CASIMIR EFFECTS: FROM THE TABLETOP TO THE STANDARD MODEL*

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Vacuum fluctuations distinguish quantum field theory from non-relativistic quantum mechanics. The phenomena that result from the modification of vacuum fluctuations by external fields or boundary conditions are known as Casimir effects. The study of Casimir effects has been plagued by divergences. In these lectures I describe the framework my collaborators and I have developed for the study of Casimir effects in the context of renormalizable quantum field theories, where divergences can be regulated, analyzed, and, for properly defined observables, removed. I discuss several examples: first a model in which quantum fluctuations stabilize a soliton, next, the physically important case of the Standard Model, where no quantum stabilized soliton has yet been discovered, and finally the "classic" Casimir effect.

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1. Introduction

These lectures, describing some new methods and new results on an old subject in quantum field theory[1]. First I describe a program to develop reliable, accurate, and efficient techniques for a variety of calculations in

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[1] The work described here is the product of a large collaboration including Eddie Farhi, Noah Graham, Peter Haagensen, Vishesh Khemani, Markus Quandt, Marco Scandurra, Oliver Schroeder and Herbert Weigel. Much of the material has been drawn from a review by Graham, Weigel, and me [1].The material on solitons in the Standard Model in §5 is largely based on the Ph. D. thesis of Vishesh Khemani.

renormalizable quantum field theories in the presence of background fields. These background field configurations need not be solutions of the classical equations of motion. Our calculations are exact to one loop, allowing us to proceed where perturbation theory or the derivative expansion would not be valid. For example, in a model with no classical soliton we can demonstrate the existence of a nontopological soliton stabilized at one-loop order by quantum fluctuations. We renormalize divergences in the conventional way: by combining counter-terms with low-order Feynman diagrams and satisfying renormalization conditions in a fixed scheme. In this way we are certain that the theory is being held fixed as the background field is varied. Our methods are also efficient and practical for numerical computation: the quantities entering the numerical calculation are cutoff independent and do not involve differences of large numbers. The numerical calculations themselves are highly convergent.

We were originally drawn into this subject by our wish to understand what becomes of a fermion in the Standard Model when its coupling to the Higgs becomes large. There are strong reasons to believe that it does not decouple, and some indications that its character changes from a point-like object to a soliton-like object stabilized by fermion quantum fluctuations. I will describe our attempt — so far unsuccessful — to find this object.

A boundary condition can be viewed as an extreme limit of the coupling of a fluctuating field to a static background. Thus our method can be adapted to the study of quantum fields in the presence of boundaries. As a simple example I examine the effect of trying to impose a Dirichlet boundary condition on a scalar field by coupling it to a static background. The zero point — or Casimir — energy of the field diverges in the limit that the background forces the field to vanish. This divergence cannot be absorbed into a renormalization of the parameters of the theory. As a result, the Casimir energy of a surface on which a Dirichlet boundary condition is imposed, and other quantities like the surface tension, which are obtained by deforming the surface, depend on the physical cutoffs that characterize the coupling between the field and the matter on the surface. In contrast, the energy density away from the surface and forces between rigid surfaces are finite and independent of these complications

Our methods are limited to one loop and, except in special cases, to static field configurations. We also require the background field configuration to have enough symmetry that the associated scattering problem admits a partial wave expansion. The one-loop approximation includes all quantum effects at order $\hbar$. It is a good approximation for strong external fields or when the number of particles circulating in the loop becomes large. It is exact for the classic Casimir effect where, apart from the boundary condition, the fluctuating field does not interact. Even when it cannot be rigorously justified, the one-loop approximation can provide insight into novel structures in the same way that classical solutions to quantum field theories have done in the past.

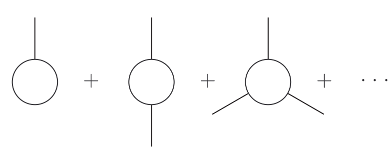
Here, time permits only the briefest introduction to our methods and a couple of examples of applications. In Section 2 I describe our method in general terms and illustrate the method with the case of a charged boson field in a bosonic background in three spatial dimensions. In Section 3 I show how the methods of dimensional regularization can be adapted to renormalize our calculations. I work in n space dimensions and show that the leading terms in the Born expansion, which diverge for integer n, can be unambiguously identified with Feynman diagrams. This approach resolves several longstanding ambiguities in Casimir calculations. In Sections 4–6 I describe three applications: In Section 4 I consider a chiral model in one space dimension and show that quantum effects of a heavy fermion can stabilize a soliton that is not present in the classical theory. In Section 5 I summarize recent work on the Standard Model. Finally, in Section 6, I give a simple, pedagogical example of the application of our methods to the "Dirichlet–Casimir" problem: the energy of a scalar field subject to the boundary condition $\phi = 0$.

2. Overview

For simplicity consider a fluctuating boson or fermion field of mass m in a static, spherically symmetric background potential $\chi(r)$ in three dimensions. Since we encounter divergences, imagine that we have analytically continued to values of the space dimension n where the integrals are convergent. Later we provide the rigorous justification for this procedure.

We take the interaction Lagrangian $\mathcal{L}_I = g\overline{\psi}\chi\psi$ for fermions ($g\psi^\dagger\chi\psi$ for bosons) where ψ is the fluctuating field. We want to compute the one-loop "effective energy," the effective action per unit time. It is given either by the sum of all one-loop diagrams with all insertions of the background $\chi(r)$, or

$$\Delta E_{\text{bare}}[\chi] =$$



(1)

equivalently by the "Casimir sum" of the shifts in the zero-point energies of all the small oscillation modes in the background χ,

$$\Delta E_{\text{bare}}[\chi] = \pm\frac{1}{2}\sum_j |\varepsilon_j| - |\varepsilon_j^0| \tag{2}$$

for bosons (+) and fermions (−), respectively. Both of these representations are divergent and require renormalization. We start from the second expression and work in the continuum. We rewrite the Casimir sum as a sum over bound states plus an integral over scattering states, weighted by the density of states $\rho(k)$. We subtract from the integral the contribution of the trivial background, which is given by the free density of states $\rho^0(k)$. Thus we have

$$\Delta E_{\mathrm{bare}}[\chi] = \pm \left(\frac{1}{2} \sum_j |\,\omega_j| + \frac{1}{2} \int_0^\infty \omega(k) \left(\rho(k) - \rho^0(k)\right) dk \right) \tag{3}$$

ω_j denotes the energy of the j^{th} bound state, and $\omega(k) = \sqrt{k^2 + m^2}$.

The density of states is related to the S-matrix and the phase shifts by

$$\rho(k) - \rho^0(k) = \frac{1}{2\pi i} \frac{d}{dk} \operatorname{Tr} \ln S(k) = \sum_\ell D^\ell \frac{1}{\pi} \frac{d\delta_\ell(k)}{dk}\,, \tag{4}$$

where ℓ labels the basis of partial waves in which S is diagonal. D^ℓ is the degeneracy factor. For example, $D^\ell = 2\ell+1$ for a boson in three dimensions. It is convenient to use Levinson's theorem to express the contribution of the bound states to Eq. (3) in terms of their binding energy. Levinson's theorem relates the number of bound states to the difference of the phase shift at $k = 0$ and ∞,

$$n_\ell^{\mathrm{bound}} = \frac{1}{\pi}(\delta_\ell(0) - \delta_\ell(\infty)) = -\int_0^\infty dk\, \frac{d\delta_\ell(k)}{dk}\,. \tag{5}$$

Subtracting mn_ℓ^{bound} from the sum over bound states in Eq. (3) and using Eqs. (5) and (4), we obtain

$$\Delta E_{\mathrm{bare}}[\chi] = \pm \left(\frac{1}{2} \sum_{j,\ell} D^\ell (|\,\omega_{j,\ell}| - m) + \int_0^\infty \frac{dk}{2\pi} \left(\omega(k) - m\right) \sum_\ell D^\ell \frac{d\delta_\ell(k)}{dk} \right), \tag{6}$$

where the sum over partial waves is to be performed before the k integration. While the phase shifts and bound state energies are finite, $\Delta E_{\mathrm{bare}}[\chi]$ is divergent because the k-integral and ℓ-sum both diverge in the ultraviolet. To better understand the origin and character of the divergences, we go back to the diagrammatic representation of the vacuum energy, Eq. (1). Since we are working with a renormalizable theory, only the first few diagrams are divergent, and these divergences can be canceled by a finite number of counter-terms. The series of diagrams gives an expansion of the effective energy in powers of the background field $\chi(r)$. Likewise, the phase shift calculation can be expanded in powers of $\chi(r)$ using the Born series,

$$\delta_\ell^N(k) = \sum_{i=1}^{N} \delta_\ell^{(i)}(k)\,, \tag{7}$$

where $\delta_\ell^{(i)}(k)$ is the contribution to the phase shift at order i in the potential $\chi(r)$. In general, the Born expansion is a poor approximation at small k, especially if the potential has bound states, when it typically does not converge. What is important for us, however, is that the contributions to $\Delta E_{\rm bare}$ from successive terms in the Born series correspond exactly to the contributions from successive Feynman diagrams. That is, the $i^{\rm th}$ term in the Born series generates a contribution to the vacuum energy which is exactly equal to the contribution of the Feynman diagram with i external insertions of χ.

This correspondence is not trivial in light of divergences. We have verified the identification for the lowest-order diagram by direct comparison in n space dimensions where both are finite. At this order the Born and Feynman contributions to $\Delta E_{\rm bare}[\chi]$ are precisely equal as analytic functions of n as we will show in Section 3. We have also performed various numerical checks to verify the identification in higher orders.

We then define the subtracted phase shift

$$\overline{\delta}_\ell^{\,N}(k) = \delta_\ell(k) - \delta_\ell^N(k)\,, \tag{8}$$

where we take N to be the number of divergent diagrams in the expansion of Eq. (1). The effect of the Born subtraction is illustrated in Fig. 1. Note that the subtracted phase shift is large at small k, so the Born approximation is very different from the true phase shift in this region. However the Born approximation becomes good at large k, so that the subtracted phase shift vanishes quickly as $k \to \infty$.

Having subtracted the potentially divergent contributions to $\Delta E_{\rm bare}[\chi]$ via the Born expansion, we add back in exactly the same quantities as Feynman diagrams, $\sum_{i=1}^{N} \mathit{\Gamma}_{\rm FD}^{(i)}[\chi]$. We combine the contributions of the diagrams with those from the counter-terms, $\Delta E_{\rm CT}[\chi]$, and apply standard perturbative renormalization conditions. We have thus removed the divergences from the computationally difficult part of the calculation and re-expressed them as Feynman diagrams, where the regularization and renormalization have been carried out with conventional methods. This approach to renormalization in strong external fields was first introduced by Schwinger [2] in his work

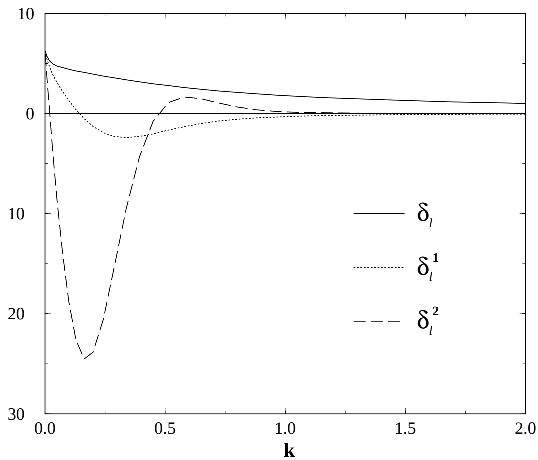


Fig. 1. Typical phase shift in three dimensions, before and after subtracting the Born approximation.

on QED in strong fields. Combining the renormalized Feynman diagrams,

$$\overline{\Gamma}^N_{\mathrm{FD}}[\chi] = \sum_{i=1}^{N} \Gamma^{(i)}_{\mathrm{FD}}[\chi] + \Delta E_{\mathrm{CT}}[\chi] \tag{9}$$

with the subtracted phase shift calculation, we obtain the complete, renormalized, one-loop effective energy,

$$\Delta E[\chi] = \pm\frac{1}{2}\sum_{\ell} D^{\ell}\left(\sum_{j}(|\omega_{j,\ell}| - m) + \int_0^\infty \frac{dk}{\pi}(\omega(k) - m)\frac{d}{dk}\overline{\delta}_\ell^{N}(k)\right) + \overline{\Gamma}^N_{\mathrm{FD}}[\chi]\,, \tag{10}$$

where the two pieces are now separately finite. Since the k integral is now convergent, we are free to interchange it with the sum over partial waves or to integrate by parts. This expression is suitable for numerical computation, since it does not contain differences of large numbers. The massless limit is also smooth, except for the case of one spatial dimension, where we expect incurable infrared divergences [3].

Eq. (10) summarizes our approach. The first line is unfamiliar to particle physicists. It sums all orders in the background field χ. Though it must be computed numerically, it is finite, unambiguous, and regulator independent. We have developed efficient methods to compute phase shifts and the Born approximation [1, 4, 5]. All the potential divergences are isolated in certain low-order Feynman diagrams, where they are canceled by counter-terms in the time honored fashion of renormalizable quantum field theory.

3. Dimensional regularization

The identification of terms in the Born series with Feynman diagrams is crucial. No arbitrariness can be tolerated in the renormalization process: if the manipulations of formally divergent quantities introduce finite ambiguities, our method is useless. There have been controversies for many years concerning how to renormalize Casimir calculations. However, we are studying a renormalizable quantum field theories. So we know that the effective energy can be calculated unambiguously. In this section, I apply the methods of dimensional regularization to scattering from a central potential and prove that the lowest-order term in the Born series is equal to the lowest-order Feynman diagram as an analytic function of n, the number of space dimensions. Since this is the most divergent diagram — quadratically divergent for $n = 3$ — we are confident that the same method will regulate all other divergences in the effective energy unambiguously.

For simplicity, I will consider the fluctuations of a single real boson, $\phi(x)$, coupled to a static, spherically symmetric background, $\chi(r)$, by $\mathcal{L}_I = |\phi(x)|^2\chi(r)$. The generalization to fermions is discussed in Ref. [6]. For $n = 1$, $\chi(r)$ reduces to a symmetric potential with even and odd parity channels. For $n \neq 1$ the S-matrix is diagonal in the basis of the irreducible tensor representations of SO(n). These are the traceless symmetric tensors of rank ℓ, where $\ell = 0, 1, 2, \ldots$. We choose a value of n between 0 and 1, where all the integrals and sums that appear in the Casimir energy converge. It is not difficult to formulate scattering theory in partial waves in n dimensions, to define phase shifts, $\delta_{n\ell}(k)$, Born approximations, $\delta^{(i)}_{n\ell}(k)$, and the density of states, $\sum_\ell \frac{1}{\pi} D^\ell_n \, d\delta_{n\ell}/dk$, where D^ℓ_n is the degeneracy of the SO(n) representation labeled by ℓ.

The first Born approximation to the phase shift is

$$\delta^{(1)}_{n\ell}(k) = -\frac{\pi}{2}\int_0^\infty \left[J_{\frac{n}{2}+\ell-1}(kr)\right]^2 \chi(r) r \, dr \tag{11}$$

and its contribution to the Casimir energy is

$$\Delta E^{(1)}_n[\chi] = \int_0^\infty \frac{dk}{2\pi}(\omega(k) - m)\sum_{\ell=0}^\infty D^n_\ell \frac{d\delta^{(1)}_{n\ell}(k)}{dk}\,. \tag{12}$$

Using the Bessel function identity

$$\sum_{\ell=0}^\infty \frac{(2q+2\ell)\Gamma(2q+\ell)}{\Gamma(\ell+1)} J_{q+\ell}(z)^2 = \frac{\Gamma(2q+1)}{\Gamma(q+1)^2}\left(\frac{z}{2}\right)^{2q} \tag{13}$$

with $q = \frac{n}{2} - 1$, we sum over ℓ in Eq. (12) and obtain

$$\Delta E_n^{(1)}[\chi\,] = \frac{\langle\chi\rangle(2-n)}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int\limits_0^\infty (\omega(k) - m)k^{n-3}\,dk = \frac{\langle\chi\rangle m^{n-1}}{(4\pi)^{\frac{n+1}{2}}}\Gamma\Big(\frac{1-n}{2}\Big) \quad (14)$$

which converges for $0 < n < 1$. Here $\langle\chi\rangle$ is the n-dimensional spatial average of $\chi(r)$,

$$\langle\chi\rangle = \int \chi(x)\,d^n x = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int\limits_0^\infty \chi(r) r^{n-1} dr\,. \quad (15)$$

The tadpole diagram is easily computed using dimensional regularization, and the result agrees precisely with Eq. (14). Thus we can be certain that our method of subtracting the first Born approximation and adding back the corresponding Feynman diagram is correct.

4. Chiral model in one dimension

As a first application of our method, I show how a quantum soliton can appear in a theory with a heavy fermion. We consider a one-dimensional chiral model in which the fermion gets its mass from its coupling to a scalar condensate. It is easy to find a spatially varying scalar background which has a tightly bound fermion level. If the classical energy of the background field plus the energy of the tightly bound fermion is less than the free fermion mass m, this configuration would appear to be a stable soliton, since it is unable to decay into free fermions. However, the energy of the lowest fermion level enters at the same order in $\hbar$ as the full one-loop fermion effective energy, since the latter simply corresponds to the shift of the zero-point energies, Eq. (2), of all the fermion modes. The question of stability can therefore only be addressed by computing the full one-loop effective energy. Here I summarize our analysis of this system and show that it supports stable solitons. More details of this calculation can be found in Ref. [7].

4.1. The model

We consider a chiral model in one dimension with a symmetry-breaking scalar potential. We couple a two-component real boson field $\vec{\phi} = (\phi_1, \phi_2)$ chirally to a fermion Ψ

$$\mathcal{L} = \tfrac{1}{2}\,\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - V(\vec{\phi}\,) + \bar{\Psi}\left\{i\partial\!\!\!/ - G\left(\phi_1 + i\gamma_5\phi_2\right)\right\}\Psi\,, \quad (16)$$

where the potential for the boson field is given by

$$V(\vec{\phi}) = \frac{\lambda}{8}\left[\vec{\phi}\cdot\vec{\phi} - v^2 + \frac{2\alpha v^2}{\lambda}\right]^2 - \alpha v^3\left(\phi_1 - v\right) + \text{const.} \quad (17)$$

$V(\vec{\phi}\,)$ has its minimum at $\vec{\phi} = (v, 0)$. Terms proportional to α break the chiral symmetry explicitly. If α we set to zero, the chiral symmetry appears to break spontaneously, but quantum fluctuations in one dimension restore the symmetry [3]. For large enough α, the classical vacuum $\vec{\phi} = (v, 0)$ is stable against quantum corrections and $m = Gv$ is the fermion mass. The coefficient c in the counter-term Lagrangian

$$\mathcal{L}_{\rm CT} = c\left(\vec{\phi}\cdot\vec{\phi} - v^2\right) \tag{18}$$

is fixed by the condition that the quantum corrections do not change the VEV of $\vec{\phi}$. This model has no stable soliton solutions at the classical level.

We are interested in the mass of the lightest state carrying unit fermion number. If its mass is less than m, this state is a stable soliton. We neglect boson loops, so that the effective energy is given by the sum of the classical and the fermion loop contributions, $E_{\rm tot}[\vec{\phi}\,] = E_{\rm cl}[\vec{\phi}\,] + E_f[\vec{\phi}\,]$. This approximation is exact in the limit where the number of independent fermion species becomes large. The fermion contribution to the effective energy is $E_f = E_{\rm Cas} + E_{\rm val}$, where $E_{\rm Cas}$ is the sum over zero-point energies, calculated with the methods we have developed. $E_{\rm val}$ is the energy required for the soliton to have unit charge. Using the methods of the previous section, we can calculate the fermion number of the background field [7,8]. If a level has crossed zero, then the background field will already carry the required fermion number and $E_{\rm val} = 0$. If the background field has zero charge, we must explicitly fill the most tightly bound level, giving $E_{\rm val} = \varepsilon_0$, where ε_0 is the energy of that level.

The scattering theory formalism must be extended to handle fermions. We choose backgrounds which preserve parity symmetry so the phase shifts can be labeled by the parity, $\delta_\pm$. Also we must sum contributions from particles and antiparticles, or in the context of the single particle Dirac equation, from positive and negative energies. So we define

$$\delta_{\rm F}(k) = \delta_+(\omega(k)) + \delta_+(-\omega(k)) + \delta_-(\omega(k)) + \delta_-(-\omega(k))\,. \tag{19}$$

Renormalization is particularly simple in this model. The first and second Born approximations corresponding to the Feynman diagrams with one and two insertions of $[\vec{\phi} - (v, 0)]$ diverge. However, the divergences are related by chiral symmetry. Both are canceled by a counter-term proportional to $\vec{\phi}^{\,2} - v^2$. It suffices to subtract the first Born approximation to $\delta(k)$ and the part of the second related to it by chiral symmetry,

$$\delta^{(1)}(k) = \frac{2G^2}{k}\int_0^\infty dx\left(v^2 - \vec{\phi}^{\,2}(x)\right)\,. \tag{20}$$

The condition that the VEV of $\vec{\phi}$ does not get renormalized requires that the counter-term exactly cancel the Feynman diagrams that are added back in compensation for the Born subtractions. Thus we have

$$E_{\rm Cas}[\vec{\phi}] = -\frac{1}{2}\sum_j (|\,\omega_j| - m) - \int_0^\infty \frac{dk}{2\pi}\,(\omega(k) - m)\,\frac{d}{dk}\left(\delta_{\rm F}(k) - \delta^{(1)}(k)\right)\,.$$

4.2. Numerical studies

We consider variational *ansätze* for the background field. As $x \to \pm\infty$, $\vec{\phi}$ must go to its vacuum value, $(v, 0)$. We find that energetically favored configurations execute a loop in the (ϕ_1, ϕ_2) with radius $R > v$ so that they enclose the origin. A simple *ansatz* with these properties is

$$\phi_1 + i\phi_2 = v\left\{1 - R + R\,\exp\left[i\pi\left(1 + \tanh\left(\frac{Gvx}{w}\right)\right)\right]\right\} \tag{21}$$

with the width (w) and amplitude (R) as variational parameters. For particular model parameters G, α, λ and v, we compute $\mathcal{B} = E_{\rm tot}/m - 1$ as a function of the variational parameters w and R. We show the resulting binding energy surface in figure 2 for one set of model parameters. The contour $\mathcal{B} = 0$ separates the region in which the effective energy of background configuration is less than m from the region in which it is larger than m. The maximal binding is indicated by a star. In figure 3 we present the profiles ϕ_1 and ϕ_2 corresponding to this variational minimum as functions of the dimensionless coordinate $\xi = xm$. This background field configuration

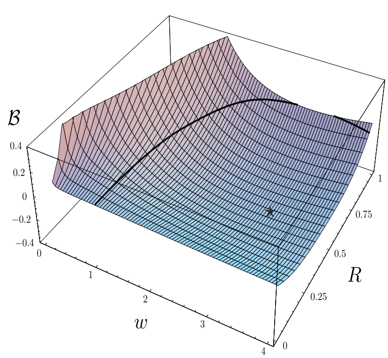


Fig. 2. $\mathcal{B}$ as a function of the *ansatz* parameters for the class of *model* parameters characterized by the relations $\alpha = 0.5G^2$, $\tilde{\lambda} = G^2$, and $v = 0.375$. A solid curve marks the contour $\mathcal{B} = 0$. The star indicates the minimum at $w = 2.808$ and $R = 0.586$.

does not carry fermion number in this case, so the most strongly bound level must explicitly be occupied. The total charge density is shown in figure 3. It receives contributions from the polarized fermion vacuum and from the explicitly occupied valence level, given by $\psi_0^\dagger(x)\psi_0(x)$ where $\psi_0(x)$ is the bound state wavefunction of the valence level.

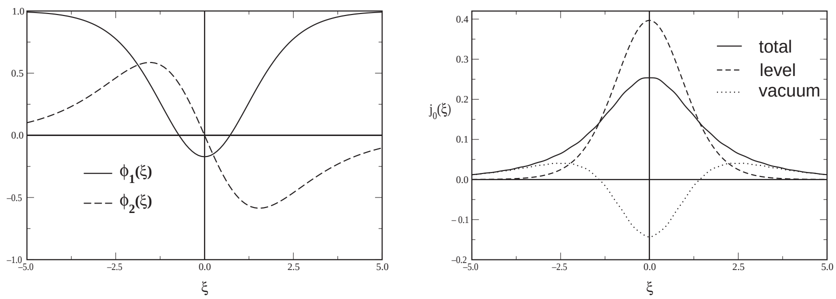


Fig. 3. ϕ_1, ϕ_2, and the fermion number density j_0 at the variational minimum. The left panel shows $\phi_1(\xi)$ and $\phi_2(\xi)$, and the right panel shows the charge density $j_0(\xi)$, which gets contributions from both the polarized fermion vacuum and the filled valence level. The model parameters are as in figure 2.

Figure 4 shows the result of repeating the binding energy calculation for various sets of model parameters. When $\mathcal{B}$ is negative, the configuration

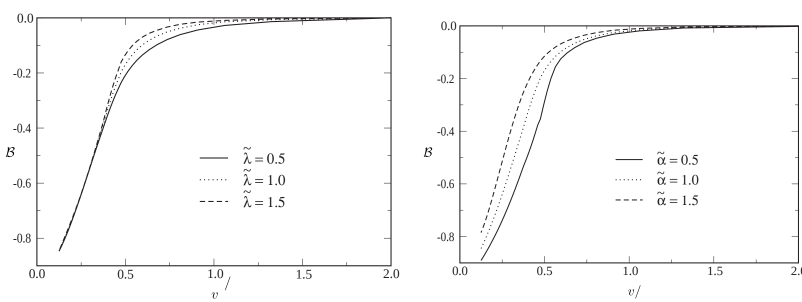


Fig. 4. The maximal binding energy as a function of the model parameters as obtained from the *ansatz* Eq. (21) in units of m. The dimensionless parameters are defined by $\tilde{\alpha} = \alpha/G^2$ and $\tilde{\lambda} = \lambda/G^2$.

is a fermion with lower energy than a fermion propagating in the trivial background. Since the true minimum of the energy will have even lower energy, we know that a soliton exists.

We have extended this analysis to a chiral Yukawa model with SU(2) symmetry in three dimensions [9]. The analysis is more complicated: rotational symmetry is replaced by grand spin, the sum of rotations in spatial SU(2) and isospin SU(2), and diagrams up to fourth order in the external field are divergent. Nevertheless, the program can still be carried out [9].

However, we do not find evidence for a bound fermionic soliton in this theory. In general, binding is weaker than in one dimension and it occurs in regions of parameter space where the model or the restriction to one fermion loop is internally inconsistent. The search for quantum stabilized solitons in the Standard Model requires the inclusion of gauge fields, which is discussed in the following section.

5. Gauge fields in three dimensions

In a chiral gauge theory[2] like the Standard Model, gauge invariance prevents fermions from having explicit mass terms. Rather, they get their mass through their coupling to a scalar field via the well-known Higgs mechanism. At tree level the fermion obtains a perturbative mass, the product of the corresponding Yukawa coupling and the vacuum expectation value of the scalar Higgs field. The decoupling of such fermions presents interesting unsolved puzzles. Ordinary decoupling arguments [11], which would show that a heavy fermion is irrelevant in the low-energy theory, break down. Increasing the mass, which causes the denominators in the fermion propagators to suppress quantum corrections, also increases the coupling, which gives a corresponding enhancement from the vertices. Furthermore, unlike an ordinary fermion with vector couplings, a chiral fermion cannot simply disappear from the theory as its mass is increased, because anomaly cancellation would be ruined. As shown in Ref. [12], gauge invariance is maintained at the level of the Lagrangian because integrating out the heavy fermion induces a Wess–Zumino term in the resulting effective Lagrangian.

For the case of Witten's non-perturbative SU(2) anomaly [13], one can analyze the theory at the level of the action along the same lines [12]. However, one can also analyze the situation from a different point of view. Ref. [14] shows that this anomaly can be understood in terms of the Hamiltonian of the theory: A theory with an odd number of left-handed fermion doublets has no gauge-invariant states. However, if the Yukawa coupling of a fermion is large enough, the perturbative fermion mass will be larger than the classical energy of the sphaleron [15], so that such fermions are no longer stable states in the spectrum of the theory. Thus, to maintain gauge invariance in the low-energy theory, there must exist either new states in the theory carrying the quantum numbers of the fermion, or a mechanism to suppress the decay of the perturbative fermion states.

Although these scenarios could rely on complicated non-perturbative physics, one simple resolution would be provided by the existence of a soliton carrying the quantum numbers of the decoupled fermion. If a localized

[2] This section is taken from Ref. [10], where the interested reader can find a much fuller discussion of the methods and the consequences of this study.

configuration of gauge and Higgs fields binds a fermion level tightly, the binding energy could outweigh the cost in classical energy to set up the background field configuration. However, to consistently include the effects of the fermion level, such calculations must also include the Casimir energy, the renormalized shift in the zero-point energies of all the other fermion modes, since both appear at the same order in $\hbar$. For a static field configuration, the Casimir energy represents the full one-loop quantum vacuum polarization energy, equivalent to summing to all orders in the derivative expansion.

To compare the quantum energy of the configuration with the sphaleron, we must also include the corresponding correction to the sphaleron's energy as well. Furthermore, we must check whether the quantum corrections induce an energy barrier between the perturbative fermion and the sphaleron, which would mean that the perturbative fermion would be quasi-stable, only able to decay by tunneling.

We have carried out such calculations in a simplified version of the Standard Model, for background fields in the spherical *ansatz*, keeping fermion vacuum fluctuations but ignoring those of the gauge and Higgs fields. We find significant quantum corrections to the height of the sphaleron barrier. As we make the fermion level heavier than the quantum corrected sphaleron, we do see evidence for a barrier suppressing its decay. For even larger Yukawa couplings, however, the barrier disappears and the fermion's decay is unsuppressed. We do not see any evidence for a soliton for any value of the Yukawa coupling, and find that including the full Casimir energy destabilizes solitons found in previous work [16].

It is not possible to do justice to the full treatment of a chiral SU(2) gauge theory in this brief survey. However, a glimpse at one class of *ansätze* will give a flavor for the results. In this case we considered an *ansatz* that interpolates between the trivial vacuum, $\Phi = v$ and a sphaleron configuration with Chern–Simons number 1/2. This is an attractive case to examine because the fermion has a zero mode (and therefore is maximally bound) in the presence of a sphaleron. The interpolating *ansatz* is given by

$$\begin{aligned} \Phi &= v(1-\xi) + \xi v U^{(1)} \,, \\ A_j &= \xi \frac{i}{g} U^{(1)} \partial_j {U^{(1)}}^\dagger \,, \end{aligned}$$

where

$$U^{(1)}(\vec{x}) = \mathrm{e}^{i f(r) \tau_j \hat{x}_j / 2}$$

and $f(r) = -2\pi \mathrm{e}^{-r/w}$. As ξ goes from zero to 1/2 the configuration goes from the trivial vacuum to a configuration halfway to winding number one. The lowest barrier on such a path is the sphaleron, so our path represents one that goes "over the sphaleron". Figure 5 shows our numerical results. The heavy solid line shows the energy of the lowest fermion number one state as a function of the interpolating parameter, ξ, for $0 \le \xi \le \frac{1}{2}$ (the figure is symmetric about $\xi = \frac{1}{2}$). Clearly there is no non-perturbative configuration with fermion number one and energy less than the perturbative fermion. For comparison several other energies are shown in the figure. The heavy dashed line is the energy of the lightest fermion number zero configuration. Note that it vanishes at $\xi = 0$ and equals the fermion number one energy at $\xi = \frac{1}{2}$ confirming the existence of a zero mode. Also shown are the energies computed without the contribution from the Casimir energy (with fermion number one (dashed) and fermion number zero (solid)). The effect is dramatic: the vacuum fluctuation energy is positive and destabilizes the solitionic fermion.

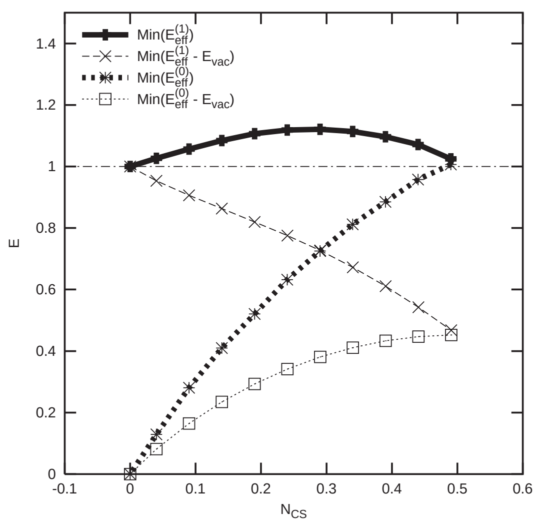


Fig. 5. A sample interpolation between the trivial vacuum and a configuration with Chern–Simons number one-half.

So far we have not found any evidence for the existence of a soliton in the spectrum of the theory. The fermion vacuum polarization contribution seems to destabilize any would-be solitons. It is possible that for a large enough Yukawa coupling, the Witten anomaly is saturated by states that do not have a particle interpretation. But we believe the anomaly puzzle could still be resolved by soliton states that exist outside the spherical *ansatz*. Studies of this possibility are underway.

6. Boundary conditions on quantum fields: the Casimir problem

Obviously boundary conditions are a very convenient idealization in field theory. Conducting boundary conditions give an excellent description of the behavior of electric and magnetic fields near good metals. Bag boundary conditions give a very useful characterization of the low modes of the quark field in hadrons. However physical materials cannot constrain arbitrarily high frequency components of a fluctuating quantum field, so the use of boundary conditions to model vacuum fluctuations, for example in the study of the Casimir effect, requires careful examination. Boundary conditions also appear in brane world scenarios for physics beyond the standard model and in lattice implementations of quantum field theories.

Our [17, 18] point of view is that interactions between fields and matter are fundamental and that boundary conditions can only be substituted when they can be shown to yield the same physics. A real material cannot constrain modes of the field with wavelengths much smaller than the typical length scale of its interactions. The interactions become negligible at wavelengths less than certain physically determined cutoffs. In contrast, a boundary condition constrains all modes. To calculate the Casimir energy it is necessary to sum over the zero point energy of all modes. This sum is highly divergent in the ultraviolet and these divergences depend on the boundary conditions. Subtraction of the vacuum energy in the absence of boundaries removes only the worst divergence (quartic in three space dimensions). We ask whether the Casimir energy and other potential observables can be defined independent of the cutoffs that characterize the actual interactions between the fluctuating fields and the matter. If so, then one can define an abstract Casimir problem, depending on the field, the boundary condition, and the geometry alone. If not, then the Casimir energy is unavoidably entangled with the detailed material physics at hand. Note that these cutoff dependences have nothing to do with the standard infinities of quantum field theory, which were removed by renormalization. Instead they arise because the background necessary to enforce the boundary condition has too much strength at high frequencies.

I do not want the point to get lost in complicated algebra, so I will discuss a case so simple that the necessary calculations are elementary. I will consider a scalar field in one dimension, obeying the Dirichlet boundary condition, $\phi = 0$, at one or two points. The limitation of this simple example is that the novel effects show up in the Casimir energy, but not in the Casimir force. In higher dimensions the effects are more dramatic and do effect measureable quantities. However the calculations are harder, so I will only report our results which are described in detail elsewhere [17,18]. Of course, this subject has been treated before[3], but not, as far as I know, in quite the way that I will describe. I will mention some of the other treatments later in my talk.

6.1. Dirichlet points — a toy model in one dimension

As a warm up, consider what it takes to enforce the boundary condition, $\phi(0) = 0$ on an otherwise non-interacting and massless scalar quantum field in one dimension. This is the problem of the "Dirichlet point". The equation of motion for ϕ coupled to some otherwise inert potential, $\sigma(x)$, is

$$-\phi'' + \lambda\sigma(x)\phi + m^2\phi = \omega^2\phi$$

for a mode with energy ω. λ is defined by normalizing σ so that $\int dx\sigma(x) = 1$.

To get all eigenmodes of ϕ to vanish at $x = 0$ it is necessary to take σ to be "sharp" and λ to be "strong":

$$\begin{aligned} \sigma(x) &\rightarrow \delta(x) \qquad \text{SHARP}\,, \\ \lambda &\rightarrow \infty \qquad\;\; \text{STRONG}\,. \end{aligned}$$

In the sharp limit ϕ' suffers a discontinuity at $x = 0$, $\Delta\phi'|_0 = \lambda\phi(0)$. A careful study shows that the boundary condition $\phi(0) = 0$ emerges for all ω in the limit $\lambda \rightarrow \infty$. *A priori* one would expect such a strong interaction to have a dramatic effect on the dynamics of ϕ, in particular on sum over zero point energies with the boundary condition compared to the energy without.

First, suppose the boundary condition is imposed at the outset on all modes. This is the standard approach [21]. The vacuum fluctuation energy is the sum over zero point energies of ϕ minus the sum without the boundary condition,

$$\tilde{E}_1 = \tfrac{1}{2}\sum\left(\hbar\omega - \hbar\omega_0\right)\,. \tag{22}$$

The situation is shown in Fig. 6(a). The solutions to the free field theory are $\phi \sim \sin kx$ and $\phi \sim \cos kx$ for $k > 0$. With the boundary condition $\phi(0) = 0$ the solutions are $\phi \sim \sin kx$ and $\phi \sim \sin|k|x$, also with $k > 0$. Since the spectra are identical, ω and ω_0 simply cancel and leave

$$\tilde{E}_1 = 0\,. \tag{23}$$

The tilde is there to remind us that the boundary condition was imposed *ab initio*. The treatment in the standard texts is more careful than this,

[3] The relationship between our work and earlier studies of the divergences of quantum field theory near boundaries [19,20] is discussed at length in Ref. [18].

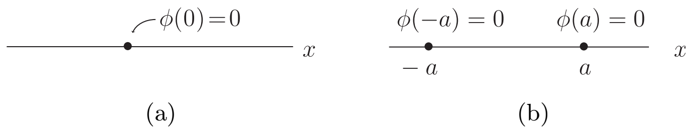


Fig. 6. Dirichlet points: (a) one, and (b) two at a separation of $2a$.

but in essence the same. Next, apply the same methods to the case of two Dirichlet points located at $x = \pm a$ (see Fig. 6(b)). The textbook result is

$$\tilde{E}_2(a) = -\frac{\pi}{48a} \quad \text{for} \quad m = 0\,. \tag{24}$$

Eqs. (23) and (24) are actually bizarre and unacceptable. First, remember that these are the *total* energies for each configuration relative to the vacuum — *we did not drop any terms.* To see the problem, consider first the limit $a \to \infty$:

$$\lim_{a\to\infty} \tilde{E}_2(a) = 0 = 2\tilde{E}_1 \tag{25}$$

which is fine. It confirms that two widely separated Dirichlet points do not interact. Now consider the limit $a \to 0$,

$$\lim_{a\to 0} \tilde{E}_2 = -\infty \stackrel{?}{=} \tilde{E}_1 \tag{26}$$

which seems to say that the energy of a single Dirichlet point, the result of two that coalesce as $a \to 0$, is infinite. Lastly, note that all of these results are suspicious because the massless scalar field theory suffers from infrared divergences in one dimension, leading us to expect log divergences where none have been found.

Now let us consider the same problem from the perspective of an interaction of ϕ with matter. We make the minimal model: we couple ϕ to a *non-dynamical* scalar background field, $\sigma(x)$, with the (super) renormalizable interaction,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \lambda\sigma(x)\phi^2 + c(\varepsilon)\sigma(x)\,, \tag{27}$$

where $c(\varepsilon)\sigma(x)$ is a counter-term and ε is a cutoff, for example the fractional part of the dimension in dimensional regularization. We could elevate σ to be a dynamical field by endowing it with an action of its own. However the core of the problem can be studied without this complication. $\mathcal{L}$ describes a renormalizable quantum field theory. In one dimension only one Feynman diagram — the tadpole — is divergent. This divergence is canceled by the counter-term. As usual there is some scheme dependence involved in renormalization. We choose the “no tadpole scheme” where $c(\varepsilon)$ is chosen to completely cancel the tadpole graph so that $\langle\sigma\rangle = 0$. Any other scheme can be related to ours by a shift in σ. This theory has infrared divergences as $m \to 0$, so we keep $m \neq 0$ throughout.

It is an straightforward to compute the renormalized energy of any configuration, $E[\sigma]$, relative to the vacuum, $\sigma = 0$, and to show that it is finite for piecewise continuous $\sigma(x)$. The crucial question is what happens to $E[\sigma]$ when we try to take σ to be *sharp* and λ to be *strong*. Do we reproduce the results obtained when the boundary condition is imposed *ab initio* or not? The *sharp* limit is benign in one dimension, so we can take $\sigma(x)$ to

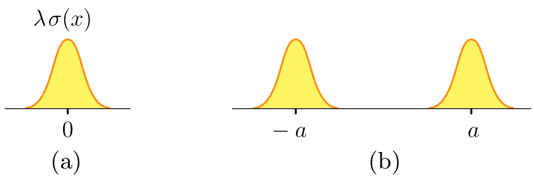


Fig. 7. Smooth but sharply peaked backgrounds approximating Dirichlet points: (a) one, and (b) two at a separation of $2a$. In the "sharp" limit the backgrounds approach delta functions.

be a "spike" $\sigma(x) = \delta(x)$ without difficulty. The renormalized energy of an isolated spike of strength λ (see Fig. 7(a)) is found to be

$$E_1(\lambda, m) = \frac{1}{2\pi}\int_m^\infty dt \frac{t\log\left(1+\frac{\lambda}{2t}\right) - \frac{\lambda}{2}}{\sqrt{t^2-m^2}}\,. \tag{28}$$

It is easy to see that this integral converges for any λ and $m \neq 0$, but diverges logarithmically as $m \to 0$. Likewise, the renormalized energy of two spikes at $x = \pm a$ is given by,

$$E_2(a, \lambda, m) = \frac{1}{2\pi}\int_m^\infty dt\, \frac{t\log\left(1+\frac{\lambda}{t}+\frac{\lambda^2}{4t^2}(1-\mathrm{e}^{-4at})\right) - \lambda}{\sqrt{t^2-m^2}}\,. \tag{29}$$

Since these results are central to the rest of the discussion, I have appended a derivation (of Eq. (28)) at the end of the talk.

Let us submit these results to the same examination as before: As $a \to \infty$ we find

$$\lim_{a\to\infty} E_2(a, \lambda, m) = 2E_1(\lambda, m) \tag{30}$$

which is fine; and as $a \to 0$,

$$\lim_{a\to 0} E_2(a, \lambda, m) = E_1(2\lambda, m) \tag{31}$$

which is much more satisfactory than Eq. (26). Also, both E_1 and E_2 diverge logarithmically as $m \to 0$, as expected. So the renormalized energy in a sharp background is finite and passes all the tests.

However, the punchline is that *both* E_1 *and* E_2 *diverge like* $-\lambda \ln \lambda$ *as* $\lambda \to \infty$. So the total *renormalized* Casimir energy diverges as the coupling constant, λ, is taken strong enough to impose the boundary condition on all modes. λ can be regarded as a cutoff, since modes with $\omega \gg \lambda$ are not affected by the interaction. So in this example the total, renormalized vacuum fluctuation energy is strongly cutoff dependent. Note that E_1 and E_2 are *renormalized* energies. There are no counter-terms available to absorb this divergence.

The two approaches disagree on the total energy, but they agree on the energy density and the force between two points. For any background $\sigma(x)$, the energy density diverges only where $\sigma(x) \neq 0$ [22]. For a spike at a the energy density remains finite for $x \neq \pm a$ in the limit $\lambda \to \infty$, and the limiting form agrees with the density calculated using the boundary condition *a priori* [21,22]. Likewise, the force between the two sharp sources agrees with the boundary condition calculation as $\lambda \to \infty$,

$$\lim_{\lambda\to\infty} -\frac{\partial}{\partial a} E_2(a,\lambda,m) = -\frac{\partial}{\partial a}\tilde{E}_2(a,m) \ \left(= -\frac{\pi}{48a^2} \quad \text{for} \quad m = 0 \right) . \tag{32}$$

In fact no measurement of the properties of the two points can detect the infinite energy stored locally on the boundary. This, however, is special to one dimension. Notice also that the vacuum fluctuation energy is negative ($\sim -\lambda \ln \lambda$ as $\lambda \to \infty$). In a more realistic context this energy would be more than overwhelmed by the positive contributions to the energy coming from the curvature of σ, which goes like $|\sigma'|^2$, and the potential energy of $\sigma(x)$ which would involve higher powers of $\sigma(x)$ beginning with the mass term $\frac{1}{2}m^2 \int dx \sigma^2(x)$. I left those terms out, not because they aren't present, but because I was trying to define an abstract problem — the "Dirichlet–Casimir" problem. Having failed, it is clear that the total energy depends not only on the vacuum fluctuation energy of ϕ, but also on the energy stored in the material represented by σ. The two cannot be separated – no abstract "Casimir energy" can be defined for the Dirichlet boundary condition in one dimension.

To summarize the results for a scalar field in one dimension:

- The renormalized vacuum fluctuation energy is well defined and finite for any (piecewise continuous) background $\sigma(x)$. It differs from the energy calculated by assuming a boundary condition *ab initio*. The renormalized energy is material (*ie.* λ), dependent, and diverges if λ is taken to infinity to impose the boundary condition on all modes.

- The change in the vacuum fluctuation energy with rigid displacement of the boundaries is finite and cutoff independent and can be calculated by imposing the boundary condition *ab initio*.

6.2. Physical effects in other dimensions

Results like those of the previous section would be a mere curiosity if they could not be measured. In higher dimensions worse divergences occur and they too are always confined to the domain where the background fields are non-zero. Thus in any experiment in which the material ($\equiv$ the background fields) are unchanged, the cutoff dependence will cancel out. A case in point is the standard Casimir force between parallel, grounded, conducting plates [23]. The force is measured by displacing the plates rigidly but not deforming them [24, 25]. The cutoff dependent terms remain unchanged as the plates are moved and the resulting force is finite. The result is the same whether you impose the boundary condition at the beginning or start with a smooth background and impose the boundary condition in a limiting process.

The situation is completely different, however, if the material must be deformed to display the physical effect. The "Casimir pressure" on a sphere is the most interesting example . To measure this it is necessary to compare the total vacuum energy of a sphere of radius R with that of a sphere of radius $R + \delta R$. If there are cutoff dependent terms associated with the material, their contribution to the energy changes as the area changes, and they contribute to the pressure. Thus, calculations of Casimir pressures will differ between the case where boundary conditions are applied *ab initio* and where they are realized as a limit of the coupling to a background field.

To examine these issues quantitatively we have studied the problem of the "Dirichlet sphere" — the boundary condition $\phi(R) = 0$ imposed on a massive (or massless) scalar field in D dimensions[4]. $D = 2$ is the "Dirichlet circle" and $D = 3$ is the sphere. As in the one dimensional example, we replace the boundary condition by the coupling to a non-dynamical scalar background, σ, with an interaction of the form

$$\mathcal{L}_{\text{int}} = \lambda\sigma(r)\phi^2 + c_1(\varepsilon)\sigma + c_2(\varepsilon)\sigma^2 + \ldots,$$

where the ... denote a finite series of counter-terms necessary to renormalized the perturbative divergences generated by the $\phi - \sigma$ interaction. For $D = 2$ and $D = 3$ only the counter-terms shown are needed. We normalized

[4] For a full account of this work, see Ref. [17, 18].

the source so its integral over space is one[5],

$$\int d^D r \sigma(r) = 1$$

so the coupling constant, λ, has dimension $[\text{mass}]^{2-D}$.

We then compute the one-loop effective energy (*ie.* the Casimir energy), $E_D(R, \mathit{\Delta}, \lambda)$, for a background σ which is peaked at $r = R$ and has a thickness $\mathit{\Delta}$. We have studied Gaußians and square barriers. The renormalized Casimir energy is finite for fixed $\mathit{\Delta}$, and λ. However it diverges in the sharp limit, $\mathit{\Delta} \to 0$ (except in one dimension, where we saw that the sharp limit was benign), so that not even the sharp limit exists for $D \geq 2$. If we keep $\mathit{\Delta}$ fixed, and take $\lambda \to \infty$ the Casimir energy diverges as well. The nature of the divergences depends on D.

- For the Dirichlet circle ($D = 2$), E_2 diverges like $\frac{\lambda^2}{R} \log \mathit{\Delta}$ at fixed λ as $\mathit{\Delta} \to 0$. Thus the tension (the two dimensional analog of pressure) diverges logarithmically as the thickness of the "circle" goes to zero.

  This is particularly nicely illustrated by examining the energy density, $\varepsilon(r)$, in a spherical shell between r and $r+dr$. The necessary formalism was developed in Ref. [22]. In Fig. 8(a) we plot $\varepsilon(r)$ for a Gaußian background of fixed strength as a function of its width (denoted w in the figures). As $w \to 0$ the energy density diverges in a non-uniform way: at any fixed $r \neq R$, it approaches a limit. However the closer to R, the slower the convergence, and as a result, the total energy diverges as $w \to 0$. The non-uniform behavior is clear in Fig. 8(b).

- For the Dirichlet sphere ($D = 3$), the leading divergence in E_3 goes like $\frac{\lambda^2}{R^2} \frac{\log \mathit{\Delta}}{\mathit{\Delta}}$. Thus the pressure diverges as the thickness of the sphere goes to zero.

We expect that these divergences signal that the actual energy depends on the physical properties of the material. $\mathit{\Delta}$ represents its thickness and λ plays the role of a frequency cutoff (*e.g.* the plasma frequency, ω_{p}) since modes with $\omega \gg \lambda$ are unaffected by σ. Since the other stresses to which materials are subject also depend on such properties of the material, we conclude that *the Casimir pressure on a surface cannot be defined in a useful way that is independent of the other dynamical properties of the material.*

Since the Casimir energy for any piecewise continuous background, σ, has an expansion in Feynman diagrams, and since the divergence structure

[5] This corresponds to the physical situation that the surface gets thinner as R is increased. Other normalizations are possible, but do not change the divergence structure. See Refs. [17, 18].

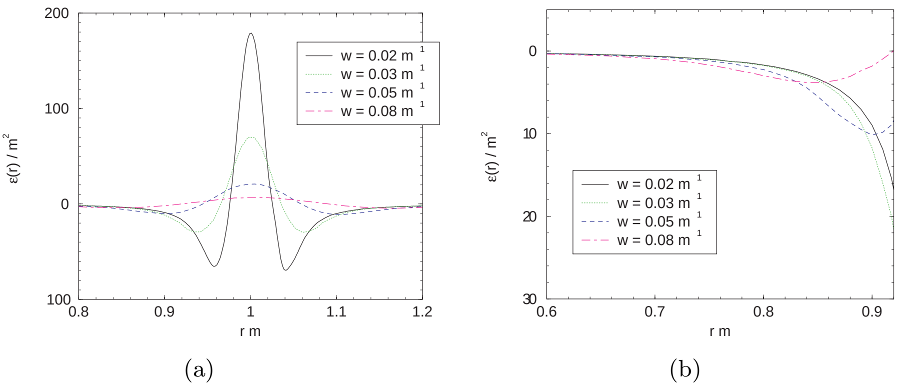


Fig. 8. Casimir energy density in a Gaußian background of width w approximating a Dirichlet circle. (a) The energy density; (b) a closeup of the behavior near $r = 0.8/m$.

of Feynman diagrams is very well understood, it is interesting to look for the origin of the divergences in the diagrams. The Casimir energy is simply the sum of all one-loop diagrams for ϕ circulating in the background σ. We find that the divergences *never* come from the loop integrations. Renormalization takes care of those. Instead the divergences come from the integration over the Fourier components of the external lines, $\tilde{\sigma}(p)$. For $D = 2$ the divergence comes from the renormalized two-point function. Since the two-point function is easy to calculate, the divergence is easy to display. For $D = 3$ the two-point function also generates the leading divergence. However even the three-point function also diverges (logarithmically) for $D = 3$. Since the loop integration for the three-point function is manifestly convergent, it provides a very clear demonstration that the divergences as $\Delta \to 0$ cannot be renormalized away by any counter-term available in the continuum theory. Instead they are physical manifestations of the cutoff dependence of the Casimir energy for the Dirichlet sphere.

Our result disagrees with calculations which assume a boundary condition *ab initio*. [26] However those calculations handle divergences in an *ad hoc* manner and obtain suspicious results: for example the Casimir energy of a Dirichlet sphere is claimed to be finite in odd dimensions ($D = 1, 3, 5, 7 \ldots$) and to diverge in even dimensions.

Although I have only discussed the Dirichlet boundary condition on a scalar field, we have also considered confining boundary conditions on a Dirac field in one dimension [27], and find similar results. Also we have studied the conducting boundary condition in three dimensions — the physically most interesting case — and believe that the pressure is cutoff dependent when the boundary condition is implemented as the limit of a renormalizable interaction between the photon and a background field.

This work has been done in collaboration with N. Graham, V. Khemani, M. Quandt, M. Scandurra, O. Schröder, and H. Weigel. The work I have reported is in large measure theirs. My work is supported in part by the U.S. Department of Energy (D.O.E.) under cooperative research agreement #DF-FC02-94ER40818.

Appendix

Here is an outline of the calculation of the vacuum fluctuation energy of an isolated "spike", Eq. (28). We start with the universally accepted formula for the sum over zero point energies of a field, ϕ, fluctuating in a background σ:

$$E[\sigma] = \tfrac{1}{2}\sum_j \omega_j + \frac{1}{2}\int_0^\infty dk\sqrt{k^2+m^2}\;\delta\rho(k)\,. \tag{33}$$

This is just the generalization to the continuum of $\frac{1}{2}\sum \hbar\omega - \hbar\omega_0$ (with $\hbar = 1$, of course). The ω_j are the energies of possible bound states. $\delta\rho(k)$ is the change in the density of states due to the background σ. $\delta\rho$ is given by another well known result:

$$\delta\rho(k) = \frac{1}{\pi}\sum_\ell \frac{d\delta_\ell}{dk}\,,$$

where $\delta_\ell(k)$ is the phase shift for scattering in the ℓ^{th} partial wave in the background σ (assumed to be symmetric enough to admit a partial wave expansion).

$E[\sigma]$ defined by Eq. (33) diverges in one dimension. To keep control of divergences we use dimensional regularization: we imagine that we are computing in D dimensions. It can be shown that Eq. (33) is finite for $0 < D < 1$. After isolating the possible divergences and renormalizing we will analytically continue to $D = 1$.

Next we use Levinson's theorem — which equates π times the number of bound states to the difference of the phase shift at $k = 0$ and $k = \infty$ (valid in D dimensions) to rewrite $E[\sigma]$ as

$$E[\sigma] = \tfrac{1}{2}\sum_j(\omega_j - m) + \frac{1}{2\pi}\int_0^\infty dk\left(\sqrt{k^2+m^2} - m\right)\sum_\ell \frac{d\delta_\ell}{dk}\,. \tag{34}$$

At this point it is possible to show that the successive terms in the Born expansion of δ_ℓ (the expansion of δ in powers of σ) can be put into one-to-one correspondence with the one loop Feynman diagrams[6]. In particular the contribution of the first Born approximation is identical to the tadpole diagram which diverges as $D \to 1$.

To renormalize, we first subtract the first Born approximation from Eq. (34) and add back the tadpole diagram to which it is equal. We then cancel the tadpole diagram against the counter-term $c(\varepsilon)$ (see Eq. (27)). The resulting *renormalized* and manifestly finite expression for the Casimir energy can now be evaluated at $D = 1$, where the sum over phase shifts includes only the symmetric and antisymmetric channels:

$$E[\sigma] = \frac{1}{2\pi} \int_0^\infty dk (\sqrt{k^2 + m^2} - m) \frac{d}{dk} \left(\delta_+(k) + \delta_-(k) - \delta^{(1)}(k) \right) , \tag{35}$$

$\delta^{(1)}(k)$ is the Born approximation to the sum of the phase shifts. To save space I have dropped the sum over bound states since there are none in the case at hand.

Now to the specific case of the δ-function background: The phase shift in the antisymmetric channel vanishes. The symmetric channel phase shift is easily computed:

$$\delta_+(k) = \tan^{-1} \frac{\lambda}{2k}$$

and the first Born approximation is

$$\delta^{(1)}(k) = \frac{\lambda}{2k} \, .$$

The resulting integral,

$$E(\lambda, m) = \frac{\lambda^3}{4\pi} \int_0^\infty dk \frac{\sqrt{k^2 + m^2} - m}{k^2(\lambda^2 + 4k^2)}$$

is most easily evaluated by rotating the contour to the positive imaginary axis and integrating by parts. The result is the expression quoted in Eq. (28).

[6] This is not true until after the Levinson's subtraction has been made. For a discussion and references, see [1].

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