\partial_- \Pi^\mu &= 0\,, \quad \partial_\pm \eta^\mu_\mp = 0\,, \quad \partial_+\partial_- X^\mu = 0\,, \quad \partial_\pm \psi^\mu_\mp = 0,\\
\Pi^2 - m^2 &= 0\,, \qquad\qquad \eta^\mu_+\eta^\mu_- - \eta^\mu_-\eta^\mu_+ = 0\,, \quad \Pi^\mu\, \eta^\mu_\pm = 0\,.
\end{aligned} \tag{32}$$

The mode expansion of X field with the appropriate boundary conditions for closed strings was given above (8). The solution of fermionic fields can be represented in the form of mode expansion as well

$$\begin{aligned}
\eta^\mu_+ &= \sum c^\mu_n \mathrm{e}^{-in\zeta^+}, \qquad \psi^\mu_+ = \sum d^\mu_n \mathrm{e}^{-in\zeta^+},\\
\eta^\mu_- &= \sum \tilde{c}^\mu_n \mathrm{e}^{-in\zeta^-}, \qquad \psi^\mu_- = \sum \tilde{d}^\mu_n \mathrm{e}^{-in\zeta^-}
\end{aligned} \tag{33}$$

with the basic anti-commutators

$$\{\eta^\mu_\pm(\zeta), \psi^\nu_\pm(\zeta^{'})\} = 2\pi \eta^{\mu\nu}\delta(\zeta - \zeta^{'})\,, \tag{34}$$

and all others equal to zero: $\{\eta^\mu_\pm(\zeta), \eta^\nu_\pm(\zeta^{'})\} = 0$, $\{\psi^\mu_\pm(\zeta), \psi^\nu_\pm(\zeta^{'})\} = 0$. Substituting the mode expansion into the anti-commutators requires the following relations between modes

$$\{c^\mu_n, d^\nu_k\} = \eta^{\mu\nu}\delta_{n+k,0}\,, \quad \{c^\mu_n, c^\nu_k\} = 0\,, \quad \{d^\mu_n, d^\nu_k\} = 0\,, \tag{35}$$

and similar ones for $\tilde{c}^\mu_n$ and $\tilde{d}^\mu_n$.

4. Enhanced $\mathcal{N} = 2$ world-sheet supersymmetry

Let us now consider conserved currents: energy momentum tensor and supercurrent

$$\begin{aligned}
T_{ab} &= \partial_{\{a}\Pi^\mu\, \partial_{b\}} X^\mu + i\bar{\eta}^\mu \rho_{\{a}\partial_{b\}}\, \Psi^\mu - \text{trace}\,,\\
J_a &= \frac{1}{2}\rho^b\rho_a \Psi^\mu \partial_b \Pi^\mu + \frac{1}{2}\rho^b\rho_a \eta^\mu \partial_b X^\mu
\end{aligned} \tag{36}$$

or in the light-cone coordinates

$$\begin{aligned}
T_{++} &= 2\partial_+\Pi^\mu\partial_+X^\mu + i\eta^\mu_+\partial_+\,\Psi^\mu_+ - \frac{i}{2}\partial_+(\eta^\mu_+\,\Psi^\mu_+)\,,\\
T_{--} &= 2\partial_-\Pi^\mu\partial_-X^\mu + i\eta^\mu_-\partial_-\,\Psi^\mu_- - \frac{i}{2}\partial_-(\eta^\mu_-\,\Psi^\mu_-)\,,\\
J_+ &= 2\partial_+\Pi^\mu\ \Psi^\mu_+ + 2\eta^\mu_+\partial_+X^\mu\,,\\
J_- &= 2\partial_-\Pi^\mu\ \Psi^\mu_- + 2\eta^\mu_-\partial_-X^\mu\,.
\end{aligned} \tag{37}$$

Substituting solution (8) into the last formulas one can get

$$
\begin{aligned}
T_{++} &= \frac{1}{2}\,\partial_+ \Pi^\mu_{\rm L}\,\partial_+ X^\mu_{\rm L} + \frac{i}{2}\eta^\mu_+ \partial_+\, \Psi^\mu_+ - \frac{i}{2}\partial_+ \eta^\mu_+\;\; \Psi^\mu_+\,,\\
T_{--} &= \frac{1}{2}\,\partial_- \Pi^\mu_{\rm R}\,\partial_- X^\mu_{\rm R} + \frac{i}{2}\eta^\mu_- \partial_-\, \Psi^\mu_- - \frac{i}{2}\partial_- \eta^\mu_-\;\; \Psi^\mu_-\,,\\
J_+ &= \partial_+ \Pi^\mu_{\rm L}\, \Psi^\mu_+ + \eta^\mu_+ \partial_+ X^\mu_{\rm L}\,,\\
J_- &= \partial_- \Pi^\mu_{\rm R}\, \Psi^\mu_- + \eta^\mu_- \partial_- X^\mu_{\rm R}\,.
\end{aligned}
\tag{38}
$$

The mode expansion of these currents is equal to

$$
\begin{aligned}
L_n &= \langle {\rm e}^{in\zeta^+} T_{++}\rangle = \sum_l :\alpha_{n-l}\cdot\beta_l: + \sum_l :(l-\frac{n}{2})c_{n-l}\cdot d_l:\,,\\
F_n &= \langle {\rm e}^{in\zeta^+} J_{+}\rangle = \sum_l \alpha_{n-l}\cdot d_l + \sum_l \beta_{n-l}\cdot c_l\,.
\end{aligned}
\tag{39}
$$

The standard computation of quantum commutation relations between these currents gives

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{D}{4} m^3 \delta_{n+m}\,,\\
[L_n, F_m] &= \left(\frac{n}{2} - m\right) F_{n+m}\,,\\
\{F_n, F_m\} &= 2L_{n+m} + Dn^2\delta_{n+m}\,.
\end{aligned}
\tag{40}
$$

This is a standard $\mathcal{N}=1$ superalgebra with the central charge twice larger than in the standard superstring theory. It is also clear from the expression for the full supercurrent J_a that its two separate pieces $G^1_a = \frac{1}{2}\rho^b\rho_a\eta^\mu\partial_b X^\mu$ and $G^2_a = \frac{1}{2}\rho^b\rho_a\Psi^\mu\partial_b\Pi^\mu$ also represent conserved currents and therefore are pointing to the fact that there should exist a higher symmetry group. In the light-cone coordinates these currents have the form

$$
\begin{aligned}
G^1_+ &= 2\eta^\mu_+\;\partial_+ X^\mu\,, \qquad & G^2_+ &= 2\partial_+\Pi^\mu\;\;\Psi^\mu_+\,,\\
G^1_- &= 2\eta^\mu_-\;\partial_- X^\mu\,, \qquad & G^2_- &= 2\partial_-\Pi^\mu\;\;\Psi^\mu_-\,.
\end{aligned}
\tag{41}
$$

Mode expansion of these currents is defined as:

$$
\begin{aligned}
G^1_n &= \langle {\rm e}^{in\zeta^+} G^1_+\rangle = \sum_l \beta_{n-l}\cdot c_l\,,\\
G^2_n &= \langle {\rm e}^{in\zeta^+} G^2_+\rangle = \sum_l \alpha_{n-l}\cdot d_l\,,
\end{aligned}
\tag{42}
$$

therefore

$$
F_n = G^1_n + G^2_n\,.
$$

These conserved currents form the following algebra

$$\begin{aligned}
[L_n, G^1_m] &= \left(\frac{n}{2} - m\right) G^1_{n+m}\,, && (J = 3/2)\,,\\
[L_n, G^2_m] &= \left(\frac{n}{2} - m\right) G^2_{n+m}\,, && (J = 3/2)\,,\\
\{G^1_n, G^1_m\} &= 0\,,\\
\{G^2_n, G^2_m\} &= 0\,.
\end{aligned} \tag{43}$$

Here J denotes the conformal spin of the corresponding operators. The anti-commutator $\{G^1_n, G^2_m\}$ cannot be computed in closed form unless we introduce additional current

$$T_a = \tfrac{1}{2}\bar{\eta}^\mu \rho_a \Psi^\mu \tag{44}$$

which appears to be also conserved as it is easy to check using equations of motion. This conserved current is connected with the U(1) invariance of the action which rotates fermionic fields. Its components are $T_+ = -\eta^\mu_+ \Psi^\mu_+$, $T_- = -\eta^\mu_- \Psi^\mu_-$, $\partial_- T_+ = \partial_+ T_- = 0$, and mode expansion is

$$T_n = \langle \mathrm{e}^{in\zeta^+} T_+ \rangle = -\sum_l : c_{n-l} \cdot d_l : \,. \tag{45}$$

Then we can compute the anti-commutator:

$$\{G^1_n, G^2_m\} = L_{n+m} + \tfrac{1}{2}(n-m) T_{n+m} + \frac{D}{2} n^2 \delta_{n+m,0}$$

and the rest of the algebra will take the form:

$$\begin{aligned}
[L_n, T_m] &= -m T_{n+m}\,, \qquad (J = 1)\,,\\
[T_n, T_m] &= D n \delta_{n+m,0}\,,\\
[T_n, G^1_m] &= -G^1_{n+m}\,,\\
[T_n, G^2_m] &= +G^2_{n+m}\,.
\end{aligned} \tag{46}$$

It is clear now that this is a well known $\mathcal{N} = 2$ superconformal algebra [36] and that initially implemented $\mathcal{N} = 1$ SUSY transformation has been naturally enhanced to $\mathcal{N} = 2$ world-sheet supersymmetry. This symmetry can also be seen if one introduces the $\mathcal{N} = 2$ superfield as follows:

$$\begin{aligned}
\hat{X}^\mu(\zeta, \vartheta_1, \vartheta_2) &= X^\mu + \vartheta_1 \Psi^\mu + \vartheta_2 \eta^\mu + \vartheta_2 \vartheta_1 \Pi^\mu\\
&= X^\mu + \vartheta_1 \Psi^\mu + \vartheta_2 \left(\eta^\mu + \vartheta_1 \Pi^\mu\right).
\end{aligned} \tag{47}$$

The important point is that there is no natural extensions of the superfield $\hat{\Omega}$ to $\mathcal{N} = 2$ superfield, simply because the constrain $\Pi^2 - m^2 = 0$ breaks the symmetry between X and Π fields. This can be seen as a sign that actually the whole system together with constraints breaks the $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. This observation makes the computation of critical dimension more subtle here. In particular it is not obvious at all that it should be two, as it is the case for $\mathcal{N} = 2$ strings.

Let us now consider the new superconstraints (II) (32) which appear in our case: $\Delta = \Pi^\mu \ \eta^\mu_\pm$

$$\Delta^\pm_{n,l} = \langle \mathrm{e}^{in\zeta^+ + il\zeta^-} : \Pi^\mu \ \eta^\mu_\pm :\rangle \tag{48}$$

or in terms of oscillators

$$\begin{aligned}
\Delta^+_{0,0} &= me \cdot c_0 + \sum_{n \neq 0} \alpha_{-n} \ c_n \,, \qquad \Delta^-_{0,0} = me \cdot \tilde{c}_0 + \sum_{n \neq 0} \tilde{\alpha}_{-n} \ \tilde{c}_n \,, \\
\Delta^+_{n,0} &= i \sum_{l \neq 0,n} \alpha_{n-l} \cdot c_l \,, \qquad \Delta^-_{0,n} = i \sum_{l \neq 0,n} \tilde{\alpha}_{n-l} \cdot \tilde{c}_l \,, \qquad n = \pm 1, \pm 2, \ldots \,, \\
\Delta^+_{n,l} &= \alpha_n \cdot \tilde{c}_l \,, \qquad \Delta^-_{n,l} = \tilde{\alpha}_n \cdot c_l \,, \qquad n, l = \pm 1, \pm 2, \ldots \,.
\end{aligned} \tag{49}$$

The important fact which uniquely defines the spectrum of this superstring theory is again the τ dependence of the operators Π^2 and $\Pi \cdot \eta_\pm$

$$\begin{aligned}
(\Pi^2 - m^2) &= k^2 \ \tau^2 + 2\{me \cdot k + k \cdot \Pi_{\mathrm{oscil}}\}\tau \\
&\quad + m^2(e^2 - 1) + 2me \cdot \Pi_{\mathrm{oscil}} + \Pi^2_{\mathrm{oscil}} \,, \\
\Pi \cdot \eta_+ &= (c_0 \cdot k + k \cdot \eta_{+\,\mathrm{oscil}})\tau + me \cdot c_0 + me \cdot \eta_{+\,\mathrm{oscil}} + \Pi_{\mathrm{oscil}} \cdot \eta_{+\,\mathrm{oscil}} \,, \\
\Pi \cdot \eta_- &= (\tilde{c}_0 \cdot k + k \cdot \eta_{-\,\mathrm{oscil}})\tau + me \cdot \tilde{c}_0 + me \cdot \eta_{-\,\mathrm{oscil}} + \Pi_{\mathrm{oscil}} \cdot \eta_{-\,\mathrm{oscil}} \,.
\end{aligned} \tag{50}$$

The first operator diverges quadratically with τ and the second one linearly in bosonic sector and we have linear divergency of first operators in fermionic sector. Therefore in order to have normalizable states in physical Hilbert space one should impose corresponding constraints. We are enforced to define the physical Hilbert space as

$$\begin{aligned}
k^2 \Psi_{\mathrm{phys}} &= 0 \,, \quad c_0 \cdot k \Psi_{\mathrm{phys}} = 0 \,, \quad \tilde{c}_0 \cdot k \ \Psi_{\mathrm{phys}} = 0 \,, \\
k \cdot \alpha_n \Psi_{\mathrm{phys}} &= 0 \,, \quad k \cdot \tilde{\alpha}_n \Psi_{\mathrm{phys}} = 0 \,, \quad k \cdot c_n \Psi_{\mathrm{phys}} = 0 \,, \quad k \cdot \tilde{c}_n \Psi_{\mathrm{phys}} = 0 \,, \\
e \cdot k \Psi_{\mathrm{phys}} &= 0 \,, \qquad n > 0 \,.
\end{aligned} \tag{51}$$

All these constraints can naturally be grouped into three systems of equations. The first three equations are nothing else but massless Klein–Gordon and Dirac equations and uniquely define the spectrum of the theory. We conclude that *all physical states with integer and half integer spins are massless.*

This is consistent with tensionless character of the theory. The second system of equations imposes important condition of transversality on fermion and boson oscillators. Finally the last equation suggests that the vector e_μ should be interpreted as polarization vector transverse to the momentum vector k_μ.

We should impose the constraints of $\mathcal{N}=2$ superconformal algebra

$$\begin{aligned}
(L_0+a)\,\Psi_{\text{phys}} &= 0\,, \\
L_n\,\Psi_{\text{phys}} &= 0\,, \qquad n=1,2,\dots\,, \\
T_0\,\Psi_{\text{phys}} &= 0\,, \\
T_n\,\Psi_{\text{phys}} &= 0\,, \qquad n=1,2,\dots\,, \\
G^1_n\,\Psi_{\text{phys}} &= 0\,, \qquad n=1,2,\dots\,, \\
G^2_n\,\Psi_{\text{phys}} &= 0\,, \qquad n=1,2,\dots\,,
\end{aligned} \tag{52}$$

together with the additional constraints $\Theta_{k,l}$ (18) and fermionic constraints (49)

$$\Delta^{\pm}_{n,l}\,\Psi_{\text{phys}} = 0\,, \qquad n,l=0,1,2,\dots\,. \tag{53}$$

One should study in great details this Hilbert space in order to learn more about content of the theory and to prove the absence of the negative norm states. We cannot also say anything certain about critical dimension of the model because we have additional symmetries and the corresponding constraints, the influence of which on the calculation of the critical dimension at the moment is not quite well understood.

In the rest of the article we shall consider a close topological model which can be constructed by twisting [24–27]. Indeed the redefinition of the energy momentum tensor by the total derivative of the U(1) current, leads to the topological theory. By this twisting operation the above $\mathcal{N}=2$ supersymmetry transforms into BRST symmetry as in [24, 27].

5. Twisted topological strings

We shall obtain the topological version of the above $\mathcal{N}=2$ theory by the redefinition of the energy momentum tensor T by $\hat{T}$ as follows:

$$\hat{T}_{ab} = T_{ab} - \frac{i}{2}\partial_a J_b\,,$$

where J_b is the U(1) current (44). In light-cone components we have

$$\begin{aligned}
\tilde{T}_{++} &= 2\partial_+\Pi^\mu\partial_+X^\mu + i\eta^\mu_+\partial_+\Psi^\mu_+\,, &\quad \tilde{T}_{--} &= 2\partial_-\Pi^\mu\partial_-X^\mu + i\eta^\mu_-\partial_-\Psi^\mu_-\,, \\
F_+ &= 2\partial_+\Pi^\mu\Psi^\mu_+ + 2\eta^\mu_+\partial_+X^\mu\,, & F_- &= 2\partial_-\Pi^\mu\Psi^\mu_- + 2\eta^\mu_-\partial_-X^\mu\,, \\
J_+ &= -\eta^\mu_+\Psi^\mu_+\,, & J_- &= -\eta^\mu_-\Psi^\mu_-\,.
\end{aligned} \tag{54}$$

Substituting solutions (8) into the last formulas one can get

$$\begin{aligned}
\tilde{T}_{++} &= \tfrac{1}{2}\partial_+ \Pi^\mu_{\rm L}\partial_+ X^\mu_{\rm L} + i\eta^\mu_+\partial_+\Psi^\mu_+ \,,\\
\tilde{T}_{--} &= \tfrac{1}{2}\partial_- \Pi^\mu_{\rm R}\partial_- X^\mu_{\rm R} + i\eta^\mu_-\partial_-\Psi^\mu_- \,,\\
F^1_+ &= \partial_+\Pi^\mu_{\rm L}\Psi^\mu_+ + \eta^\mu_+\partial_+ X^\mu_{\rm L} \,,\\
F^1_- &= \partial_-\Pi^\mu_{\rm R}\Psi^\mu_- + \eta^\mu_-\partial_- X^\mu_{\rm R} \,,\\
J_+ &= -\eta^\mu_+\Psi^\mu_+ \,,\\
J_- &= -\eta^\mu_-\Psi^\mu_- \,.
\end{aligned} \tag{55}$$

Mode expansion of these currents is defined as:

$$\begin{aligned}
L_n &= \langle {\rm e}^{in\zeta^+}\tilde{T}_{++}\rangle = \sum_l :\alpha_{n-l}\cdot\beta_l: + \sum_l :l\ c_{n-l}\cdot d_l:,\\
F^1_n &= \langle {\rm e}^{in\zeta^+} F_+\rangle = \sum_l \alpha_{n-l}\cdot d_l + \sum_l \beta_{n-l}\cdot c_l \,,\\
F^2_n &= \langle {\rm e}^{in\zeta^+} F^2_+\rangle = \sum_l \alpha_{n-l}\cdot d_l - \sum_l \beta_{n-l}\cdot c_l \,,\\
J_n &= \langle {\rm e}^{in\zeta^+} J_+\rangle = -\sum_l :c_{n-l}\cdot d_l:,
\end{aligned} \tag{56}$$

where we have introduced a new operator F^2_+ which is equal to the following expression

$$F^2_+ = \partial_+\Pi^\mu_{\rm L}\Psi^\mu_+ - \eta^\mu_+\partial_+ X^\mu_{\rm L} \,. \tag{57}$$

The necessity of introducing this operator comes from the fact that again when we compute the algebra between operators T, F^1, J the algebra is closed only if we introduce this new operator. For these four operators the algebra is closed

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} \,,\\
[L_n, F^1_m] &= \left(\frac{n}{2}-m\right)F^1_{n+m} - \frac{n}{2}F^2_{n+m} \,,\\
[L_n, F^2_m] &= \left(\frac{n}{2}-m\right)F^2_{n+m} - \frac{n}{2}F^1_{n+m} \,,\\
[L_n, J_m] &= -mJ_{n+m} - \frac{D}{2}n(n+1)\delta_{n+m,0} \,,\\
[J_n, J_m] &= Dn\delta_{n+m,0} \,,\\
[J_n, F^1_m] &= F^2_{n+m} \,,\\
[J_n, F^2_m] &= F^1_{n+m} \,,
\end{aligned}$$

$$
\begin{aligned}
\{F^1_n, F^1_m\} &= 2L_{n+m} + (n+m)J_{n+m} + Dn^2\delta_{n+m,0}\,, \\
\{F^1_n, F^2_m\} &= -(n-m)J_{n+m} + D\ n\ \delta_{n+m,0}\,, \\
\{F^2_n, F^2_m\} &= -2L_{n+m} - (n+m)J_{n+m} - Dn^2\delta_{n+m,0}\,.
\end{aligned} \tag{58}
$$

There is no defined conformal dimensions for the fermion operators F^1 and F^2, they also do not have defined charges with respect to the U(1) group. As it is easy to see the linear combination of these operators do have defined charges. Indeed we should introduce the linear combination of supercurrents as we did in the previous model

$$
2G_n = F^1_n - F^2_n\,, \qquad 2Q_n = F^1_n + F^2_n\,, \tag{59}
$$

in order to have diagonal form of supercurrent with respect to the conformal operator L_n. In coordinate space they look as (41). For these conserved currents the algebra takes the form

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m}\,, && (J=2)\,, \\
[L_n, G_m] &= (n-m)G_{n+m}\,, && (J=2)\,, \\
[L_n, Q_m] &= -mQ_{n+m}\,, && (J=1)\,, \\
[L_n, J_m] &= -mJ_{n+m} - \frac{D}{2}n(n+1)\delta_{n+m,0}\,, && (J=1)\,, \\
[J_n, J_m] &= Dm\delta_{n+m,0}\,, \\
[J_n, G_m] &= -G_{n+m}\,, \\
[J_n, Q_m] &= +Q_{n+m}\,, \\
\{G_n, G_m\} &= 0\,, \\
\{Q_n, Q_m\} &= 0\,, \\
\{G_n, Q_m\} &= L_{n+m} + mJ_{n+m} + \frac{D}{2}n(n+1)\delta_{n+m,0}
\end{aligned} \tag{60}
$$

which is well known in topological conformal field theory [27]. One can see that we have here zero central charge and two nilpotent fermion operators G and Q which form the $N=2$ world-sheet supersymmetry. Two conserved supercurrents which appear above should have come from the explicit fermion symmetry of the theory. As we shall see in a moment these symmetries can be justified. One can check that the system is invariant under fermion transformation laws δ and $\bar{\delta}$ defined as follows [35]:

$$
\begin{array}{lll}
\delta X^\mu = 0\,, & \bar\delta X^\mu = 0\,, & \delta\omega = i\varepsilon_+\xi_-\,,\\
\delta\Psi_-^\mu = -2\varepsilon_+\partial_- X^\mu\,, & \bar\delta\Psi_-^\mu = 0\,, & \delta\xi_- = 0\,,\\
\delta\Psi_+^\mu = 0\,, & \bar\delta\Psi_+^\mu = -2\varepsilon_-\partial_+ X^\mu\,, & \delta\xi_+ = -\varepsilon_+\Omega\,,\\
\delta F^\mu = -2i\varepsilon_+\partial_-\Psi_+^\mu\,, & \bar\delta F^\mu = 2i\varepsilon_-\partial_+\Psi_-^\mu\,, & \delta\Omega = 0\,,\\
\delta\Pi^\mu = i\varepsilon_+\eta_-^\mu\,, & \bar\delta\Pi^\mu = i\varepsilon_-\eta_+^\mu\,, & \bar\delta\omega = i\varepsilon_-\xi_+\,,\\
\delta\eta_-^\mu = 0\,, & \bar\delta\eta_-^\mu = \varepsilon_-\Phi^\mu\,, & \bar\delta\xi_- = \varepsilon_-\Omega\,,\\
\delta\eta_+^\mu = -\varepsilon_+\Phi^\mu\,, & \bar\delta\eta_+^\mu = 0\,, & \bar\delta\xi_+ = 0\,,\\
\delta\Phi^\mu = 0\,, & \bar\delta\Phi^\mu = 0\,, & \bar\delta\Omega = 0\,.
\end{array}
\tag{61}
$$

The algebra of these fermionic symmetries is nilpotent and is very similar to BRST transformations

$$
\delta_\varepsilon\delta_{\acute\varepsilon}(H) = \bar\delta_\varepsilon\bar\delta_{\acute\varepsilon}(H) = 0\,, \qquad (\delta_\varepsilon\bar\delta_{\acute\varepsilon} - \bar\delta_{\acute\varepsilon}\delta_\varepsilon)(H) = 0\,, \tag{62}
$$

where H is any of the fields $(X, \Psi, F, \Pi, \eta, \Phi, \omega, \xi, \Omega)$.

From (62) it follows that the action is invariant under fermionic symmetries (61). We can compute the current corresponding to this fermion symmetry. The variation of the action is

$$
\begin{aligned}
\delta S = \frac{2}{\pi}\int d^2\zeta \Big\{ &-\ 2i\varepsilon_+\eta_-^\mu\partial_+\partial_- X^\mu - i\varepsilon_+\Phi^\mu\partial_-\Psi_+^\mu + i\eta_-^\mu\partial_+(-2\varepsilon_+\partial_- X^\mu)\\
&-\ \frac{1}{2}(-2i\varepsilon_+\partial_-\Psi_+^\mu)\Phi^\mu\Big\} = -\frac{2i}{\pi}\int d^2\zeta\ G_-\ \partial_+(\varepsilon_+)
\end{aligned}
\tag{63}
$$

and supercurrent $G_- = 2\eta_-^\mu\partial_- X^\mu$ coincides with the one which appeared in the previous section. The important fact now is that the Lagrangian is a variation of the super-potentials W and $\bar W$

$$
W = \Pi^\mu\ \partial_+\Psi_-^\mu + \frac{1}{2}\eta_+^\mu\ F^\mu, \qquad \bar W = \Pi^\mu\ \partial_-\Psi_+^\mu - \frac{1}{2}\eta_-^\mu\ F^\mu, \tag{64}
$$

so that

$$
\delta W = \varepsilon_+\mathcal{L}\,, \qquad\qquad \bar\delta\ \bar W = \varepsilon_-\mathcal{L}\,. \tag{65}
$$

It is also true that there exists a potential V such that

$$
\delta V = -i\varepsilon_+\bar W\,, \qquad \bar\delta V = i\varepsilon_- W\,, \qquad V = \frac{1}{2}\Pi^\mu F^\mu, \tag{66}
$$

thus

$$
i\delta\bar\delta V = \varepsilon_+\varepsilon_-\mathcal{L}\,, \qquad i\bar\delta\delta V = \varepsilon_+\varepsilon_-\mathcal{L}\,.
$$

The constrains (II) can also be represented by the $\bar{\delta}\delta$ transformation and therefore the full Lagrangian in (29) can be represented as

$$2\varepsilon_+\varepsilon_-\mathcal{L}_{\rm tot} = i\bar{\delta}\delta\left(\varPi^\mu F^\mu + \omega(\varPi^2 - m^2)\right). \tag{67}$$

Thus the action can be represented as BRST commutator $\mathcal{L} = \{G_+, W\} = \{G_-, \bar{W}\}$. The above fermion symmetry allows to express some important operators as variation of others. In particular the energy momentum tensor is a variation of second supercurrent Q, but only up to the total derivative of U(1) current J_a

$$\begin{aligned} \bar{\delta}\ Q_+ &= -2\varepsilon_- T_{++} + 2i\varepsilon_-\partial_+ J_+\,, \\ \delta\ Q_- &= -2\varepsilon_+ T_{--} + 2i\varepsilon_+\partial_- J_-\,. \end{aligned} \tag{68}$$

Instead, the supercurrent G introduced above is total variation of the vector $v_a = \varPi^\mu \partial_a X^\mu$

$$\begin{aligned} \bar{\delta}\ v_+ &= i\varepsilon_- G_+, & v_+ &= 2\varPi^\mu\partial_+ X^\mu\,, \\ \delta\ v_- &= i\varepsilon_+ G_-, & v_- &= 2\varPi^\mu\partial_- X^\mu\,. \end{aligned} \tag{69}$$

Let us consider the second fermion symmetry of the action

$$\begin{array}{lll}
\delta X^\mu = -i\varepsilon_-\varPsi^\mu_+\,, & \bar{\delta}X^\mu = -i\varepsilon_+\varPsi^\mu_-\,, & \delta\omega = 0\,, \\
\delta\varPsi^\mu_- = -\varepsilon_- F^\mu\,, & \bar{\delta}\varPsi^\mu_- = 0\,, & \delta\xi_- = -\varepsilon_-\omega\,, \\
\delta\varPsi^\mu_+ = 0\,, & \bar{\delta}\varPsi^\mu_+ = \varepsilon_+ F^\mu\,, & \delta\xi_+ = 0\,, \\
\delta F^\mu = 0\,, & \bar{\delta}F^\mu = 0\,, & \delta\varOmega = i\varepsilon_-\xi_+\,, \\
\delta\varPi^\mu = 0\,, & \bar{\delta}\varPi^\mu = 0\,, & \bar{\delta}\omega = 0\,, \\
\delta\eta^\mu_- = 0\,, & \bar{\delta}\eta^\mu_- = 2\varepsilon_+\partial_-\varPi^\mu\,, & \bar{\delta}\xi_- = 0\,, \\
\delta\eta^\mu_+ = 2\varepsilon_-\partial_+\varPi^\mu\,, & \bar{\delta}\eta^\mu_+ = 0\,, & \bar{\delta}\xi_+ = \varepsilon_+\omega\,, \\
\delta\varPhi^\mu = -2i\varepsilon_-\partial_+\eta^\mu_-\,, & \bar{\delta}\varPhi^\mu = 2i\varepsilon_+\partial_-\eta^\mu_+\,, & \bar{\delta}\varOmega = i\varepsilon_+\xi_-\,.
\end{array} \tag{70}$$

The algebra of these fermionic symmetries is nilpotent and is very similar to BRST transformations

$$\delta_\varepsilon\delta_{\acute{\varepsilon}}(H) = \bar{\delta}_\varepsilon\bar{\delta}_{\acute{\varepsilon}}(H) = 0\,, \qquad (\delta_\varepsilon\bar{\delta}_{\acute{\varepsilon}} - \bar{\delta}_{\acute{\varepsilon}}\delta_\varepsilon)(H) = 0\,, \tag{71}$$

where H is any of the fields $(X, \varPsi, F, \varPi, \eta, \varPhi, \omega, \xi, \varOmega)$. Let us compute the corresponding current. The variation of the action is

$$\begin{aligned} \delta S &= \frac{2}{\pi}\int d^2\zeta\Big\{ -2\varPi^\mu\partial_+\partial_-(-i\varepsilon_-\varPsi^\mu_+) + 2i\varepsilon_-\partial_+\varPi^\mu\partial_-\varPsi^\mu_+ \\ &\quad + i\eta^\mu_-\partial_+(-\varepsilon_- F^\mu) - \frac{1}{2}F^\mu(-2i\varepsilon_-\partial_+\eta^\mu_-)\Big\} = -\frac{2i}{\pi}\int d^2\zeta Q_+(\partial_-\varepsilon_-) \end{aligned} \tag{72}$$

and $Q_+ = 2\partial_+ \Pi^\mu \Psi^\mu_+$ also appeared in the previous section. The important fact now is that the energy momentum tensor is BRST commutator with respect to the second fermion symmetry

$$\begin{aligned} \delta G_+ &= \delta(2\eta^\mu_+ \partial_+ X^\mu) = 2\varepsilon_- T_{++} \,, \\ \delta G_- &= \delta\ (2\eta^\mu_- \partial_- X^\mu) = 2\varepsilon_+ T_{--} \,, \end{aligned} \tag{73}$$

and supercurrent Q is a variation of $U(1)$ current J_a

$$\begin{aligned} \delta J_+ &= \delta(-\eta^\mu_+ \Psi^\mu_+) = -\varepsilon_- Q_+ \,, \\ \delta J_- &= \delta(-\eta^\mu_- \Psi^\mu_-) = -\varepsilon_+ Q_- \,. \end{aligned} \tag{74}$$

Thus the energy momentum tensor is a BRST commutator $T_{++} = \{Q_+, G_+\}$ and its central charge vanishes, the model transforms into topological theory.

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REFERENCES

[1] J.M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998); *Int. J. Theor. Phys.* **38**, 1113 (1999).

[2] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *Phys. Lett.* **B428**, 105 (1998).

[3] E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).

[4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, Y. Oz, *Phys. Rep.* **323**, 183 (2000).

[5] A.M. Polyakov, *Int. J. Mod. Phys.* **A17S1**, 119 (2002).

[6] D. Berenstein, J.M. Maldacena, H. Nastase, *J. High Energy Phys.* **0204**, 013 (2002).

[7] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *Nucl. Phys.* **B636**, 99 (2002).

[8] P. Haggi-Mani, B. Sundborg, *J. High Energy Phys.* **0004**, 031 (2000).

[9] B. Sundborg, *Nucl. Phys. Proc. Suppl.* **102**, 113 (2001).

[10] E. Witten, Talk at the John Schwarz 60-th Birthday Symposium, http://theory.caltech.edu/jhs60/witten/1.html

[11] D. Gross, *Phys. Rev. Lett.* **60**, 1229 (1988).

[12] G.W. Moore, `hep-th/9310026`.

[13] J. Isberg, U. Lindstrom, B. Sundborg, G. Theodoridis, *Nucl. Phys.* **B411**, 122 (1994).

[14] S. Ferrara, C. Fronsdal, *Phys. Lett.* **B433**, 19 (1998).

[15] A.K. Bengtsson, I. Bengtsson, L. Brink, *Nucl. Phys.* **B227**, 41 (1983).

[16] A.K. Bengtsson, I. Bengtsson, L. Brink, *Nucl. Phys.* **B227**, 31 (1983).

[17] M.A. Vasiliev, `hep-th/9910096`.

[18] G.K. Savvidy, *Phys. Lett.* **B438**, 69 (1998).

[19] E. Sezgin, P. Sundell, *J. High Energy Phys.* **0109**, 036 (2001).

[20] A. Mikhailov, `hep-th/0201019`.

[21] I.R. Klebanov, A.M. Polyakov, *Phys. Lett.* **B550**, 213 (2002).

[22] D. Francia, A. Sagnotti, *Phys. Lett.* **B543**, 303 (2002).

[23] G.K.Savvidy, K.G.Savvidy, *Mod. Phys. Lett.* **A8**, 2963 (1993); G.K. Savvidy, *J.High Energy Phys.* **0009**, 044 (2000); R.V. Ambartzumian *et al.*, *Phys. Lett.* **B275**, 99 (1992); G.K. Savvidy, K.G. Savvidy, *Int. J. Mod. Phys.* **A8**, 3993 (1993); R. Manvelian, G. Savvidy, *Phys. Lett.* **B533**, 138 (2002).

[24] E. Witten, *Comm. Math. Phys.* **117**, 353 (1988); E. Witten, *Comm. Math. Phys.* **118**, 411 (1988).

[25] J.M. Labastida, M. Pernici, E. Witten, *Nucl. Phys.* **B310**, 611 (1988).

[26] D. Montano, J. Sonnenschein, *Nucl. Phys.* **B313**, 258 (1989).

[27] T. Eguchi, S.-K. Yang, *Mod. Phys. Lett.* **A5**, 1693 (1990).

[28] A.M. Polyakov, *Nucl. Phys.* **B268**, 406 (1986); H. Kleinert, *Phys. Lett.* **174B**, 335 (1986); T. Curtright, P. van Nieuwenhuizen, *Nucl. Phys.* **B294**, 125 (1987); U. Lindström, M. Roček, P. van Nieuwenhuizen, *Phys. Lett.* **B199**, 219 (1987); U. Lindström, M. Roček, *Phys. Lett.* **B201**, 63 (1988).

[29] G.K. Savvidy, *Phys. Lett.* **B552**, 72 (2003).

[30] H.J. De Vega, A. Nicolaidis, *Phys. Lett.* **B295**, 214 (1992).

[31] A. Clark, A. Karch, P. Kovtun, D. Yamada, `hep-th/0304107`.

[32] P. Ramond, *Phys. Rev.* **D3**, 2415 (1971).

[33] A. Neveu, J. Schwarz, *Nucl. Phys.* **B31**, 86 (1971).

[34] J.L. Gervais, B. Sakita, *Nucl. Phys.* **B34**, 632 (1971); Y. Iwasaki, K. Kikkawa, *Phys. Rev.* **D8**, 440 (1973); B. Zumino, in *Renormalisation and Invariance in QFT*, ed. E. Caianiello, Plenum Press, 1974, pp. 367–381; L. Brink, P. Di Vecchia, P. Howe, *Phys. Lett.* **65B**, 471 (1976); S. Deser, B. Zumino, *Phys. Lett.* **65B**, 369 (1976); A.M. Polyakov, *Phys. Lett.* **103B**, 207 (1981); *Phys. Lett.* **103B**, 211 (1981).

[35] A. Nichols, R. Manvelyan, G.K. Savvidy, `hep-th/0212324`.

[36] M. Ademollo, A. D'Adda, R. D'Auria, E. Napolitano, P. Di Vecchia, F. Gliozzi, S. Sciuto, *Nucl. Phys.* **B77**, 189 (1974); *Nucl. Phys.* **B114**, 297 (1976); *Phys. Lett.* **62B**, 105 (1976); L. Brink, J.H. Schwarz, *Nucl. Phys.* **B121**, 285 (1977).

[37] A. D'Adda, F. Lizzi, *Phys. Lett.* **191B**, 85 (1987).

[38] H. Ooguri, C. Vafa, *Mod. Phys. Lett.* **A5**, 1389 (1990).

[39] H. Ooguri, C. Vafa, *Nucl. Phys.* **B361**, 469 (1991).

[40] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, *Commun. Math. Phys.* **165**, 311 (1994).

[41] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, *Nucl. Phys.* **B405**, 279 (1993).

[42] H. Ooguri, C. Vafa, *Nucl. Phys.* **B451**, 121 (1995).

[43] M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theory* Vol. 1,2, Cambridge University Press, Cambridge 1997.

[44] J. Polchinski, *String Theory* Vol.1,2, Cambridge University Press, Cambridge 1998.

[45] M. Fierz, W. Pauli, *Proc. Roy. Soc.* **A173**, 211 (1939); W. Rarita, J. Schwinger, *Phys. Rev.* **60**, 61 (1941).