RECENT PROGRESS IN SUPERSYMMETRIC YANG–MILLS QUANTUM MECHANICS IN VARIOUS DIMENSIONS*

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We review the last year progress in understanding supersymmetric SU(2) Yang–Mills quantum mechanics in the $D = 4$ and 10 space-time dimensions. The four dimensional system is now well under control and the precise spectrum is obtained in all channels. In $D = 10$ some new results are also available.

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1. Introduction

Supersymmetric Yang–Mills quantum mechanics (SYMQM) shares many properties of advanced field theories thereby it provides a useful laboratory to study explicitly some of their properties [1, 2]. It emerges from the dimensional reduction (in space) of the full supersymmetric Yang–Mills field theory defined in the D dimensional space-time. Depending on D it covers a wide range of interesting phenomena. $D = 2$ system is exactly soluble with manifestly supersymmetric continuous spectrum and analytically calculable Witten index [1, 3]. For $D = 4$ the model is nontrivial and posesses both localized and non localized eigenstates, which have been constructed only recently in all fermionic sectors. On the other hand in the zero fermion sector it reduces to the zero volume limit of the pure Yang–Mills theory well studied in the context of lattice field theory [4–6]. Finally the $D = 10$, SU(∞) system, with its threshold bound state and the continuum of scattering states, is considered as a model of M-theory [7] and has attracted a lot of interest (for recent review see *e.g.* [8]).

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In this talk I would like to review the current status of the programme which attempts to solve these models in various dimensions and for various gauge groups [9]. The four dimensional ($D = 4$) system with the SU(2) gauge group, where the main progress has been achieved, will be discussed in the next Section. In Section 3, a new approach to the $D = 10$, SU(2) model, together with some preliminary results for the purely bosonic sector will be presented.

2. $D = 4$ supersymmetric Yang–Mills quantum mechanics

Reduction of the D dimensional supersymmetric SU(2) Yang–Mills field theory to a single point in the $d = D - 1$ dimensional space leads to the quantum mechanical system which for $D = 4$ is described by nine bosonic coordinates $x_a^i(t)$, $i = 1, 2, 3; a = 1, 2, 3$ and six independent fermionic coordinates contained in the Majorana spinor $\psi_a^\alpha(t)$, $\alpha = 1, ..., 4$. Equivalently (in $D = 4$) one could impose the Weyl condition and work with Weyl spinors. Hamiltonian reads [2]

$$\begin{aligned} H &= H_B + H_F\,, \\ H_B &= \frac{1}{2}p_a^i p_a^i + \frac{g^2}{4}\epsilon_{abc}\epsilon_{ade}x_b^i x_c^j x_d^i x_e^j\,, \\ H_F &= \frac{ig}{2}\epsilon_{abc}\psi_a^T \Gamma^k \psi_b x_c^k\,, \end{aligned} \tag{1}$$

where ψ^T is the transpose of the real Majorana spinor, and Γ in $D = 4$ are just the standard Dirac α matrices.

After the reduction of the three dimensional space to a single point, the rotational symmetry of the original theory becomes the internal spin(3) symmetry. It is generated by the angular momentum

$$J^i = \epsilon^{ijk}\left(x_a^j p_a^k - \frac{1}{4}\psi_a^T \Sigma^{jk}\psi_a\right), \tag{2}$$

with

$$\Sigma^{jk} = -\frac{i}{4}[\Gamma^j, \Gamma^k]. \tag{3}$$

The model has also a gauge invariance with the generators

$$G_a = \epsilon_{abc}\left(x_b^k p_c^k - \frac{i}{2}\psi_b^T \psi_c\right), \tag{4}$$

and is invariant under the supersymmetry transformations generated by

$$Q_\alpha = (\Gamma^k \psi_a)_\alpha p_a^k + ig\epsilon_{abc}(\Sigma^{jk}\psi_a)_\alpha x_b^j x_c^k. \tag{5}$$

The bosonic potential (written in the vector notation in the color space)

$$V = \frac{g^2}{4}\Sigma_{jk}(\vec{x}^j \times \vec{x}^k)^2, \tag{6}$$

exhibits the famous flat directions responsible for a rich structure of the spectrum.

2.1. Original approach

We shall now review a simple way to calculate automatically algebraic expressions for matrix elements of a wide class of quantum observables and to construct numerically the complete spectrum of a given system [9]. The method implements literarily in the computer the rules which govern the quantum world.

Quantum mechanics inside a PC

For a moment consider a single bosonic degree of freedom. Generalization to fermions and to more variables readily follows. Action of any polynomial observable can be easily implemented in an algebraic program if we use the discrete eigenbasis of the occupation number operator $a^\dagger a$

$$\{|n\rangle\}, \quad |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\,. \tag{7}$$

Since the bosonic coordinate and momentum operators are

$$x = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad p = \frac{1}{i\sqrt{2}}(a - a^\dagger), \tag{8}$$

a typical quantum observable can be represented as the multiple actions of the basic creation and annihilation operators[1].

A quantum state is a superposition of arbitrary number, n_s, of elementary states $|n\rangle$

$$|\mathrm{st}\rangle = \Sigma_I^{n_s} a_I |n^{(I)}\rangle, \tag{9}$$

and will be represented as a Mathematica list

$$\mathrm{st} = \left\{n_s, \{a_1, \ldots, a_{n_s}\}, \{n^{(1)}\}, \{n^{(2)}\}, \ldots, \{n^{(n_s)}\}\right\}, \tag{10}$$

with $n_s + 2$ elements. The first element specifies the number of elementary states entering the linear combination, Eq. (9). The second element is the list itself and contains all complex amplitudes $a_I, I = 1, \ldots, n_s$. Remaining n_s

[1] The method can be also extended to non polynomial potentials.

sublists give the occupation numbers of elementary, basis states. According to this convention an elementary state $|n\rangle$ is represented by $\{1,\{1\},\{n\}\}$.

Next, we implement basic operations defined on states: addition, multiplication by a number and the scalar product. They are simply programmed as definite operations on Mathematica lists transforming them in accord with the principles of quantum mechanics. Creation and annihilation operators are then defined as a list-valued functions on above lists. According to Eq. (8) the action of the position and momentum operators becomes also defined. Then we define any quantum observable: Hamiltonian, angular momentum, generators of gauge transformations, supersymmetry generators, *etc.*

Further procedure is now clear: given a particular system, define the list corresponding to the empty state, then generate a finite basis of $N_{\rm cut}$ vectors and calculate matrix representations of the Hamiltonian and other operators using above rules. Next, the complete spectrum and its various symmetry properties is obtained by the numerical diagonalization.

By studying the cutoff dependence of the spectrum one can estimate the systematic errors induced by restricting the Hilbert space. In many systems studied so far convergent results were obtained before the basis grew too large.

Creation and annihilation operators for SYMQM

Supersymmetric Hamiltonian (1) is polynomial in momenta and coordinates, hence above idea can be readily applied. To this end rewrite bosonic and fermionic variables in terms of the creation and annihilation operators of simple, normalized harmonic oscillators

$$[a^i_a, a^{k\dagger}_b] = \delta^{ik}_{ab}, \quad \{f^\rho_a, f^{\sigma\dagger}_b\} = \delta^{\rho\sigma}_{ab}, \quad \rho,\sigma = 1,2, \tag{11}$$

such that the canonical (anti)commutation relations

$$[x^i_a, p^k_b] = i\delta^{ik}\delta_{ab}, \quad \{\psi^\alpha_a, \psi^\beta_b\} = \delta^{\alpha\beta}_{ab} \tag{12}$$

are preserved. Standard extensions of Eq. (8) for bosonic variables read

$$x^i_a = \frac{1}{\sqrt{2}}(a^i_a + a^{i\dagger}_a), \quad p^i_a = \frac{1}{i\sqrt{2}}(a^i_a - a^{i\dagger}_a). \tag{13}$$

For fermions, the following representation for a quantum Hermitean Majorana spinor was used

$$\psi_a = \frac{1+i}{2\sqrt{2}} \begin{pmatrix} -f^1_a - if^2_a + if^{1\dagger}_a + f^{2\dagger}_a \\ +if^1_a - f^2_a - f^{1\dagger}_a + if^{2\dagger}_a \\ -f^1_a + if^2_a + if^{1\dagger}_a - f^{2\dagger}_a \\ -if^1_a - f^2_a + f^{1\dagger}_a + if^{2\dagger}_a \end{pmatrix}. \tag{14}$$

The basis and the cutoff

The complete Hilbert space is spanned by all independent polynomials of creation operators $a_b^{i\dagger}$ and $f_c^{\sigma\dagger}$ acting on the empty state

$$|(0,0,0),(0,0,0),(0,0,0),(0,0,0),(0,0,0)\rangle, \tag{15}$$

which in the Mathematica "representation" reads

$$\{1,\{1\},\{\{0,0,0\},\{0,0,0\},\{0,0,0\},\{0,0,0\},\{0,0,0\}\}\}. \tag{16}$$

By construction, the first three vectors (in color) specify bosonic, and the last two fermionic, occupation numbers. In practical applications we shall work in the restricted Hilbert space containing at most B bosonic quanta in total. Hence the gauge and rotationally invariant cutoff $N_{\rm cut}$ is defined as

$$\Sigma_{i,b} a_b^i a_b^{i\dagger} \equiv B \leq B_{\max} \equiv N_{\rm cut}. \tag{17}$$

Since the Pauli principle admits only six Majorana fermions in this system, there is no need to restrict the fermion number F.

Local gauge invariance is taken into account by constructing only the physical, *i.e.* gauge invariant basis. To create all independent, gauge invariant states at fixed F and B consider all possible contractions of color indices in a creator of (F,B) order

$$a_{a_1}^{i_1\dagger}...a_{a_B}^{i_B\dagger} f_{b_1}^{\sigma_1\dagger}..f_{b_F}^{\sigma_F\dagger} \tag{18}$$

for all values of the spatial indices i and σ. All color contractions fall naturally into different *gauge invariant classes*. Creators from different classes differ by color contractions *between* bosonic and fermionic operators. For example

$$a_a^{i\dagger} a_a^{j\dagger} a_b^{k\dagger} a_b^{l\dagger} f_c^{\sigma\dagger} f_c^{\rho\dagger} \tag{19}$$

and

$$a_a^{i\dagger} a_a^{j\dagger} a_b^{k\dagger} a_c^{l\dagger} f_c^{\sigma\dagger} f_b^{\rho\dagger} \tag{20}$$

are in different gauge invariant classes. Creators of odd order are constructed with one triple contraction. For example

$$\epsilon_{cde} a_a^{i\dagger} a_a^{j\dagger} a_b^{k\dagger} a_b^{l\dagger} a_c^{m\dagger} f_d^{\sigma\dagger} f_e^{\rho\dagger} \tag{21}$$

and

$$\epsilon_{cde} a_a^{i\dagger} a_a^{j\dagger} a_b^{k\dagger} a_c^{l\dagger} a_d^{m\dagger} f_e^{\sigma\dagger} f_b^{\rho\dagger} \tag{22}$$

also belong to different gauge invariant classes. To select linearly independent states we used again the rules of "quantum algebra". Therefore, at fixed F and B the final procedure is as follows: *(i)* identify all gauge invariant classes of creators, *(ii)* loop over all values of spatial indices and for each $i_1, ..., i_B, \sigma_1, ..., \sigma_F$ create corresponding state from the empty state, Eq. (15), *(iii)* identify and reject linearly dependent vectors, *(iv)* orthonormalize the remaining set of states.

Given the basis it was then a simple matter to calculate automatically matrix representation of the Hamiltonian and other observables.

Early results

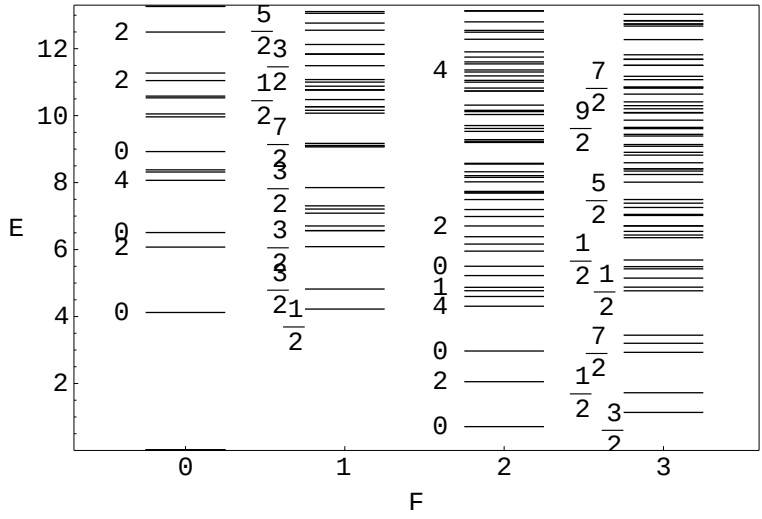

Fig. 1. Spectrum of supersymmetric Yang–Mills quantum mechanics.

In Fig. 1 we quote the spectrum obtained in Ref. [9] in all four independent fermionic sectors[2]. The cutoff in this calculation was varying between $B_{\max} = 8$ for $F = 0$ and $B_{\max} = 5$ for the largest (*i.e.* most difficult) $F = 3$ sector. The angular momentum of a sample of states is also shown. It was deduced from the degeneracy of the SO(3) multiplets and independently from the explicit representation of the angular momentum in our basis.

Later on this results were upgraded, with the considerable numerical effort, to $B_{\max} = 8$ in all sectors [10]. Fig. 2 is the outcome of the latter calculations which took a couple of months of a 600 MHz ALPHA workstation. The cutoff dependence displayed in this Figure confirmed and quantified the general expectation that the system has both continuous and discrete spec-

[2] The Hamiltonian has also the particle-hole symmetry.

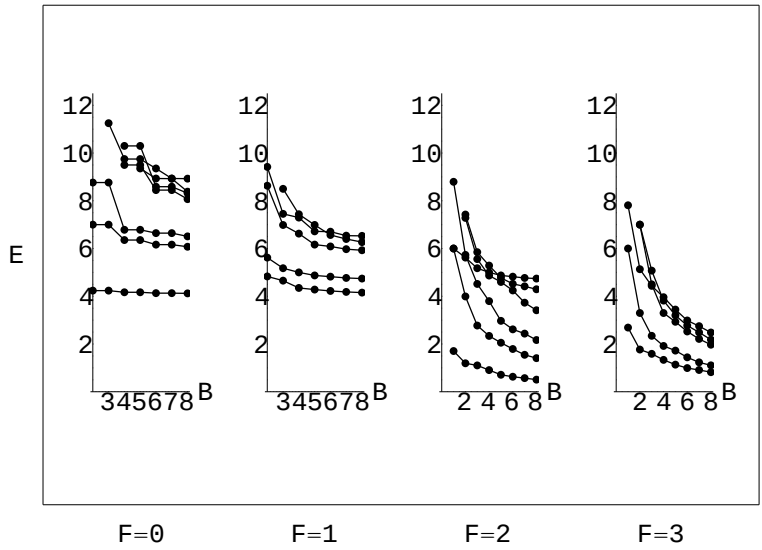

Fig. 2. Cutoff dependence of the first few eigenenergies of the Hamiltonian (1).

trum. Discrete, localized states (which manifest themselves here as quickly convergent levels) appear for $F = 0$ and $F = 1$ while the non localized states, with substantially slower convergence, are seen in the "fermion rich" sectors with $F = 2$ and 3. This was expected from the flat valley nature of the potential, Eq. (6), combined with the supersymmetry [11]. A series of other results about the supersymmetric structure of the system, Witten index, *etc.* was also obtained.

Further progress was achieved recently by exploiting analytically the symmetries of the system.

2.2. New developments

Separation of variables

Above results have been beautifully confirmed and extended with the aid of the "almost analytical" approach pioneered by Savvidy [12] and developed by van Baal in slightly different context [13]. Decomposing solution of the nine dimensional Schrödinger equation, in the $F = 0, J = 0$ and $F = 2, J = 0$ channels into covariant tensors, he reduced the problem to a numerically tractable set of coupled ordinary differential equations. When adapted to our case his method can push the cutoff as high as $B_{\max} = 39$ in these two channels, see Fig. 3. Now the discrete, localized, and quickly convergent with the cutoff states with $F = 0$ are clearly seen, their energy determined with a very high precision. Moreover, the intricate nature of the solutions with

two fermions is also evident. The flat lines signal again the localized bound states of two gluinos, while slowly falling with the cutoff levels correspond to the non localized states from the continuous spectrum. Some of the bound states in $F = 0$ and $F = 2$ sectors have degenerate energies as required by supersymmetry. Other supersymmetric partners must then be located in fully fermionic ($F = 1, 3$) sectors for which the method has not been yet generalized.

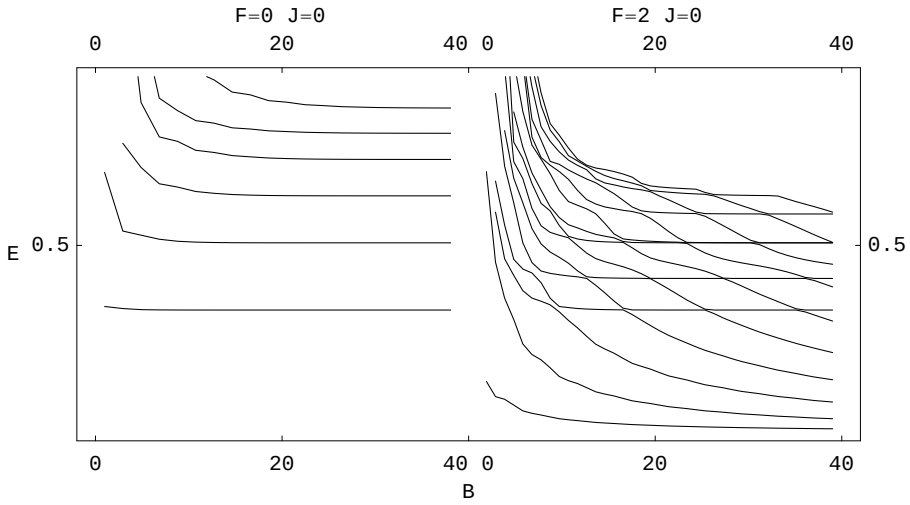

Fig. 3. High cutoff results from van Baal approach.

High cutoff solutions in all channels

Similar high precision results are now available for all angular momenta and in all fermionic sectors [14]. To this end we have extended the recursive approach first applied to the $D = 2$ system [3]. In the two dimensional case recursions, which relate the matrix elements in a bigger basis with those in the smaller basis, were simple enough to be solved analytically. For the $D = 4$ this is not the case any more. However they speed up dramatically computations allowing to reach high precision spectra in all fermionic sectors and all angular momentum channels. Second, rotational invariance is now fully employed, by projecting into channels of given angular momentum. Consequently there is no need to diagonalize very large matrices. Both these improvements result not only in the more precise spectra, but in much more satisfactory understanding of the supersymmetric structure of the $D = 4$ system, including complete classification of all supermultiplets.

Analytical study of continuous spectra in cut systems

Finally, in a recent development a new scaling in the continuous spectra of a cut quantum systems was found [15]. This provides an important tool for studying the continuum limit of non localized states and their interactions.

To summarize this section, the technology to study quantitatively $D = 4$, SU(2), SYMOM is now available and one can readily attack more subtle problems like scattering [18] or thermodynamics of this system [19].

We now turn to, much more difficult, $D = 10$ model whose quantitative study with presently discussed methods is just beginning (see [2,16] for other attempts).

3. TWS scheme

In higher dimensions only the original approach is available and the spectrum of lowest zero-volume glueballs was obtained for all $4 \leq D \leq 10$ in the $F = 0$ sector [17]. The number of states grows rapidly with D and consequently at $D = 10$ only $B \leq 4$ were reached in practice. In fermionic sectors situation is yet worse. The difficulties are two-fold: (a) bases become too large to store, and (b) time required for calculating the full Hamiltonian matrix grows prohibitively large. The TWS scheme, presented below, offers substantial reduction of both parameters.

Consider the (B, F) sector of the Hilbert space, *i.e.* the one with B bosons and F fermions. The basis vectors before orthonormalization can be uniquely characterized by the compound index

$$I = \{C, i^1, \ldots, i^B, \sigma^1, \ldots, \sigma^F\}, \tag{23}$$

since given I the actual state can be readily created by the compound creator from the gauge invariant class C (*cf.* (19)–(22)). Many properties of the mixing matrix and of the Hamiltonian matrix can be deduced from the index I alone without need to create the actual state. Therefore we shall abandon where possible the lengthy Mathematica representation of states and use instead much more compact indices (23) . This virtually eliminates the first problem. Second difficulty is also reduced by observing that for polynomial Hamiltonians most of the matrix elements are zero and many of non-zero ones are actually the same. We shall therefore expose the sparse nature of the Hamiltonian matrix and identify equal matrix elements. This can be done using indices (23) only, and will lead to the new classification of basis states.

We begin with the properties of the mixing matrix $M_{IK} = \langle I|K\rangle$, also referred to as the norm matrix.

3.1. Norm matrix and new classification of states

For the sake of simplicity we restrict the analysis to the $F = 0$ sector. Generalization for fermions is possible. Consider the generic element of the mixing matrix

$$M_{IK} = \langle I|K\rangle = \langle C_I, i^1, \ldots, i^{B_I}|C_K, k^1, \ldots, k^{B_K}\rangle. \tag{24}$$

Obviously only states which contain the same quanta can mix, therefore

$$M_{IK} \neq 0 \text{ if } B_I = B_K = B, \{i^1, \ldots, i^B\} \sim \{k^1, \ldots, k^B\}, \tag{25}$$

where the "$\sim$" means that the values of all { i } and { k } indices coincide up to a permutation. On the other hand states from different gauge invariant classes can mix in general.

Words

This simple observation suggests to group all states with given B into *words*. A word W is defined as a set of all states which contain the same quanta. For example in a $B = 4$ sector states $\{C, 1, 2, 2, 5\}$ and $\{D, 1, 2, 5, 2\}$ belong to the same word while states $\{C', 4, 3, 3, 8\}$ and $\{D', 1, 2, 3, 5\}$ are from different words. All gauge invariant classes, which can be constructed for particular set of spatial indices, are included in a given word by definition. Hence a word is uniquely labelled by an *ordered* set of B integers, *e.g.* $W = (1225)$ defines the word considered above. How many states are in a word? As many as there are inequivalent permutations of spatial indices which give linearly independent states in all contributing gauge invariant classes. Now Eq. (25) can be stated as the following

THOREM 1: Only states in the same word mix.

Hence the whole mixing matrix splits into blocks of small $(N_W \times N_W)$ mixing matrices for each word W with N_W being the number of states in a word W.

Types

Words come in different *types*. A type T characterizes the pattern of indices irrespectively of their actual value. For example words (1225) and (3448) belong to the same type which we denote as [211], and (1234) or (2227) belong to the two different types [1111] and [31] respectively. The number of words in a given type, N_T is equal to the number of realizations of the pattern by different values of the indices. For our first example ($d = D - 1$)

$$N_{211} = d(d-1)(d-2)/2, \tag{26}$$

and similarly for other types. Types can be identified with the partitions of B. Therefore the total number of types in a given bosonic sector is just equal to the number of partitions, P_B, of B.

Since neither the number of states in a word nor the number of types in a sector depend on the dimensionality of the problem, the main complexity lies in the number of words in a given type, *e.g.* Eq. (26) which grows rapidly with d for larger B. However the big simplification occurs due to the following consequence of the Wick theorem.

THEOREM II: All mixing matrices in the same type are equal.

To see this rewrite M_{IK} as the sum over all contractions

$$M_{IK} = \langle I|K\rangle = \langle 0|a^{i_1}\dots a^{i_B}a^\dagger_{k_1}\dots a^\dagger_{k_B}|0\rangle = \Sigma_{\text{contractions}}\Pi\delta_{i.k.}, \qquad (27)$$

where the product on the right hand side denotes symbolically given contraction. It is evident, even from this schematic expression, that M_{IK} does not depend on the particular *values* of the indices, but depends only on their *configurations* or patterns. Hence the theorem follows.

Therefore the calculation of the full mixing matrix is simply reduced to calculating P_B small matrices and using the same copy for all words in a given type. Moreover, the tedious orthonormalizaton procedure also decouples into small blocks, corresponding to words, and has to be done only for one word in a type with the rest being the exact copies of the results from the first word. This simplification prompts us to introduce the new scheme of organizing states in a basis — the TWS scheme which groups states according to the following hierarchy:

$$S(tates) \to W(ords) \to T(ypes) \to Sectors. \qquad (28)$$

With this organization any state in a basis is labelled by four integers

$$|I\rangle = |b, t, w, s\rangle, \qquad (29)$$

with b, t, w, s enumerating sectors, types, words and states respectively.

It turns out that the structure of the Hamiltonian is also much more transparent in this representation as discussed below.

3.2. SYMQM Hamiltonian

As above we shall consider only bosonic sectors $(B, F) = (B, 0)$ where only H_B of Eq. (1) contributes. Since H_B contains terms of the second and fourth order in bosonic creation and annihilation operators, it can only induce transitions with $\Delta B = 0, 2, 4$. Each transition can be interpreted as

TABLE I

Effective transitions induced by the bosonic Hamiltonian (1) in nine space dimensions.

ΔB	rule	action
	1	diagonal
0	2	$2 \to 2$ new
	3	$2 \to 2$ old
	4	add 2 new
2	5	add 2 old
	6	add 2×2 new-new
4	7	add 2×2 new-old
	8	add 2×2 old-old

the effective operator creating/annihilating 0,2 or 4 indices in our TWS representation. They are listed in Table I and will be referred to as *rules* since each transition implies a rule how to generate from the initial type/word the final type/word with nonzero matrix element. For example $\Delta B = 0$

TABLE II

Structure of the bosonic Hamiltonian in the TWS representation, b and t are defined in Table III.

b		1		2			3							4						
	t	1	1	2	1	2	3	4	5	1	2	3	4	5	6	7	8	9	10	11
1	1	1	4				6													
	1	4	12		5		4					7				6				
2	2			1		5		4					8		7					
	1		5		1		2			5		4								
	2			5		13		2			5		5		4					
3	3	6	4		2		12					5				4				
	4			4		2		12						5	5			4		
	5								1								5		4	
	1				5					1		2								
	2					5					1		3		2					
	3		7		4		5			2		123				2				
	4			8		5					3		1		2					
	5							5						1	3			2		
4	6			7		4		5			2		2	3	123			2		
	7		6				4					2				12				
	8								5								13		2	
	9							4						2	2			12		
	10								4								2		12	
	11																			1

transition, labelled as rule 2, annihilates two indices in the initial state and creates 2 indices with a common value which is new, *i.e.* does not exist in the initial state. Rule 3 has similar action but the common value of the created pair coincides with one of the values already existing in the initial state. Rule 7 adds two pairs of identical indices one being new and one already existing, *etc.* It follows from Eq. (6) that we cannot create/annihilate a pair of different indices. Similarly there are no terms creating four the same indices.

We emphasize that above classification should be used only to identify all nonzero matrix elements of the Hamiltonian. The actual values of matrix elements will be calculated within the original approach.

TABLE III

Labelling of sectors and types in Table II.

b	t	B	partition
1	1	0	
	1		2
2	2	2	1
	1		4
	2		31
3	3	4	22
	4		211
	5		1111
	1		6
	2		51
	3		42
	4		33
	5		411
4	6	6	321
	7		222
	8		3111
	9		2211
	10		21111
	11		111111

Table II shows the resulting structure of the Hamiltonian in the TWS representation for the first four sectors. Labelling of rows and columns is explained in Table III. The Hamiltonian is block tri-diagonal in bosonic sectors (odd and even sectors are decoupled bacuse of the parity conservation). Each nontrivial sector-to-sector block is again sparse with only few nonzero subblocks connecting initial and final types according to the rules listed in Table I. For simplicity they are labelled by these rules. Each such type-to-type subblock is again a $N_{\mathrm{Tf}} \times N_{\mathrm{Ti}}$ matrix with entries connecting different

words of the initial and final types. Again this matrix is in general sparse — from given initial word one can reach, with a given rule, only few final words. This is not the end, the entries of this matrix are again matrices labelled by the individual states belonging to initial and final words. Only at this level the orthogonalization of basis states enters. Since elementary states in a single word are not orthogonal, the word-to-word blocks must be corrected by the square roots of the corresponding norm matrices. This is, however, much less time consuming than the global orthonormalization.

Moreover one can show that the word-to-word subblocks in a given type-to-type block are independent of the initial word. Similarly to the mixing matrix case, it suffices to calculate transitions from one initial word in each type and substitute them for other initial words. The proof is more technical and will not be given here.

The whole structure, even though little involved at first glance, can be readily formalized and leads to much faster calculations of the whole matrix.

3.3. Preliminary results

The complexity of the problem is summarized in Table IV where dimensions of the Hilbert space for various D and B are displayed. All bases were generated and orthogonalized with the original approach (*cf.* Section 4.1). Mathematica programs took about 2 days to produce the last entry, thereby the cutoff $B = 7$–8 seems to be a practical maximum for the brute force method at $D = 10$. Calculation and diagonalization of the Hamiltonian is yet more demanding — for D=10 one can reach only $B = 4$ within the reasonable computing time [17].

TABLE IV

Sizes of bases in the $F = 0$ sector for space-time dimensions $4 \leq D \leq 10$. N_s is the number of basis vectors with given number of bosonic quanta, B.

D	4	5	6	7	8	9	10
B				N_s			
0	1	1	1	1	1	1	1
1	–	–	–	–	–	–	–
2	6	10	15	21	28	36	45
3	1	4	10	20	35	56	84
4	21	55	120	231	406	666	1035
5	6	36	126	336	756	1512	2772
6	56	220	680	1771	4060	8436	16215
7	21	180	855	2976	8478	20952	
8	126	714	3045	10521			

The TWS approach allows to reach one to two orders of magnitude bigger bases, which at $D = 10$ translates for $B_{\max} = 6$–8. This rather modest, in terms of B, improvement is nevertheless quite relevant for the precision of the energy levels. Fig. 4 shows the first five eigenenergies as a function of the cutoff. Solid lines are drawn just to guide the eye. Levels are labelled by corresponding SO(9) representations. Since our cutoff respects the rotational invariance, resulting spectrum of a cut Hamiltonian should have full SO(9) symmetry [3], and indeed we observe first few SO(9) multiplets as indicated in the figure [4]. The ordering of the levels (singlet, tensor, singlet) turns out to be the same as in the well known $D = 4$ case [5] and in fact remains unchanged for all $4 \leq D \leq 10$ [17] . The energy of the lowest singlet state has already converged within 3%. For second and third state the relative change with B dropped from 13% to 7% and from 17% to 10% respectively. Cutoff $B_{\max} = 8$ can be reached with the reasonable computing effort.

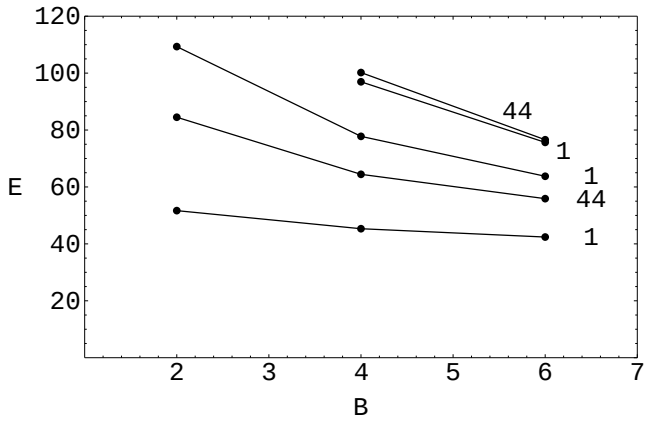

Fig. 4. Lower levels of the spectrum of the pure Yang–Mills D=10 system.

4. Summary and outlook

In the last year there has been a substantial progress in solving supersymmetric Yang–Mills quantum mechanics with the SU(2) gauge group. In four dimensions we can reach such high cutoffs that the restriction of the Hilbert space becomes irrelevant for most purposes. Precise spectrum in all channels has been obtained and the intricate pattern of continuous and discrete states found and clarified. Supersymmetric structure of the system can be now studied in detail. In particular the complete classification of supersymmetric multiplets is now available.

[3] Only in the purely bosonic sector.

[4] Incidentally this provides one of the tests of the whole TWS scheme.

In the ten dimensional system developments are obviously slower but neverheless steady. In the bosonic sector one can now reach the cutoffs (hence the precision) comparable to the $D = 4$ model studied with the original technique. The new TSW scheme is yet to be generalized to fermionic sectors. The ongoing puzzle of nonconserved massless Majorana fermions calls for further study. On the other hand reported here progress in the four dimensional case opens some new possibilities also for the ten dimensional system.

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REFERENCES

[1] M. Claudson, M. B. Halpern, *Nucl. Phys.* **B250**, 689 (1985).
[2] M. B. Halpern, C. Schwartz, *Int. J. Mod. Phys.* **A13**, 4367 (1998).
[3] M. Campostrini, J. Wosiek, *Phys. Lett.* **B550**, 121 (2002).
[4] M. Lüscher, *Nucl. Phys.* **B219**, 233 (1983).
[5] M. Lüscher, G. Münster, *Nucl. Phys.* **B232**, 445 (1984).
[6] P. van Baal, *Acta Phys. Pol. B* **20**, 295 (1989).
[7] T. Banks, W. Fishler, S. Shenker, L. Susskind, *Phys. Rev.* **D55**, 6189 (1997).
[8] W. Taylor, *Rev. Mod. Phys.* **73**, 419 (2001).
[9] J. Wosiek, *Nucl. Phys.* **B644**, 85 (2002).
[10] J. Kotanski, J. Wosiek, *Nucl. Phys. B* (*Proc. Suppl.*) **119**, 932 (2003).
[11] B. de Wit, M. Lüscher, H. Nicolai, *Nucl. Phys.* **B320**, 135 (1989).
[12] G.K. Savvidy, *Phys. Lett.* **B159**, 325 (1985).
[13] P. van Baal, *The Witten Index Beyond the Adiabatic Approximation,* in: *Michael Marinov Memorial Volume, Multiple Facets of Quantization and Supersymmetry*, eds. M. Olshanetsky and A. Vainshtein, World Scientific, Singapore, 2002, p.556.
[14] M.Campostrini, J. Wosiek, in preparation.
[15] M. Trzetrzelewski, J. Wosiek, Jagellonian University preprint, `hep-th/0308007`.
[16] U. H. Danielsson, G. Ferretti, B. Sundborg, *Int. J. Mod. Phys.* **A11**, 5463 (1996).
[17] J. Wosiek, *Supersymmetric Yang–Mills Quantum Mechanics*, in: *Proceedings of the NATO Advanced Research Workshop on Confinement, Topology and Other Non-Perturbative Aspects of QCD*, eds. J. Greensite and S. Olejnik, Kluwer AP, Dordrecht 2002.
[18] K. Becker, M. Becker, *Nucl. Phys.* **B506**, 48 (1997).
[19] D. Kabat, G. Lifschytz, D.A. Lowe, *Phys. Rev.* **D64**, 124015 (2001).