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DIFFERENTIAL EQUATIONS FOR THE 2-LOOP EQUAL MASS SUNRISE*

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The differential equations for the 2-loop sunrise graph, at equal masses but arbitrary momentum transfer, are used for the analytic evaluation of the coefficients of its Laurent-expansion in the continuous dimension d.

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1. Introduction

The differential equations in the squared external momentum p^2 for the Master Integrals (MIs) of the 2-loop sunrise graph with arbitrary masses m_1, m_2, m_3 of Fig. 1 were written in [1]. They were used for obtaining

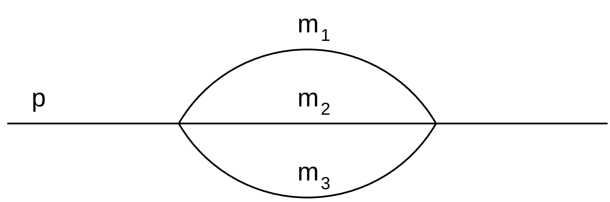

Fig. 1. The 2-loop sunrise graph.

analytically particular values and behaviours at zero and infinite momentum transfer [1], at pseudothresholds [2] and threshold [3], as well as for direct numerical integration [4]. In this contribution I will report on some progress in the anlytic study of the solutions of the equations for arbitrary momentum

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transfer in the equal mass limit $m_i = 1$. A more complete account will be given elsewhere [5]; while the algebraic burden in the arbitrary mass case will surely be much heavier, there are indications [6] that the approach can be extended to the arbitrary mass case as well.

In the equal mass limit the 2-loop sunrise has two MIs, which in the usual d-continuous regularization scheme can be written as

$$
\begin{aligned}
S(d,p^2) &= \frac{1}{\Gamma^2\left(3-\frac{d}{2}\right)}\int\frac{d^dk_1}{4\pi^{\frac{d}{2}}}\int\frac{d^dk_2}{4\pi^{\frac{d}{2}}}\frac{1}{(k_1^2+1)(k_2^2+1)[(p-k_1-k_2)^2+1]}\,,\\
S_1(d,p^2) &= \frac{1}{\Gamma^2\left(3-\frac{d}{2}\right)}\int\frac{d^dk_1}{4\pi^{\frac{d}{2}}}\int\frac{d^dk_2}{4\pi^{\frac{d}{2}}}\frac{1}{(k_1^2+1)^2(k_2^2+1)[(p-k_1-k_2)^2+1]}\,.
\end{aligned}
\tag{1}
$$

Let us put $p^2 = z$ (z is positive when p is Euclidean); the two MIs then satisfy the following linear system of first order differential equations in z

$$
\begin{aligned}
z\frac{d}{dz}S(d,z) &= (d-3)S(d,z)+3S_1(d,z)\,,\\
z(z+1)(z+9)\frac{d}{dz}S_1(d,z) &= \frac{1}{2}(d-3)(8-3d)(z+3)S(d,z)\\
&+\frac{1}{2}\left[(d-4)z^2+10(2-d)z+9(8-3d)\right]S_1(d,z)\\
&+\frac{1}{2}\frac{z}{(d-4)^2}\,.
\end{aligned}
\tag{2}
$$

The system can be rewritten as a second order differential equation for $S(d,z)$ only

$$
\begin{aligned}
&z(z+1)(z+9)\frac{d^2}{dz^2}S(d,z)\\
&+\frac{1}{2}\left[(12-3d)z^2+10(6-d)z+9d\right]\frac{d}{dz}S(d,z)\\
&+\frac{1}{2}(d-3)\left[(d-4)z-d-4\right]S(d,z) = \frac{3}{2}\frac{1}{(d-4)^2}\,.
\end{aligned}
\tag{3}
$$

As the second MI $S_1(d,z)$ can be written in terms of $S(d,z)$ and its first derivative

$$
S_1(d,z) = \frac{1}{3}\left[-(d-3)+z\frac{d}{dz}\right]S(d,z)\,,
\tag{4}
$$

we can take from now on $S(d,z)$ and its derivative $dS(d,z)/dz$ as the effective MIs.

2. From near 4 to near 2 dimensions

We want to expand $S(d,z)$ around $d=4$ as Laurent series in $(d-4)$ and then to obtain analytically the values of the coefficients of the expansion by solving the relevant differential equations. It was found *a posteriori* that all the formulae are much simpler when expanding around $d=2$. To give the relations between the two expansions, let us recall that acting on any scalar Feynman integral in d dimensions with a suitable differential operator, one obtains the same integral in $(d-2)$ dimensions. times a numerical factor depending on d [7]. Acting on the MIs in d dimensions (or, in our case, on the two functions $S(d,z)$ and $dS(d,z)/dz$), one obtains the same MIs in $d-2$ dimensions in terms of mass derivatives of the MIs in d dimensions, which can be expressed again in terms of MIs in d dimensions; solving the linear system for the d-dimensional MIs and replacing finally d by $d+2$ one obtains

$$\begin{aligned} S(2+d,z) &= \frac{1}{3(d-1)(3d-2)(3d-4)} \\ &\quad \times\Bigg\{ -\frac{9}{(d-2)^2}+\frac{3z-63}{4(d-2)} \\ &\quad +(z+1)(z+9)\left[1+(z-3)\frac{d}{dz}\right]S(d,z) \\ &\quad +(d-2)(87+22z-z^2)S(d,z)\Bigg\}\,. \end{aligned} \tag{5}$$

Quite in general, if

$$A(2+d)=B(d)\,,$$

one can set $d=2+\eta$ and Laurent-expand in η; one obtains

$$\sum_k \eta^k A^{(k)}(4) = \sum_k \eta^k B^{(k)}(2)\,.$$

The Laurent expansion in η of $S(d,z)$ for $d=4+\eta$ begins with a double pole in η and reads

$$S(4+\eta,z)=\frac{1}{\eta^2}S^{(-2)}(4,z)+\frac{1}{\eta}S^{(-1)}(4,z)+S^{(0)}(4,z)+\eta S^{(1)}(4,z)+\ldots$$

while $S(2+\eta,z)$ has no singularities in η, and its expansion is

$$S(2+\eta,z)=S^{(0)}(2,z)+\eta S^{(1)}(2,z)+\ldots\;.$$

By inserting the two expansions in Eq. (5), one gets the required coefficients $S^{(k)}(4,z)$ of the Laurent expansion in η of $S(4+\eta,z)$ around 4 in terms of the coefficients $S^{(k)}(2,z)$ of the expansion of $S(2+\eta,z)$ around 2. As $S(d,z)$ is regular at $d=2$, the poles in η are not hidden in $S(d,z)$ but are explicitly exhibited by the $1/(d-2)$ factors in the r.h.s. of Eq. (5). Working out the algebra, one finds at once

$$\begin{aligned} S^{(-2)}(4,z) &= -\frac{3}{8} \,, \\ S^{(-1)}(4,z) &= \frac{9}{16} + \frac{z}{32} \,. \end{aligned} \tag{6}$$

3. The expansion at $d = 2$ of the differential equation

By expanding systematically in $(d-2)$ all the terms appearing in Eq. (3), one obtains a set of chained equations of the form

$$\begin{aligned} \Bigg\{ \frac{d^2}{dz^2} &+ \left[\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right] \frac{d}{dz} \\ &+ \left[\frac{1}{3z} - \frac{1}{4(z+1)} - \frac{1}{12(z+9)}\right]\Bigg\} S^{(k)}(2,z) = N^{(k)}(2,z) \,, \end{aligned} \tag{7}$$

where the homogeneous part is the same for any order k, and the first few inhomogeneous terms are

$$\begin{aligned} N^{(0)}(2,z) &= \frac{1}{24z} - \frac{3}{64(z+1)} + \frac{1}{192(z+9)} = \frac{3}{8z(z+1)(z+9)} \,, \\ N^{(1)}(2,z) &= \left(-\frac{1}{2z} + \frac{1}{z+1} + \frac{1}{z+9}\right) \frac{dS^{(0)}(2,z)}{dz} \\ &+ \left(\frac{5}{18z} - \frac{1}{8(z+1)} - \frac{11}{72(z+9)}\right) S^{(0)}(2,z) \\ &+ \frac{1}{24z} - \frac{3}{64(z+1)} + \frac{1}{192(z+9)} \,, \\ N^{(2)}(2,z) &= \ldots \,. \end{aligned} \tag{8}$$

Equations (8) are chained, in the sense that the inhomogeneous term of order k involves lower terms, of order $(k-1)$ (for $k > 0$) and $(k-2)$ (for $k > 1$) in the expansion of $S(2,z)$, as can be seen from Eq. (3) and is shown explicitly in Eqs. (8).

The system Eq. (7) is to be solved bottom up in k, starting from $k = 0$ (in which case the inhomogeneous term is completely known) and then proceeding to higher values increasing k by one, so that at each step the inhomogeneous term is known from the solution of the previous equations. The chained equations can then be solved by using Euler's method of the variation of the constants. The homogeneous equation is the same for all the values of k,

$$\left\{\frac{d^2}{dz^2} + \left[\frac{1}{z}+\frac{1}{z+1}+\frac{1}{z+9}\right]\frac{d}{dz} + \left[\frac{1}{3z}-\frac{1}{4(z+1)}-\frac{1}{12(z+9)}\right]\right\}\Psi(z) = 0\,; \tag{9}$$

if $\Psi_1(z), \Psi_2(z)$ are two independent solutions of the homogeneous equation, $W(z)$ the corresponding Wronskian

$$W(z) = \Psi_1(z)\frac{d\Psi_2(z)}{dz} - \Psi_2(z)\frac{d\Psi_1(z)}{dz} \tag{10}$$

according to Euler's method the solutions of Eqs. (7) are given by the integral representations

$$\begin{aligned} S^{(k)}(2,z) &= \Psi_1(z)\left(\Psi_1^{(k)} - \int_0^z \frac{dw}{W(w)}\Psi_2(w)N^{(k)}(2,w)\right) \\ &+ \Psi_2(z)\left(\Psi_2^{(k)} + \int_0^z \frac{dw}{W(w)}\Psi_1(w)N^{(k)}(2,w)\right)\,, \end{aligned} \tag{11}$$

where $\Psi_1^{(k)}, \Psi_2^{(k)}$ are two integration constants.

Eq. (11) at this moment is just a formal representation of the solutions for the coeffcients $S^{(k)}(2,z)$; it becomes a substancial (not just formal!) formula only when all the ingredients – the two solutions of the homogeneous equation $\Psi_i(z)$, their Wronskian $W(z)$ and the two integration constants $\Psi_i^{(k)}$ are known explicitly.

Although the Wronskian is defined in terms of the $\Psi_i(z)$, it can be immediately obtained (up to a multiplicative constant) from Eq. (9). An elementary calculation using the definition Eq. (10) and the value of the second derivatives of the $\Psi_i(z)$, as given by Eq. (9) of which they are solutions, leads to the equation

$$\frac{d}{dz}W(z) = -\left(\frac{1}{z}+\frac{1}{z+1}+\frac{1}{z+9}\right)W(z)\,,$$

which gives at once

$$W(z) = \frac{9}{z(z+1)(z+9)} \ , \tag{12}$$

where the multiplicative constant has been fixed anticipating later results.

Finding the two $\Psi_i(z)$ requires much more work.

4. Solving the homogeneous equation at the singular points

By inspection, the singular points of Eq. (9) are found to be

$$z = 0, -1, -9, \infty \ ;$$

at each of those points one has two independent solutions, the first regular and the second with a logarithmic singularity. The expansions of the solutions around each of the singular points is immediately provided by the differential equation itself.

Around $z = 0$ the two solutions of Eq. (9) can be written as

$$\begin{aligned} \Psi_1^{(0)}(z) &= \psi_1^{(0)}(z) \, , \\ \Psi_2^{(0)}(z) &= \ln z \; \psi_1^{(0)}(z) + \psi_2^{(0)}(z) \, , \end{aligned} \tag{13}$$

where the $\psi_i^{(0)}(z)$ are power series in z. Imposing $\psi_1^{(0)}(0) = 1$, one finds

$$\begin{aligned} \psi_1^{(0)}(z) &= 1 - \frac{1}{3}z + \frac{5}{27}z^2 + \ldots \ , \\ \psi_2^{(0)}(z) &= -\frac{4}{9}z + \frac{26}{81}z^2 + \ldots \ ; \end{aligned} \tag{14}$$

the coefficients are given recursively (hence up to any reuired order) by the equation. The radius of convergence is 1 (the next singularity is at $z = -1$) and the two solutions are real for positive z (spacelike momentum transfer). The continuation to the timelike region is done by giving to z the value $z = -(u + i\epsilon)$; for $0 < u < 1$ one has $\ln z = \ln u - i\pi$ and $\Psi_2^{(0)}(z)$ develops an imaginary part $-i\pi\psi_1^{(0)}(z)$.

Similarly, around $z = -1$ the 2 independent solutions can be written as

$$\begin{aligned} \Psi_1^{(1)}(z) &= \psi_1^{(1)}(z) \, , \\ \Psi_2^{(1)}(z) &= \ln(z+1)\psi_1^{(1)}(z) + \psi_2^{(1)}(z) \end{aligned} \tag{15}$$

with Eq. (9) providing recursively the coefficients of the expansions in powers of $(z+1)$ of the two $\psi_i^{(1)}(z)$ once the initial condition is given. If $\psi_1^{(1)}(0) = 1$

one has

$$\begin{aligned}\psi_1^{(1)}(z) &= 1+\frac{1}{4}(z+1)+\frac{5}{32}(z+1)^2+\dots\ , \\ \psi_2^{(1)}(z) &= +\frac{3}{8}(z+1)+\frac{33}{128}(z+1)^2+\dots\end{aligned} \qquad (16)$$

with radius of convergence 1 (up to the singularity at $z=0$) *etc.*

The other two singular points $z=-9$ and $z=\infty$ can be treated in the same way, the corresponding formualae are not given due to lack of space.

5. The interpolating solutions

Having the solutions piecewise is not sufficient, one must build two solutions in the whole $-\infty < z < \infty$ range by suitably joining the above expressions of the solutions at the singular points. A hint is provided by the knowledge of the imaginary part of the original Feynman integral $S(d,p^2)$ Eq. (1) at $d=2$ dimensions; as already observed in [8], the Cutkosky–Veltman rule gives for the imaginary part of $S(d,p^2)$ at $d=2$ and $u=-z\geq 9$ (and up to a multiplicative constant) the integral representation

$$J(u) = \int_4^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{R_4(u,b)}}\ , \qquad (17)$$

where $R_4(u,b)$ stands for the polynomial (of 4th order in b and 2nd order in u)

$$R_4(u,b) = b(b-4)(b-(\sqrt{u}-1)^2)(b-(\sqrt{u}+1)^2)\ ,$$

and the b integration runs between two adjacent zeros of $R_4(u,b)$. As the inhomogeneous part of Eq. (3) cannot develop an imaginary part, the imaginary part of the Feynman integral in $d=2$ dimensions, $J(u)$ of Eq. (17), is necessarily a solution of the associated homogeneous equation at $d=2$, *i.e.* of Eq. (9) — a fact which can also be checked explicitly.

One is then naturally lead to consider all the b-integrals of $1/\sqrt{R_4(u,b)}$ between any two adjacent roots for all possible values of u. The details of the analysis cannot be reported here again for lack of space. As a result, one finds for instance that when u is in the range $0<u<1$ the roots of $R_4(u,b)$ are ordered as

$$0 < (\sqrt{u}-1)^2 < (\sqrt{u}+1)^2 < 4$$

and the associated b-integrals are

$$
\begin{aligned}
J_1^{(0,1)}(u) &= \int_0^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{-R_4(u,b)}} \ , \\
J_2^{(0,1)}(u) &= \int_{(\sqrt{u}-1)^2}^{(\sqrt{u}+1)^2} \frac{db}{\sqrt{R_4(u,b)}} \ , \\
J_3^{(0,1)}(u) &= \int_{(\sqrt{u}+1)^2}^{4} \frac{db}{\sqrt{-R_4(u,b)}} \ .
\end{aligned}
$$

The three integrals are all real (and positive) due to the choice of the sign in front of $R_4(u,b)$ in the square roots; more important, they all satisfy Eq. (9) — therefore they cannot be all independent. With standard changes of variables, they can be brought in the form of Legendre's complete elliptic integrals of the first kind [8]; in that way one finds for instance

$$
J_1^{(0,1)}(u) = J_3^{(0,1)}(u) = \frac{1}{\sqrt{(1+\sqrt{u})^3(3-\sqrt{u})}} K\left(\frac{(1-\sqrt{u})^3(3+\sqrt{u})}{(1+\sqrt{u})^3(3-\sqrt{u})}\right) \ ,
$$

showing in particular that the first and third integrals are indeed equal. A similar (but different!) formula holds for the second integral.

Although usually one can not do very much for expressing elliptic integrals in terms of other more familiar functions (such as logarithms or the like), in the limiting cases $u \to 0$ and $u \to 1$ two of the 4 roots of $R_4(b,u)$ become equal, and the by now elementary b-integrations give

$$
\begin{aligned}
\lim_{u\to 0^+} J_1^{(0,1)}(u) &= \frac{1}{\sqrt{3}}\left(-\frac{1}{2}\ln u + \ln 3\right) \ , \\
\lim_{u\to 0^+} J_2^{(0,1)}(u) &= \frac{\pi}{\sqrt{3}} \ , \qquad (18) \\
\lim_{u\to 1^-} J_1^{(0,1)}(u) &= \frac{\pi}{4} \ , \\
\lim_{u\to 1^-} J_2^{(0,1)}(u) &= -\frac{3}{4}\ln(1-u) + \frac{9}{4}\ln 2 \ . \qquad (19)
\end{aligned}
$$

One has now all the information needed for defining two solutions $\Psi_i(z)$ of Eq. (9) in an interval which contains the two singular points $z = 0$ and $z = -1$. Let us start by defining, for $z > 0$,

$$
\begin{aligned}
\Psi_1(z) &= \Psi_1^{(0)}(z) \ , \\
\Psi_2(z) &= \Psi_2^{(0)}(z) \ .
\end{aligned}
$$

That fixes the multiplicative constant in the Wronskian as well, giving the result already anticipated in Eq. (12). From Eqs. (13), (14) we easily read the behaviours of the $\Psi_i(z)$ for $u = -z$ small and positive; but in the range $0 < u < 1$ the solutions can also be expressed in terms of the $J_i^{(0,1)}(u)$; by matching the behaviours $u \to 0^+$ of the $J_i^{(0,1)}(u)$, Eqs. (18) to the behaviours of the $\Psi_i(z)$, one finds, in the interval $0 < u < 1$,

$$\begin{aligned}\Psi_1(z-i\epsilon) &= \frac{\sqrt{3}}{\pi}J_2^{(0,1)}(u)\,,\\ \Psi_2(z-i\epsilon) &= -2\sqrt{3}J_1^{(0,1)}(u) + \frac{\sqrt{3}}{\pi}(2\ln 3 - i\pi)J_2^{(0,1)}(u)\,.\end{aligned} \tag{20}$$

One can now compare Eqs. (19) with Eqs. (15), (16) and express the $J_i^{(0,1)}(u)$ in terms of the $\Psi_i^{(1)}(z)$; substituting in Eqs. (20) one finds for the solutions $\Psi_i(z)$, for z around -1, the values

$$\begin{aligned}\Psi_1(z-i\epsilon) &= \frac{9\sqrt{3}}{4\pi}\ln 2\;\Psi_1^{(1)}(z-i\epsilon) - \frac{3\sqrt{3}}{4\pi}\Psi_2^{(1)}(z-i\epsilon)\;,\\ \Psi_2(z-i\epsilon) &= \frac{\sqrt{3}}{4}\left(\frac{18}{\pi}\ln 2\ln 3 - 2\pi - i9\ln 2\right)\Psi_1^{(1)}(z-i\epsilon)\\ &\quad + \frac{3\sqrt{3}}{4\pi}(-2\ln 3 + i\pi)\Psi_2^{(1)}(z-i\epsilon)\;.\end{aligned}$$

One can then move to the next interval $1 < u < 9$ and so on till the $\Psi_i(z)$ are expressed, in the whole range $-\infty < z < \infty$, in terms of the $\Psi_i^{(k)}(z)$, each known within the convergence radius of the expansions given by Eq. (9), as well as in terms of the elliptic integrals $J_i^{(k,l)}(u)$.

6. The integration constants

$S(d,p^2)$ Eq. (1) is known to be real for $u = -p^2$ below the threshold at $u = 9$. Take the solution as given by Euler's formula Eq. (11); in the region $0 < u < 1$, or $0 > z > -1$ the argument w of the inhomogeneous term $N^{(k)}(w)$ varies in the interval $0 > w > z > -1$, and is therefore real ($N^{(k)}(w)$, Eq. (8) involves either real algebraic fractions or lower order terms of the expansion in $(d-2)$ of $S(d,z)$, which are real in that region) — therefore an imaginary part, if any, can come only from the $\Psi_i(z)$ and the $\Psi_i(w)$. By using the values of the $\Psi_i(z)$ as given by Eqs. (20), one finds for $u = -z$ in the range $0 < u < 1$

$$\mathrm{Im}S^{(k)}(2,z) = -\sqrt{3}\Psi_2^{(k)}J_2^{(0,1)}(u)$$

implying, for any k,

$$\Psi_2^{(k)} = 0 \ .$$

The argument can be repeated in the interval $1 < u < 9$ (between pseudothreshold and threshold), where the $\Psi_i(z)$, in analogy with Eqs. (20), are expressed in terms of

$$\begin{aligned} J_1^{(1,9)}(u) &= \int_0^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{-R_4(u,b)}} \,, \\ J_2^{(1,9)}(u) &= \int_{(\sqrt{u}-1)^2}^{4} \frac{db}{\sqrt{R_4(u,b)}} \ . \end{aligned}$$

One finds for z in the interval $-1 > z > -9$, *i.e.* $1 < u < 9$

$$\mathrm{Im} S^{(k)}(2,z) = 3\frac{\sqrt{3}}{\pi} J_1^{(1,9)}(-z) \left(\Psi_1^{(k)} + 2\sqrt{3} \int_0^{-1} \frac{dw}{W(w)} J_1^{(0,1)}(-w) N^{(k)}(2,w) \right) \ ;$$

as the imaginary part must vanish in that interval, the other integration constant is given by

$$\Psi_1^{(k)} = -2\sqrt{3} \int_0^{-1} \frac{dw}{W(w)} J_1^{(0,1)}(-w) N^{(k)}(2,w) \,.$$

7. Conclusions

The values of all the quantities entering in Eq. (11), namely the two $\Psi_i(z)$, $W(z)$ and the two integration constants $K_i^{(k)}$ have been obtained, so that the previously formal expression given by Eq. (11) became a substancial formula giving the functions $S^{(k)}(2,z)$ in closed analytic form. Indeed, from the explicit knowledge of the singularities and the relevant expansions of the $\Psi_i(z)$, the singularities and the relevant expansions of the $S^{(k)}(2,z)$ are immediately obtained — and from the explicit knowledge of the singularities and the relevant expansions of the $S^{(k)}(2,z)$ the fast and precise numerical routines for their evaluation can in turn be obtained.

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