THE RENORMALIZATION GROUP AND THE COLOR GLASS CONDENSATE*

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The Color Glass Condensate is the matter which controls the high energy limit of strongly interacting particles. I qualitatively describe the nature and origin of this matter, and the renormalization group equations which allow for a computation of its properties.

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1. Implications of high gluon density

In Fig. 1, I show a slice of a hadron at small x, that is the transverse distribution of the low longitudinal momentum degrees of freedom (wee partons). At small x, this density grows and at very small x satisfies

$$\Lambda^2 = \frac{1}{\pi R^2}\frac{dN}{dy} \gg \Lambda^2_{\mathrm{QCD}}\,. \tag{1}$$

This means the typical separation between partons is small, and α_{s} evaluated at this scale is weak.

What can stabilize the density distribution of partons? Instabilities are typically driven by negative mass squared terms proportional to the phase space density ρ. They are typically stabilized by interactions proportional to $\alpha_{\mathrm{s}}\rho^2$. The stable density is $\rho \sim 1/\alpha_{\mathrm{s}}$, and when α_{s} is weak, this is large. This is closely related to the phenomena of Bose condensation and superconductivity. The quantum occupation numbers for this state are large and it is highly coherent. Hence the name condensate. This implies the gluon density distribution should saturate [1].

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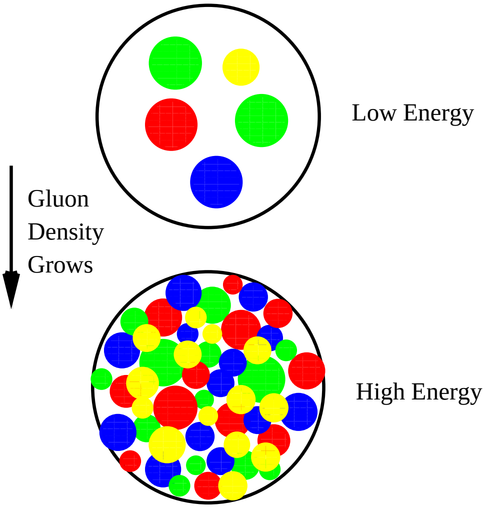

Fig. 1. The transverse distribution of partons inside a hadron.

The partons which generate this condensate come from higher values of x in the frame where the condensate is at rest. These higher x degrees of freedom are Lorentz time dilated compared to their natural time scales, and this time dilation is transferred to the low x degrees of freedom. Since the color distribution comes from partons at very many different values of x, one expects the distribution of color in the transverse space to be random. These properties are similar to ordinary glasses, and in fact the theory one writes down to describe this is the same type used to describe spin glasses. Hence the name glass.

The gluons which make up this distribution are colored. It is therefore a scientifically accurate name that this low x high density gluon matter be called Color Glass Condensate.

In order to parameterize the Color Glass Condensate, we introduce the saturation momentum,

$$Q^2_{\rm sat} \sim \alpha_{\rm s} \Lambda^2 \,. \tag{2}$$

This is the largest momentum where the phases space density remains of order $1/\alpha_{\rm s}$.

To compute the dependence of this momentum on x, it turns out we need to understand the gluon distribution in an intermediate range of phase space density, where $1/\alpha_{\rm s} \gg \rho \gg 1$. One can analyze the BFKL equation in this range of momentum, and compute the dependence of the saturation momentum on x [2]. Over a wide range of energy, one finds [3]

$$Q_{\rm sat}^2 \sim 1\ {\rm GeV}^2 \left(\frac{x_0}{x}\right)^{0.3} . \tag{3}$$

This result is quite close to the value found by an analysis of deep inelastic scattering and diffraction by Golec-Biernat–Wustoff [4].

In fact, Golec-Biernat, Kwieciński and Stasto made a very important discovery based on the HERA data [5]. They discovered that over a very wide range of momentum of the virtual photon, Q^2 in deep inelastic scattering, the cross section

$$\sigma_{\gamma^* p} = F\left(\frac{Q^2}{Q_{\rm sat}^2}\right) . \tag{4}$$

This works for $x < 10^{-2}$ and $Q^2 < 400$ GeV^2. One can understand this result for small $Q^2 \sim Q_{\rm sat}^2$, but it is a surprise it works well at such large Q^2. In fact one finds that another scale appears in the problem $Q_{\rm sat}^4/\Lambda_{\rm QCD}^2$. In this intermediate scaling region, $Q_{\rm sat}^2 \ll Q^2 \ll Q_{\rm sat}^4/\Lambda^2$ correlation function scale as powers of Q^2, that is anomalous dimensions which are computable within BFKL dynamics. In this region, the phase space density is not so large, and the time scales for evolution of matter are more or less normal time scales, but the correlations are non-trivial. The description of the matter is quantum, not classical [2]. This region has been called the Quantum Color Fluid by my good friend Kharzeev [6].

We can now draw a phase diagram of QCD. In Fig. 2, the regions of phase space occupied by the Color Glass Condensate, the Quantum Color Fluid, and the ordinary parton gas is shown in the $\ln(1/x) - \ln(Q^2)$ plane. We

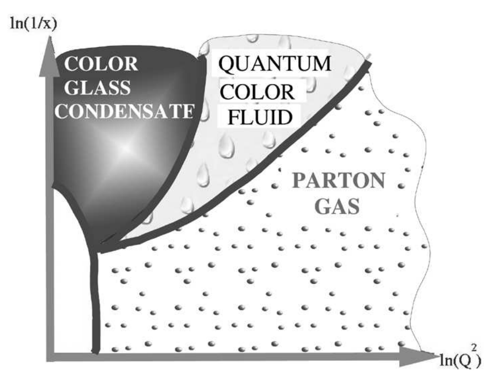

Fig. 2. The phase diagram for high density QCD.

believe we have a semi-quantitative theory of this matter, and this theory should become precise at asymptotically small x. This is because the density becomes large at small, and therefore the coupling is weak, and we can compute the properties of weakly coupled theories.

2. Universality

The concept of universality is built into our theoretical description of the Color Glass Condensate. The properties of the matter depend only upon the density of partons per unit area, independent of the nature of the original parton. Because all matter is made from CGC at high energies, the properties of hadrons relevant for high energy processes are universal. The parton distributions themselves can be computed as a property of this matter.

There is a deeper sense in which this matter is universal which arises from renormalization group ideas. To understand this, one needs to know about the property of limiting fragmentation. If one plots the distribution

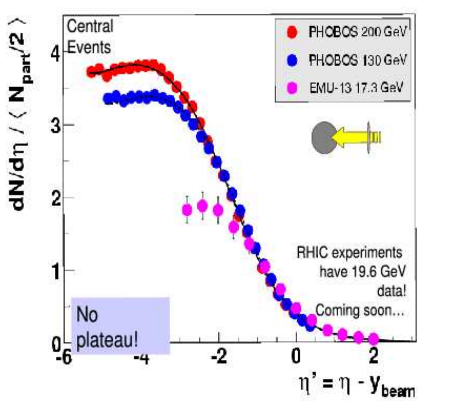

Fig. 3. Limiting fragmentation in heavy ion collisions.

of produced particles as a function of rapidity measured from the rapidity of one of the colliding particles, except for a few units of rapidity which corresponding to the slow moving particles (the wee partons) in the center of mass frame, the distributions are the same at different energies. It is as if the only effect of going to higher energy is to add in new low momentum degrees of freedom. The high momentum degrees of freedom are frozen out.

This means that there should be an effective action for the low momentum degrees of freedom, and going to higher energy integrates out these degrees of freedom, and results in a new theory for the yet lower momentum degrees of freedom at the higher energy scale. In fact these low momentum degrees of freedom are integrated out to generate sources for the wee partons at higher energy. This process is a renormalization group [1, 7].

Unlike the ordinary renormalization groups we are familiar with from perturbative field theory, this renormalization group turns out to be a functional differential equation. It is essentially an infinite dimensional diffusion equation for the wavefunction which describes the small x gluons. Because of its diffusive nature, going to smaller values of x implies that the wavefunction spreads, and this spreading is the origin of the non-trivial growth of the gluon distribution function and the saturation momentum. One can write the functional differential equations explicitly, and see that all known renormalization group equations which include DGLAP and BFKL are reproduced. In addition one gets a complete description of both the Color Glass Condensate and the Quantum Color Fluid regions of the QCD $\ln(1/x) - \ln(Q^2)$ phase diagram. The solutions at small x appear to be universal [8]. For a complete review of these and other topics see the recent review of Iancu and Venugopalan [9].

What I am claiming here is very radical: We have a complete understanding of the high energy limit of QCD in terms of a universal form of matter. At high enough energy, we have the tools at our disposal to explicitly compute the properties of QCD.

3. The renormalization group equations

3.1. The renormalization group

The Color Glass Condensate is described by classical gluon fields which are produced by a light cone current,

$$J^{+} = \delta^{\mu +} \rho(x^{-}, x_{\mathrm{T}})\,. \tag{5}$$

The current has no dependence upon x^{+} because of Lorentz time dilation. This current describes the gluonic degrees of freedom with x values larger than the scale at which we are studying our gluon fields. An effective action for the Color Glass Condensate gives a path integral representation of the form

$$\int [dA][d\rho] \exp\left(iS[A,\rho] - F[\rho]\right)\,. \tag{6}$$

In this equation S is the action for the gluon fields in the presence of a light cone current described by the charge density ρ. (To define it properly one must provide a manifestly gauge invariant action.) Once one solves for the fields in terms of ρ, then one is required to average over the source with a weight function $F[\rho]$ which in the McLerran–Venugopalan model is taken as a Gaussian [1]. Implicit in the path integral is a longitudinal momentum cutoff. The fields have momenta below this cutoff, and the effect of integrating out the fields above the cutoff is included in the source ρ and the integration over various values of ρ.

The question arises: How does one determine $F[\rho]$? It turns out that F is determined by renormalization group equations generated by varying the longitudinal momentum cutoff [1, 7]. The reason why the renormalization group treatment is essential follows from trying to solve for physical quantities, such as the gluon distribution function within the CGC approach. In lowest order one computes the classical field associated with the source ρ, inserts it into an expression for the operator of interest, and then averages over ρ. The lowest order corrections to this involve Gaussian fluctuations around this classical solution. If there is some scale associated with the process of physical interest, say p^+, one finds that the first order corrections are of order $\alpha_{\rm s} \ln(\Lambda^+/p^+)$, where Λ^+ is the longitudinal momentum cutoff. The coupling constant α is small because we evaluate it at the saturation momentum scale. The quantum corrections to the lowest order result are therefore small so long as

$$\mathrm{e}^{-c/\alpha_{\rm s}} \Lambda^+ \ll p^+ \leq \Lambda^+ \, . \tag{7}$$

In order to go to lower momenta, it is easiest to change the longitudinal momentum cutoff to a smaller value. To do this, we have to integrate out the degrees of freedom between the old longitudinal momentum cutoff scale and the new one. This can be done in Gaussian approximation since the coupling is weak, and so long as the ratio of the various cutoff scales satisfies $\alpha_{\rm s} \ln(\Lambda^+/\Lambda^{+\prime}) \ll 1$. It turns out that this integration does not change the action for the interaction of the gluon fields with the source. All that changes is the weight function for integration over the source fields. If we let

$$dy = \ln\left(\frac{\Lambda^+}{\Lambda^{+\prime}}\right) \tag{8}$$

the renormalization group equation becomes

$$\frac{d}{dy}\mathrm{e}^{-F[\rho]} = -H\left(\rho, \frac{d}{d\rho}\right)\mathrm{e}^{-F[\rho]} \, . \tag{9}$$

It turns out that H is second order in $d/d\rho$, real, and positive semidefinite. Therefore H can be interpreted as a Hamiltonian for a $2+1$ dimensional quantum system.

The Hamiltonian above has an unusual property. If there was a potential for the Hamiltonian H with a unique minimum, then at large times the solution of the above equation would tend towards the ground state. In the Color Glass Condensate H, there is in fact zero potential. The system never tends to the ground state, and there is quantum diffusion. To see how this works consider a 1 dimensional example.

$$\frac{d}{dy}Z = \frac{-p^2}{2}Z\,. \tag{10}$$

This has the solution

$$Z = \frac{1}{\sqrt{2\pi y}}\exp\left(-\frac{x^2}{2y}\right)\,. \tag{11}$$

As the Euclidean time y increases, the wavefunction spreads, corresponding to diffusion. This is unlike the situation where there is a potential

$$\frac{d}{dy}Z = \left(\frac{-p^2}{2} - V(x)\right)Z\,. \tag{12}$$

Here as we evolve in time, the coordinate x settles into the minimum of V, and has small excursions around it. The solution for Z becomes time independent.

The consequences of this simple observation are enormous. For the case of diffusion, physical quantities are never independent of rapidity, even at the smallest values of x. The non-triviality of the small x limit is a consequence of the lack of a potential in the renormalization group evolution equation!

The interested reader is referred to the growing literature on this subject for details. Suffice it to say that one can use the renormalization group equation above, and the explicit form for H which has been computed to reproduce all known renormalization group equations. The explicit form of the equations exists, and various approximate solutions have been constructed. The picture which results agrees with the phenomenology of small x physics. Understanding and solving these equations provides a rich area for future research.

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