CALCULATION OF FEYNMAN INTEGRALS BY DIFFERENCE EQUATIONS*

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In this paper we describe a method of calculation of master integrals based on the solution of systems of difference equations in one variable. Various explicit examples are given, as well as the generalization to arbitrary diagrams.

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1. Introduction

Calculating Feynman integrals is not easy, especially when diagrams with many loops are considered. It is by now usual to use integration by parts [1,2] identities for reducing a given generic Feynman integral to a linear combination of a small number of irreducible integrals, the so-called "master integrals". For these an explicit calculation is required. In this paper we illustrate a recently developed method of calculation based on the solution of difference equations. An extended exposition of this method and its applications can be found in Refs. [3–6].

2. Difference equations

Probably the reader is more accustomed to objects like differential equations, which express relation between functions and their derivatives. *Difference equations* are a particular kind of functional equations which express relation between values of functions with argument shifted by integers.

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Formally they are identical to "recurrence relations" but, instead of being only a mean to calculate the value of a function from its neighborhoods, they are functional equation in an unknown function; in some sense they may be considered as a "discretized" form of differential equations. In particular, we will encounter *linear* difference equations. Difference equation are a well-known mathematical topic (G. Boole ∼1850, N.E. Norlund 1929, L.M. Milne-Thomson 1933). A good exposition of the theory and methods of solution can be found in the beautiful book [7].

2.1. Fibonacci's numbers

Let us start with a very basic example: the sequence of Fibonacci's integers $I(n) = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$; these numbers satisfy the recurrence relation

$$I(n+2) = I(n+1) + I(n).\tag{1}$$

Now let us think of this relation as a *second*-order linear difference equation with <u>constants</u> coefficients. The general solution of this equation has the form $I(n) = \mu^n$; by substituting it into the recurrence relation one obtains the *characteristic equation*

$$\mu^2 - \mu - 1 = 0$$

from which we work out the values of μ

$$\mu_1 = \frac{1 + \sqrt{5}}{2}, \quad \mu_2 = \frac{1 - \sqrt{5}}{2}.$$

Therefore, the general solution of Eq. (1) is an arbitrary linear combination of the elementary solutions μ_1^n and μ_2^n. In order to reproduce the Fibonacci's sequence of integers we need to fix the constants, by using the initial conditions $I(1) = I(2) = 1$. One finds

$$I(n) = \frac{1}{\sqrt{5}}\mu_1^n - \frac{1}{\sqrt{5}}\mu_2^n.$$

3. One-loop vacuum integral

Now let us jump directly to the calculation of the simplest master integral: the Euclidean integral in D dimensions (with unity mass for simplicity)

$$J(n) = \pi^{-D/2} \int \frac{d^D k}{(k^2 + 1)^n} \equiv \underset{n}{\bigcirc}.\tag{2}$$

By using the identity obtained by integrating by parts [1, 2]

$$\delta_{\mu\nu} \int d^D k \frac{\partial}{\partial k_\nu} \frac{k_\mu}{(k^2 + 1)^n} = 0$$

one finds the recurrence relation

$$(n - 1) \; \bigcirc_{n} \; - (n - 1 - D/2) \; \bigcirc_{n-1} \; = 0 \,, \tag{3}$$

which is a *first*-order difference equation with polynomial coefficients for the function J. A solution of Eq. (3) can be written in the form of an *expansion in factorial series*:

$$
\begin{aligned}
J(n) &= \mu^n \sum_{s=0}^{\infty} a_s \frac{\Gamma(n + 1)}{\Gamma(n - K + s + 1)} \\
&= \mu^n \frac{\Gamma(n + 1)}{\Gamma(n - K + 1)} \left[a_0 + \frac{a_1}{n - K + 1} + \frac{a_2}{(n - K + 1)(n - K + 2)} + \dots \right],
\end{aligned}
\tag{4}
$$

where μ, K, a_s are to be determined.

3.0.1. Factorial series

Factorial series play for difference equations the same role that power series play for differential equations [7]. Factorial series of *first kind* like Eq. (4) are similar to asymptotic expansions in $1/n$, but with the big advantage of be convergent! For example, let us consider the second derivative of the Gamma function

$$\Psi'(n) \equiv \frac{d^2}{dn^2} \ln \Gamma(n) \,.$$

It satisfies the nonhomogeneous first order difference equation

$$\Psi'(n + 1) - \Psi'(n) = -\frac{1}{n^2} \,.$$

If $\Psi'(n)$ is expanded in power series of $1/n$, one obtains an *asymptotic* series, not convergent for any value of n,

$$\Psi'(n) = \frac{1}{n} + \frac{1}{2n^2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{n^{2k+1}} \,, \qquad B_{2k} \approx \frac{2(2k)!}{(2\pi)^{2k}} \quad \text{if } k \to \infty \,,$$

where B_{2k} are the Bernoulli's numbers; but if one expands it in factorial series, one obtains a series comfortably convergent for $n > 0$:

$$\Psi'(n) = \sum_{s=1}^{\infty} \frac{\Gamma(s)}{s} \frac{\Gamma(n)}{\Gamma(n + s)} \,.$$

3.1. Solution of the difference equation

By substituting the expansion (4) into the difference equation (3) (by using the Boole's operators [7]) one finds the values $\mu = 1$ and $K = -D/2$, and the recurrence relation between the coefficients a_s

$$sa_s = a_{s-1}\left(s + \frac{D}{2}\right)\left(s + \frac{D}{2} - 1\right).$$

The first coefficient a_0 is determined by comparing the large-n behaviors of the integral (2) and the factorial series (4):

$$\pi^{-D/2}\int \frac{d^D k}{(k^2 + 1)^n} \approx \pi^{-D/2}\int d^D k\, e^{-nk^2} = n^{-D/2},$$

$$\mu^n \sum_{s=0}^{\infty} \frac{a_s \Gamma(n+1)}{\Gamma(n - K + s + 1)} \approx a_0 n^K \mu^n = a_0 n^{-D/2} \quad \Rightarrow \quad a_0 = 1.$$

For $s \to \infty$ the term of the factorial series behaves as

$$\frac{a_s}{\Gamma(n + D/2 + s + 1)} \approx \frac{s!\, s^{D-1}}{s!\, s^{D/2+n}} = s^{-1+D/2-n};$$

$n = D/2$ is the *abscissa of convergence* of the factorial series, in fact:

- if $n > D/2$ the series is convergent, and it converges quickly for $n \gg 1$;

- if $n \le D/2$ the series does not converge, but we can calculate $J(n+i)$ for some large integer i by using the series, and then to use repeatedly the recurrence relation (3) in order to obtain $J(n+i-1)$, $J(n+i-2)$, ..., $J(n)$.

4. On-mass-shell self-mass master integral

$$I(n) = \pi^{-D/2}\int \frac{d^D k}{(k^2 + 1)^n (k^2 - 2p \cdot k)} \equiv \underset{n}{\bigcirc}.$$

By combining identities obtained integrating by parts one finds a *nonhomogeneous second* order difference equation

$$(n - D)\underset{n-2}{\bigcirc} + (2n - D - 1)\underset{n-1}{\bigcirc} - 3(n-1)\underset{n}{\bigcirc} = (1 - \tfrac{D}{2})\underset{n-1}{\bigcirc}. \quad (5)$$

The general solution of this equation has the form

$$I(n) = C_1 I_1(n) + C_2 I_2(n) + I_3(n),$$

where I_1 and I_2 are solutions of the homogeneous equation, C_1 and C_2 arbitrary constants, and I_3 is a particular solution of the nonhomogeneous equation. As before, we look for solutions in the form of expansions in factorial series

$$I_j(n) = \mu_j^n \sum_{s=0}^{\infty} \frac{b_s^{(j)} \Gamma(n+1)}{\Gamma(n - K_j + s + 1)} , \quad j = 1, 2, 3 . \tag{6}$$

By substituting Eq. (6) into Eq. (5) one finds the values

$$\mu_1 = 1, \qquad\qquad \mu_2 = -1/3, \qquad\qquad \mu_3 = 1,$$
$$K_1 = -D/2 + 1/2, \qquad K_2 = -D/2 + 1/2, \qquad K_3 = -D/2 ,$$

and the recurrence relations

$$4s b_s^{(1)} - (\bar{s} - \tfrac{1}{2})(7\bar{s} - D + \tfrac{1}{2}) b_{s-1}^{(1)} + 3(\bar{s} - \tfrac{3}{2})(\bar{s} - \tfrac{1}{2})^2 b_{s-2}^{(1)} = 0 , \tag{7}$$

$$(4s+2) b_s^{(3)} - \bar{s}(7\bar{s} - D + 4) b_{s-1}^{(3)} + 3\bar{s}^2(\bar{s} - 1) b_{s-2}^{(3)} = (\tfrac{D}{2} - 1)(a_s - \bar{s} a_{s-1}) , \tag{8}$$

where $\bar{s} = s + \frac{D}{2} - 1$, $a_i = b_i = 0$ for $i < 0$ and $b_0^{(1)} = b_0^{(2)} = 1$.

The constants C_1 and C_2 are determined by comparing the behaviors for $n \to \infty$, $I_1(n) \approx n^{-D/2+1/2}$, $I_2(n) \approx (-1/3)^n n^{-D/2+1/2}$ and $I_3(n) \approx (1/2 - D/4) n^{-D/2}$ with

$$\underset{n}{\bigcirc} \approx \pi^{-D/2} \int \frac{d^D k \ \mathrm{e}^{-nk^2}}{k^2 - 2p \cdot k} \approx \frac{\sqrt{\pi}}{2} n^{-D/2+1/2} ;$$

one finds $C_1 = \frac{\sqrt{\pi}}{2}$, $C_2 = 0$, therefore the solution I_2 can be discarded. As in the previous case, the factorial series converge quickly for $n \gg 1$, and the recurrence relations are used for the values of n such that the series converges slowly or does not converge.

5. General case

Now we sketch the generalization of the method used above to the calculation of master integrals of arbitrary diagrams. Given an L-loop integral, we put the exponent n on the first denominator (arbitrarily chosen)

$$B(n) = \pi^{-DL/2} \int d^D k_1 \ldots d^D k_L \frac{N}{D_1^n D_2 D_3 \ldots D_{N_d}} ;$$

the numerator N in general contain products of powers of some irreducible scalar products involving loop momenta. By combining by suitable algorithms the identities obtained by integration by parts

$$\delta_{\mu\nu} \int d^D k_1 \ldots d^D k_L \frac{\partial}{\partial k_{i\nu}} \frac{\{k_{1\mu}; k_{2\mu}; \ldots; p_{1\mu}; \ldots\} N_j}{D_1^n D_2^{\alpha_2} D_3^{\alpha_3} \ldots D_L^{\alpha_L}} = 0$$

one finds a nonhomogeneous difference equation of order r (typically $r \sim 2$–10)

$$p_0(n)B(n) + p_1(n)B(n+1) + \ldots + p_r(n)B(n+r) = T(n);$$

the right-hand side $T(n)$ contains new integrals B_1, B_2,... which have one or more denominators missing with respect to B, as an example

$$B_1(n) = \pi^{-DL/2} \int d^D k_1 \ldots d^D k_L \frac{N}{D_1^n D_3 D_4 \ldots D_{N_d}},$$

and which are simpler. By building difference equations for B_1, B_2, ... one obtains a system of difference equations in triangular form. The system is solved one equation at once, beginning with that corresponding to the simplest master integral, with the minimum number of denominators. The solutions

$$B_l(n) = C_1 I_1^{\mathrm{OM}}(n) + C_2 I_2^{\mathrm{OM}}(n) + \ldots C_r I_r^{\mathrm{OM}}(n) + I^{\mathrm{NO}}(n)$$

are expanded in series

$$I_j^{\mathrm{OM}}(n) = \mu_j^n \sum_{s=0}^{\infty} \frac{a_s^{(j)} \Gamma(n+1)}{\Gamma(n - K_j + s + 1)},$$

$$I^{\mathrm{NO}}(n) = \sum_{j=1}^{S} (\mu_j^{\mathrm{NO}})^n \sum_{s=0}^{\infty} \frac{a_s^{(j,\mathrm{NO})} \Gamma(n+1)}{\Gamma(n - K_j^{\mathrm{NO}} + s + 1)};$$

μ_j^{NO}, K_j^{NO} and $a_s^{(j,\mathrm{NO})}$ are obtained from the expansion of $T(n)$ which is supposed to be known. By substituting the expansions into the difference equation one finds the characteristic equation and the values of $\mu_1, \mu_2, \ldots, \mu_r$, the r indicial equations and the values of $K_1, \ldots, K_r$, and the recurrence relations for the coefficients $a_s^{(j)}$ which have usually order higher than Eqs. (7)–(8)(~ 10–20). By comparing the behaviors for $n \to \infty$ the values of the constants C_1, C_2, ..., C_r are obtained (usually most of constants are null). For example, the large-n behavior of an Euclidean massive integral is:

$$B(n) \approx \left(\frac{1}{m_1^2}\right)^{n-D/2} n^{-D/2} F(0),$$

where $D_1 = k_1^2 + m_1^2$, $m_1 \neq 0$. $F(0)$ is the value of the integral corresponding to the diagram with one loop less obtained by setting $k_1 = 0$, graphically

$$\approx \left(\frac{1}{m_1^2}\right)^{n-D/2} n^{-D/2}$$

.

6. System of difference equations in the single-scale case

To give to the reader an idea of how the difference equations for more complicated diagrams look, we list here the whole system of difference equations for the two-loop self-mass diagram in the particular case with all masses equal to 1 and on-shell external momentum. As above, the point on a line means that the corresponding denominator is raised to the power indicated. The factorizable two-loop subdiagrams are decomposed into the product of one-loop diagrams.

$$(n-1) \quad - (n-1-D/2) \quad = 0 \,,$$

$$(n-D) \quad + (2n-D-1) \quad - 3(n-1) \quad = (1-\tfrac{D}{2}) \,,$$

$$(n-D) \quad + (2n-D-1) \quad - 3(n-1)$$

$$= (2-D) \quad \times \quad ,$$

$$(n-D)(2n-3D+2) \quad + 2[7n^2 - n(10D+1) + 3D^2 + 3D - 2]$$

$$-16(n-1)(n-D+1) \quad = 2(D-2)^2 \quad \times \quad ,$$

$$(2n-3D+4) \quad + (4n-D-2) \quad - 6(n-1)$$

$$= -(D-2) \quad - 2(D-2) \quad \times \quad ,$$

$$48(n-D+1)\left[(n-D)\,\underset{n-2}{\text{[diagram]}}+(2n-D-1)\,\underset{n-1}{\text{[diagram]}}-3(n-1)\,\underset{n}{\text{[diagram]}}\right]$$

$$=-5(n-D)(2n-3D+2)\,\underset{n-2}{\text{[diagram]}}-[-10n^2+n(28D-58)-18D^2$$

$$+46D+12]\,\underset{n-1}{\text{[diagram]}}-(D-2)(-32n+6D+52)\,\underset{n-1}{\text{[diagram]}}\times\text{[diagram]},$$

$$48(n-D+1)\left[(n-D)\,\overset{n-2}{\text{[diagram]}}+(2n-D-1)\,\overset{n-1}{\text{[diagram]}}-3(n-1)\,\overset{n}{\text{[diagram]}}\right]$$

$$=-5(n-D)(2n-3D+2)\,\underset{n-2}{\text{[diagram]}}-[-10n^2+n(28D-58)-18D^2+46D$$

$$+12]\,\underset{n-1}{\text{[diagram]}}-24(D-2)(n-D+1)\,\underset{n-1}{\text{[diagram]}}\times\text{[diagram]}$$

$$-2(D-2)(8n-9D+2)\,\underset{n-1}{\text{[diagram]}}\times\text{[diagram]},$$

$$36(n-D+1)\left[2(n-D+1)\,\overset{n-2}{\text{[diagram]}}+(2n-D)\,\overset{n-1}{\text{[diagram]}}-4(n-1)\,\overset{n}{\text{[diagram]}}\right]$$

$$=+12(n-D+1)\left[(2n-3D+4)\,\overset{n-2}{\text{[diagram]}}-2(n-D+1)\,\overset{n-1}{\text{[diagram]}}\right.$$

$$-(n-D)\,\underset{n-2}{\text{[diagram]}}+(n-2D+4)\,\underset{n-1}{\text{[diagram]}}-(n-D)\,\overset{n-2}{\text{[diagram]}}$$

$$\left.+(n-2D+4)\,\underset{n-1}{\text{[diagram]}}\right]-(n-D)(2n-3D+2)\,\underset{n-2}{\text{[diagram]}}$$

$$-(-2n^2-16nD+46n+18D^2-70D+60)\,\underset{n-1}{\text{[diagram]}}$$

$$+12(n-D+1)\left[-(n-D)\,\underset{n-2}{\text{[diagram]}}\times\text{[diagram]}+(n-2D+4)\,\underset{n-1}{\text{[diagram]}}\times\text{[diagram]}\right.$$

$$+(D-2)\,\underset{n-1}{\text{[diagram]}}-(D-2)\,\underset{n-1}{\text{[diagram]}}\times\text{[diagram]}-(D-2)\,\underset{n-1}{\text{[diagram]}}\times\text{[diagram]}\right]$$

$$+2(D-2)(8n-3D-10)\,\underset{n-1}{\text{[diagram]}}\times\text{[diagram]}.$$

7. The program SYS

A generic integral, through the solution of a system of algebraic identities between integrals obtained integrating by parts, is transformed into a linear combination of a small number of "master integrals", chosen typically with numerator unity and denominators raised to power one.

The method of calculation based on the solution of finite difference equations is applied only to these "master integrals".
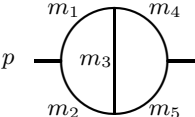
Considered the number and the complexity of the systems of difference equations for diagrams with more than 2 loops, we have written on purpose a comprehensive program, called SYS, with the following features:

- C program, about 23000 lines.

- The program automatically determines the master integrals of a diagram, it builds and solves the systems of difference equations.

- Input: description of the diagram, number of terms of the expansion in $D - 4$.

- The program contains a simplified algebraic manipulator, used to solve systems of identities among integrals with this kind of coefficients: arbitrary precision integers, rationals, ratios of polynomials in one and two variables (for example D and x) with integer coefficients.

- Efficient management of systems of identities of size up to the limit of disk space (tested up to 5×10^6 identities).

- Numerical solution of systems of difference and differential equations up to 500 equations, using arbitrary precision floating point complex numbers and truncated series in ε.

- All the coefficients of the expansions in ε are worked out in numerical form, even those of divergent terms.

- Floating number precision: up to 1200 digits (see [6]) (essentially one sums expansions in *one* variable).

- Arithmetic libraries which deal with operations on integers, polynomials, rationals, floating point numbers and truncated series in ε were written on purpose by the author.

8. Applications of the method

1. High-precision calculation of vacuum (2–4 loops), self-mass (1–3 loops), vertex and box diagrams (1–2 loops) in the case of equal masses and on-shell external momenta [3, 4, 6].

2. High-precision calculation of electron $g - 2$ master integrals, at 3 loops [5] and at 4 loops [8] (actually only a few, work in progress).

3. High-precision calculation of integrals for *arbitrary* values of mass and momenta (work in progress).

Here is an example of high-precision (60 digits) result obtained with the program SYS: the expansion in $\varepsilon = (4 - D)/2$ of the following two-loop self-mass integral with these values of masses and momenta: $p^2 = -1$, $m_1^2 = 2$, $m_2^2 = 3$, $m_3^2 = 4$, $m_4^2 = 5$, $m_5^2 = 6$,

$$= \Gamma(1 + \varepsilon)^2 \big[$$

$$+0.213887190522735619987564582191165767549812066097826018573$$
$$-0.918487358981510992237303382863013109752582619839173727799\varepsilon$$
$$+2.395114573687568040190458275515297803861359807588836500079\varepsilon^2$$
$$-5.204471820133273025267916826530760950863731824065596549403\varepsilon^3$$
$$+10.627554296376428061053900350381893577369805111976033879388\varepsilon^4 + O(\varepsilon^5)\big].$$

9. Conclusions

1. Method of calculation applicable to diagrams with any topology.

2. The calculation of an integral is reduced to the sum of convergent series in *one* variable. As a consequence, the number of operations needed to obtain a precision of the results of N digits grows linearly with N and *does not depend on the complexity of the diagram*; calculations with $N \sim 10^3$ are still feasible [6].

3. Easiness of cross-checks, due to possibility of writing different difference equations for each topologically distinct line of a given diagram.

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