# STRING/GAUGE DUALITY: (RE)DISCOVERING THE QCD STRING IN AdS SPACE*

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These lectures trace the origin of string theory as a theory of hadronic interactions (predating QCD itself) to the present ideas on how the QCD string may arise in Superstring theory in a suitably deformed background metric. The role of 't Hooft's large $N_c$ limit, Maldacena's String/Gauge duality conjecture and lattice spectral data are emphasized to motivate and hopefully guide further efforts to define a fundamental QCD string.

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## 1. Preface: Not by accident

Sting theory, contrary to conventional lore, was discovered (or invented) not by accident but by a systematic program to build a relativistic quantum theory of the hadronic (or strong) interactions without resorting to the use of local fields. The approach, referred to as "$S$ matrix theory", sought to impose a minimal set of consistency conditions directly on the $S$ matrix [1]. At the time, it appeared absurd to consider the known light hadrons (*e.g.* pions) as "elementary" fields, particularly with the realization that they were just the first member of a Regge family of increasingly high mass and spin ($J \simeq \alpha' m_J^2 + \alpha_0$). In the language of low energy effective field theory, the problem for a finite quantum theory of hadrons and gravity are analogous. The effective low energy theory of hadrons (*i.e.* the pions) is the chiral Lagrangian,

$$S[U] = \int d^4x \left\{ \frac{F_\pi^2}{4} \mathrm{Tr}[\partial_\mu \Sigma^\dagger \partial_\mu \Sigma] - \frac{\langle \bar\Psi \Psi \rangle}{2N_f} \mathrm{Tr}[\mathcal{M}\Sigma^\dagger + \mathcal{M}^\dagger \Sigma] + \cdots \right\}, \quad (1.1)$$

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and for gravity, the Einstein–Hilbert Lagrangian,

$$S[g] = \frac{M_{\mathrm{P}}^2}{16\pi^2} \int d^4x \{\sqrt{-g}\,(R+\Lambda) + \cdots\}\,. \tag{1.2}$$

Both are beautiful geometric quantum theories, but they are non-renormalizable with dimensionful coupling constants inversely proportional to the mass ($1/F_\pi$ and $1/M_{\mathrm{P}}$). In each order of the loop expansion, one must cancel UV infinities with new high dimensional counter terms. With the discovery of QCD the analogy appears to be lost. But it is the goal of these lectures to argue that this is not the case. The only weak coupling limit for QCD (in the infrared) is the 't Hooft expansion for small $1/N_c$ at fixed QCD scale $\Lambda_{\mathrm{QCD}}$. This leads to a distinctly string-like hadronic phenomenology. However the central question of these lectures is **not** the existences of a phenomenological QCD string. Rather it is the question:

> Is the Yang–Mills theory for QCD exactly equivalent (*i.e.* dual) to a fundamental String Theory?

This question goes beyond the existence of a confining QCD vacuum with stringy electric flux tubes to the question of a mathematically precise identity between QCD and string theory in the same sense that the Sine Gordon and Massive Thirring quantum theories are equivalent. In this example not only does duality exchange strong and weak coupling expansions, but after all non-perturbative effects are included they have identical $S$ matrix.

The recent progress in superstring theory associated with Maldacena's AdS/CFT conjecture, backed up by almost 5 years of consistency checks, strongly supports the existence of an exact Gauge/String duality between some (super) Yang–Mills theories and superstrings in non-trivial (asymptotically AdS) background. Consequently at long last we have some mathematical support for such dualities. Naturally this has revived the search for a QCD string and brought many features into much clearer focus. These lecture will briefly review the history and recent progress in this ancient quest for the QCD string. It should be added that constructing a hadronic string is not only of interest in gaining a deeper understanding of QCD but, if successful, a major step in understanding what constitutes a string theory itself.

## 2. Lecture one: Ancient lore

### *2.1. Empirical basis*

The discovery of string theory in the late 1960's followed from a detail study of the phenomenology of hadronic scattering, specifically finite energy sum rules constrained by Regge theory at high energies. The Regge limit for the simplest amplitude, $\pi^+\pi^-$ elastic scattering in Mandelstam variable $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$, was traditionally parameterized as

$$A_{\pi^+\pi^-\to\pi^+\pi^-}(s,t) \simeq g_o^2 \Gamma[1 - \alpha_\rho(t)](-\alpha' s)^{\alpha_\rho(t)} \,. \tag{2.1}$$

The Gamma function prefactor gives cross channel poles for rho exchange at J=1 and higher spins states. Taking the ratio of width to mass for the rho ($\Gamma_\rho/m_\rho \simeq 0.1$) as a small parameter, one sought a new perturbative expansions starting with a zero width approximation. This was traditionally enforced for all resonance states by using an exactly linear rho trajectory ($\alpha(t) = \alpha' t + \alpha_0$) so that "resonance" poles at integer $J = \alpha(m^2)$ had real masses [2].

In 1968 Veneziano [3] realized that exact $s, t$ symmetry could be imposed by assuming an amplitude of the form,

$$A_{\pi^+\pi^-\to\pi^+\pi^-}(s,t) = g_o^2 \frac{\Gamma[1-\alpha_\rho(t)]\Gamma[1-\alpha_\rho(s)]}{\Gamma[1-\alpha_\rho(s)-\alpha_\rho(t)]} \,, \tag{2.2}$$

the so called **dual** resonance model. Here duality referred to Dolan–Horn–Schmid duality which states that sum over s-channel resonances poles interpolates the exchanged Regge power behavior,

$$\sum_r \frac{g_r^2(t)}{s - (M_r - i\Gamma_r)^2} \simeq \beta(t)(-\alpha' s)^{\alpha(t)} \,. \tag{2.3}$$

This property is easily derived for the dual pion scattering amplitude (2.2). The Regge limits follows from the Stirling's approximation as $s \to \infty$ and the resonance expansion follows from the integral representation for the Beta function,

$$A_{\pi^+\pi^-\to\pi^+\pi^-}(s,t) = -g_o^2 \alpha_\rho(t) \int_0^1 dx\, x^{-\alpha_\rho(s)} (1-x)^{-1-\alpha_\rho(t)} \,. \tag{2.4}$$

Expanding at small x we get,

$$A_{\pi^+\pi^-\to\pi^+\pi^-}(s,t) = -g_o^2 \sum_{J=1}^{\infty} \frac{(\alpha_\rho(t))(\alpha_\rho(t)+1)\cdots(\alpha_\rho(t)+J-1)}{(J-1)!} \int_0^1 dx x^{-1-\alpha(s)+J}$$

$$\Rightarrow \sum_{J=1}^{\infty} \frac{g_o^2 A_J(\alpha' t)}{\alpha_\rho(s) - J} \simeq g_o^2 \Gamma(-1 - \alpha_\rho(t))(-\alpha' s)^{\alpha_\rho(t)}, \tag{2.5}$$

where $A_J$ is a polynomial of order $J$. In fact the initial enthusiasm for this model included a surprising feature of chiral symmetry. In the soft pion limit $p_1 \to 0$, the Adler zero,

$$A_{\pi^+\pi^-\to\pi^+\pi^-}(s,t) = (1-\alpha_\rho(s)-\alpha_\rho(t))\frac{\Gamma[1-\alpha_\rho(t)]\Gamma[1-\alpha_\rho(s)]}{\Gamma[2-\alpha_\rho(s)-\alpha_\rho(t)]} \sim \alpha'(s)+(t)\to 0\,, \tag{2.6}$$

is imposed if we take the phenomenologically reasonable values for the rho trajectory intercept: $\alpha_\rho(0) = 0.5$! Veneziano's amplitude is the 4 point function of the NS superstring — ignoring the conformal constraint on the Regge intercept and the dimension of space time which was not understood at the time. This led to Neveu–Schwarz's seminal paper [4]on the $N$-point generalization, entitled "Factorizable dual model of pions".

As we will explain this initial enthusiasm was premature.

### 2.2. Covariant string formulation

It is surprisingly easy to generalize the 4 point Beta function to get the $N$ point dual resonance amplitudes and the covariant quantization of the Bosonic string. The argument goes as follows. Consider the 4 point function for tachyon scattering [6] in a symmetric form

$$\int_0^1 x^{-1-\alpha(s)}(1-x)^{-1-\alpha(t)}dx = \int_{x_1}^{x_3}\frac{dx_2}{(x_4-x_3)(x_4-x_1)(x_3-x_1)} \times \prod_{1\le i<j\le 4}(x_j-x_i)^{2\alpha' p_j p_i}\,, \tag{2.7}$$

where $\alpha(s) = \alpha' s + 1 = 2\alpha' p_1 p_2$ for $\alpha' p_i^2 = -1$ and the three dummy variables maybe fixed at $x_1 = 0, x_3 = 1, x_4 = \infty$. Since the integrand is invariant under Möbius transformations $x_i \to (ax_i+b)/(cx_i+d)$, this does not spoil cyclic symmetry. Now there is an obvious guess for the $N$ point open string tachyon amplitude,

$$A_N(p_1,\cdots,p_N) = g_o^{N-2}\int\frac{dx_2dx_3\cdots dx_{N-2}}{(x_N-x_{N-1})(x_N-x_1)(x_{N-1}-x_1)} \times \prod_{1\le i<j\le N}(x_j-x_i)^{2\alpha' p_j p_i}\,. \tag{2.8}$$

The integration region is restricted to be $x_1 \le x_2 \le x_3 \le \cdots \le x_N$. Modern string theory lecture notes generally require hundreds of pages of derivation to write down this amplitude if they bother to do it all. (This is not to imply that you should not learn the fundamentals of string path integral quantization but the discovery string theory was in large part due to the simplicity for the final answer for the tree amplitude.)

One can also follow the pioneers of the field and write down the Old Covariant Quantized string working "backward" from the answer. One needs to factorize the $N$-point function, *i.e.* introduce a complete set of states. Short circuiting the full derivation, it amounts to a free (string) field expansion,

$$X^\mu(0,\tau) = \hat{q}^\mu + i\hat{p}^\mu\tau + \sum_n \frac{1}{\sqrt{n}}\left(a_n^\mu \exp[\tau] + a_n^{\mu\dagger}\exp[-\tau]\right), \tag{2.9}$$

into normal mode oscillators,

$$[\hat{q}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu} \text{ and } [a_n^\mu, a_m^{\nu\dagger}] = \eta^{\mu\nu}\delta_{n,m}\,, \tag{2.10}$$

acting on the ground state tachyon at momentum $p$,

$$\hat{p}^\mu|0,p\rangle = p^\mu|0,p\rangle \text{ and } a_n^\mu|0,p\rangle = 0\,. \tag{2.11}$$

Then a short algebraic exercise will convince you that the integrand for the $N$-point function does factorize as,

$$\langle 0,p_1|V(x_2,p_2)V(x_3,p_3)\cdots V(x_{N-1},p_{N-1})|0,p_N\rangle = \prod_{1\le i<j\le N}(x_j - x_i)^{p_j p_i}\,, \tag{2.12}$$

with

$$\begin{aligned} V(x,p) &= :\exp[ipX]: = \exp[ip\hat{q}]\exp[p\hat{p}\log(x)]\,\exp\left[ip\sum_n \frac{a_n^\dagger x^n}{\sqrt{n}}\right] \\ &\quad\times\exp\left[ip\sum_n \frac{a_n x^{-n}}{\sqrt{n}}\right], \end{aligned} \tag{2.13}$$

and $x \equiv \exp[-\tau]$. To calculate the matrix element (*i.e.* amplitude) one merely normal orders the operators giving factors,

$$\exp\left[-\sum_n \frac{p_i p_j}{n}(\frac{x_i}{x_j})^n\right] = \exp\left[p_i p_j \log(1 - x_i/x_j)\right] = \left(1 - \frac{x_i}{x_j}\right)^{p_i p_j}, \tag{2.14}$$

for each pair of vertex insertions. The stringy interpretation follows from identification of world sheet surface co-ordinates $(\sigma,\tau)$ so with $z = \exp[-\tau - i\sigma]$ the general expansion for its space-time position, $X^\mu(\sigma,\tau)$,

$$X^\mu(z,\bar{z}) = \hat{q}^\mu + \hat{p}^\mu log(z\bar{z}) + \sum_n \frac{1}{\sqrt{n}}(a_n^\mu z^n + b_n^{\dagger\mu}\bar{z}^{-n})\,, \tag{2.15}$$

is the normal mode expansion of the free 2-d conformal equations of motion for a free string,

$$\partial_\tau^2 X^\mu + \partial_\sigma^2 X^\mu = 0\,, \tag{2.16}$$

in Euclidean world sheet metric.

Nambu–Gotto took this one step further by noticing that this is a gauge fixed form of the equation of motion for a general co-ordinate invariant world sheet (Nambu–Gotto) action,

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'_{\rm s}} \int d^2\xi \sqrt{-\det(h)}\,, \text{ where } h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu\,. \tag{2.17}$$

At the classical level this is also equivalent to the Polyakov form,

$$S_{\rm P} = -\frac{1}{2\pi\alpha'_{\rm s}} \int d^2\xi \sqrt{-\det(\gamma)}[\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu] \tag{2.18}$$

with an auxiliary "Lagrange multiplier" 2-d metric, $\gamma_{ij}$. However the Polyakov form is easier to gauge fix and quantize using BRST technology. To get a feeling for the dynamics of the open string, it is interesting to write down a few classical solutions.

### *2.3. Two open string solutions*

In the static ($t = X^0 = i\tau$) orthogonal gauge ($h_{12} = h_{11} + h_{22} = 0$), the Euler–Lagrange equations for the Nambu–Gotto string action is linear,

$$\partial_t^2 X^\mu - \partial_\sigma^2 X^\mu = 0\,, \tag{2.19}$$

with the non-linear Varasoro constraints,

$$\partial_\sigma X^\mu \partial_t X_\mu = 0\,, \quad \partial_t X^k \partial_t X_k + \partial_\sigma X^k \partial_\sigma X_k = 1\,. \tag{2.20}$$

Classical solutions must satisfy both Eq. (2.19) and Eqs. (2.20).

**Solution # 1:** The string stretch along the 3rd axis with (fixed) Dirichlet boundary conditions, $\sigma \in [0, L]$: All spatial components $X^k = 0$ except

$$X^3 = \sigma\,, \tag{2.21}$$

with energy $E_0 = T_0 L$ exhibiting linear confinement with the string tension $T_0 = 1/2\pi\alpha'_{\rm s}$. For future reference the exact quantum solution has energy

$$E_n = T_0 L \sqrt{1 - \frac{\pi(D-2)}{12 T_0 L^2} + \frac{2\pi a_n^\dagger a_n}{T_0 L^2}} \tag{2.22}$$

for $D$ space-time dimensions.

**Solution # 2:** The free string rotating in the $(X_1, X_2)$ plane with Neumann boundary conditions, $\sigma \in [0, \pi L/2]$: All spatial components $X^k = 0$ except

$$X^1 + iX^2 = \frac{L}{2}\cos\frac{2\sigma}{L}\exp\frac{i2t}{L}\,, \tag{2.23}$$

with energy $E = \pi L T_0/2$ and total angular moment (spin) $J = \alpha'_{\rm s} E^2$. This latter result, which is the key requirement for QCD Regge phenomenology, is a rather non-trivial property of a relativistic massless string. The end points always travel at the speed of light, so as the energy increases the string gets longer ($E \sim L$) BUT the angular velocity decreases ($\omega = 2/L = 1/(\pi T_0 E)$)! Nonetheless the angular moment increases quadratically because the total stored energy grows linearly in $L$ and the moment of inertia grows as a cubic $L^3$. This is in stark contrast with a rigid non-relativist bar where $J \sim E^{1/2}$. Clearly the linear Regge trajectory supports the general picture of a massless "flux" tube with energy coming entirely from its tension. Again for future reference the exact quantum state for this leading trajectory is

$$\left(a^1_{(1)} + ia^2_{(1)}\right)^J |0,p\rangle\,. \tag{2.24}$$

### *2.4. Failure of the old QCD string*

We should now take a break from this discourse and learn all of rules of superstring perturbations theory. With the help of anomaly cancellation, we would discover 5 consistent perturbation expansions — free of tachyons and negative norm (*i.e.* ghost) states. The resulting phenomenology for perturbative superstrings (in flat space-time) has 4 disasters from the view point of a QCD string:

1. Zero mass states (*i.e.* $1^-$ gauge/ $2^{++}$ graviton)
2. Supersymmetry
3. Extra dimension: $4 + 6 = 10$
4. No Hard Scattering Processes

An abject failure for QCD strings — albeit a very interesting framework for a theory of quantum gravity interacting with matter. A theory of Everything perhaps. There are two possibilities for the QCD string, either it has nothing to do with a fundamental superstring or there are dramatic new effects when non-trivial background metrics are considered.

## 3. Lecture two: Gauge/string duality

In a sense the modern era of the QCD string begins almost immediately after the discover of QCD itself with 't Hooft analysis of the large $N_c$ limit in 1974. One had to understand how the picture of valence quarks attached to the strings of the dual resonance model might arise, even after you assume electric confinement. Apparently there must be some rather small parameter to explain the zero width approximation. This insight continues to guide the attempt to define a QCD string today.

### *3.1. Large $N_c$ topology*

SU(3) Yang–Mills theory has no free dimensionless parameters, except the mass of the quarks relative to the intrinsics QCD scale $m_q/\Lambda_{\rm QCD}$ and the $\theta$ parameter. Note that there is in fact no coupling constant in the quantum theory because by dimensional transmutation (or breaking of conformal symmetry at zero mass for the quarks) this is eliminated in favor of $\Lambda_{\rm QCD}$. Thus there is no conventional weak coupling expansion, except due to "asymptotic freedom" as a formal method of expanding in the UV for large "energies", $E \gg \Lambda_{\rm QCD}$.

't Hooft asked the question on whether the rank of the group for SU$(N_c)$ Yang–Mills theory could be used as a formal expansion parameter $g_{\rm s} \sim 1/N_c$. In perturbation theory there is a well defined limit for fixing $g^2_{\rm YM}N_c$, the renormalized 't Hooft coupling. (Here the "coupling" is just a short hand for the loop expansion in $\hbar$ and fixing $g^2_{\rm YM}N_c$ short hand for working at fixed $\Lambda_{\rm QCD}$.) The result is the famous topological restructuring of the loop expansion as sum over Riemann surfaces. Starting from the action

$$S = \frac{1}{g^2_{\rm YM}}{\rm Tr}[(\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu])^2] + \frac{1}{g^2_{\rm YM}}\bar{\Psi}(\gamma_\mu\partial_\mu - iA_\mu)\Psi\,, \quad (3.1)$$

and writing down Feynman diagrams in the "double line" form, we get

$$\begin{aligned} \text{Gluon Loops}\ &:\ \delta^r_r = N_c \;\Rightarrow O(N_c^F)\,, \\ \text{Gluon \& Quark Prop}\ &:\ g^2_{\rm YM} = g^2_{\rm YM}N_c \times \frac{1}{N_c} \;\Rightarrow\; O(N_c^{-E})\,, \\ \text{Vertices}\ &:\ \frac{1}{g^2_{\rm YM}} = \frac{1}{g^2_{\rm YM}N_c} \times N_c \Rightarrow\; O(N_c^V)\,. \end{aligned} \quad (3.2)$$

Using Euler's theorem the factors of $N_c$ for gluon loops (faces $F$), propagators (edges $E$), quark loops ( boundaries $B$) and interactions ( vertices $V$) is rewritten,

$$N_c^{F-E+V-B} = N_c^{\chi} = N_c^{2-2H-B}\,, \quad (3.3)$$

depending only on the topology of the graph as function of the number of non-planar glueballs propagators ( handles $H$) and the quark loops (boundaries $B$). This is precisely the topological expansion of string theory in terms of the genus of the world sheet. Perhaps more significant this topology can also be shown to hold on the lattice in the confined phase. On the lattice the strong coupling expansion is a sum over surfaces of electric flux so in spite of the extreme breaking of Lorenz invariance due the lattice the physical mechanism is clearly string-like flux tubes. The argument is quite analogous to weak coupling. For illustration consider the Wilson form of the pure gauge action,

$$\begin{aligned} S &= \frac{1}{g_{\rm YM}^2}\sum_{\rm P} {\rm Tr}[2 - U_{\rm P} - U_{\rm P}^\dagger]\,, \\ U_{\rm P} &= U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)\,, \\ U_\mu &= \exp[iaA_\mu]\,. \end{aligned}$$

In strong coupling the action is expanded in a power series and each link is integrated over its Haar measure, $dU_\mu(x)$. Every link must be paired with (at least) one anti-link ($U \to U^\dagger$) to get a non-zero result. The $N_c$ counting is

$$\begin{aligned} \text{Plaquettes} &: \frac{1}{g_{\rm YM}^2} = \frac{1}{g_{\rm YM}^2 N_c} \times N_c \Rightarrow O(N_c^F)\,, \\ \text{Links} &: \int dU\, U_{r_1}^{l_1} U_{l_2}^{\dagger r_2} = \frac{1}{N_c}\delta_{l_2}^{l_1}\delta_{r_1}^{r_2} \Rightarrow O(N_c^{-E})\,, \\ \text{Sites} &: \delta_r^r = N_c \Rightarrow O(N_c^V)\,. \end{aligned} \tag{3.4}$$

Quark loops create boundaries just as before. Thus using Euler's theorem again the strong coupling expansion (ignoring self-intersections of surfaces) yields **exactly** the same topological result as in weak coupling. However it should be realized that the meaning is very different. The vertices give the index sums, the faces are now field strengths and edges are not propagators. The topology expansion for large $N_c$ Yang–Mills is indeed a robust feature in need of a deeper explanation.

In a real sense the large $N_c$ limit **defines** the QCD string perturbative expansion. But to go beyond this statement of faith and actually take the large $N_c$ limit to give a mathematical tractable definition of the perturbative QCD string, even at the lowest order in the string coupling, $g_{\rm s} \sim 1/N_c$, has proven frustrating, except in two dimensional QCD. Also it is interesting to note that there is more than one large $N_c$ limit. For SU(3), one replace the quark field by an anti-symmetric color tensor , $\Phi^{ij} = \epsilon^{ijk}\psi_k$. If one takes the large $N_c$ limit of $N_f = 1$ flavor QCD with this tensor representation for quark fields the fermion loop is no longer subdominant. In fact the leading term can be shown to be precisely the same as the large $N_c$ limit of $\mathcal{N} = \infty$ SUSY Yang–Mills theory! Should we be alarmed at this in view of the glib statement that the large $N_c$ limit defines string perturbation theory. I think not. In fact the full non-perturbative QCD string theory might well have more than one weak coupling string expansion, just as is the conventional view of superstrings in 10-d.

### *3.2. AdS/CFT correspondence for superstrings*

String theory has undergone a tremendous transformation in the last 35 years. In the "First String Revolution" perturbative string vacua were restricted to five alternatives ( IIA, IIB, I, H0, HE) by the requirement to remove tachyons, ghosts and cancel anomalies. This appeared to restrict dramatically the space of possible string theory. In the "Second String Revolution" non-perturbative dualities related these 5 cases (and M theory) into a single connected manifold. However, that is not the end of the story. Solitonic objects called $D$-brane give rise to a tremendous explosion of possible vacua so in the infrared the physics of strings in non-trivial backgrounds are seen to mimic a plethora of effective fields theories.

In 1998 Maldacena realized that at least under certain circumstances string theories had to be dual (*i.e.* equivalent) to Yang–Mills theory. While this is technically still a conjecture consistency relations are now so extensive that the Sting/Gauge duality is hard to doubt.

Maldacena's first example is IIB superstrings (or in the low energy limit IIB supergravity) propagating in an $\mathrm{AdS}^5 \times \mathrm{S}^5$ 10-d manifold,

$$ds^2 = \frac{r^2}{R^2} \sum_{\mu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^2} \left(dr^2 + r^2 d^2 \Omega_5\right), \tag{3.5}$$

which is dual to 4-d $\mathcal{N} = 4$ $\mathrm{U}(N_c)$ super Yang–Mills theory. The five co-ordinates, $(x_\mu, r)$, forms an AdS manifold with radius $R$ and $d^2\Omega_5$ is the metric on $\mathrm{S}^5$. To motivate this duality, one must identify it as the near horizon limit of a black brane metric due $N_c$ parallel massive $D3$-branes at $r = 0$ (see Fig. 1). Evidence had accumulated that there are two equivalent ways to describe the dynamics of the $D3$-branes. First by considering short open strings attached to the branes which at low energies is SUSY Yang–Mills theory and second by the near horizon fluctuations of closed superstrings or at low energy supergravity. The leap of faith was to conjecture that in the near horizon limit these are exactly equivalent. In sense this is a new manifestation of the old perturbative open/closed string duality.

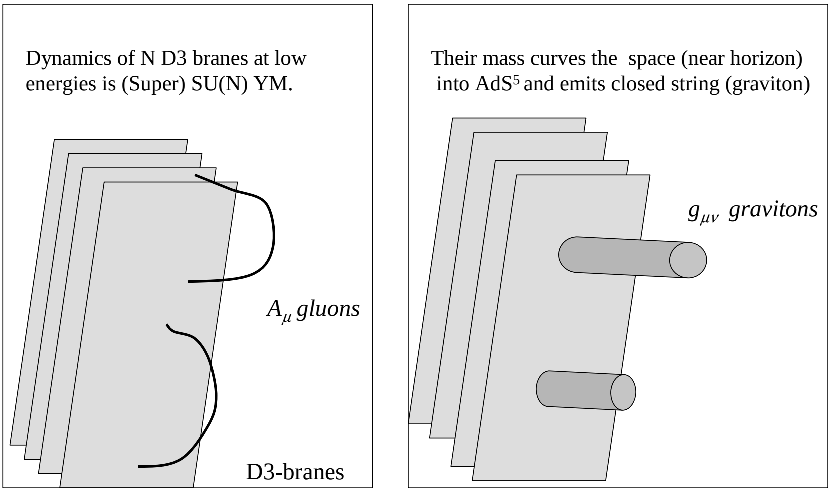


Fig. 1. Open/closed string duality on $N_c$ $D3$-branes.

In this dual correspondence the string (or gravity) correlation functions as you approach the boundary of $\mathrm{AdS}^5$ ($r \to \infty$) are equivalent to gauge invariant correlators in SYM theory. The discrete "Kaluza–Klein" modes in $\mathrm{S}^5$ give the multiplets under SU(4) $R$ symmetry . Note the combination of holography in r and the Kaluza–Klein mechanism on $\mathrm{S}^5$ explains how a 10-d string can be dual to a 4-d field theory. There is no loss of degrees of freedom. The 't Hooft gauge coupling is $g^2_{\mathrm{YM}} N_c = R^4/\alpha'^2_{\mathrm{s}}$ where the intrinsic string length scale is $\sqrt{\alpha_{\mathrm{s}}} = l_{\mathrm{s}}$. Consequently strong 't Hooft coupling gauge theory is weak coupling gravity gravity ($l_{\mathrm{s}} \sim l_{\mathrm{Planck}}$) and the $1/N_c$ plays the role of the closed string coupling constant $g_{\mathrm{s}} = g^2_{\mathrm{YM}}/4\pi \sim 1/N_c$ as one would expect from the large $N_c$ topological discussion for QCD above.

Although Maldacena's String/Gauge duality is believed to hold for general coupling and general $N_c$, it is difficult to quantize string theory in this background which includes a non-zero Ramond–Ramond flux even in the perturbative limit ($N_c \to \infty$). Consequently most analytic results rely on the strong 't Hooft coupling limit ($g_{\mathrm{s}} N_c \sim g^2_{\mathrm{YM}} N_c \to \infty$), where the theory becomes classical IIB gravity. Other special cases, such as the *pp*-wave limit and semiclassical limits are tractable as well.

### *3.3. Confinement*

One may view the correspondence in holographic terms. The Yang–Mills UV (short distance) degrees of freedom are dual to excitations near to the edge at $r \to \infty$, while the IR (long distance physics) is represented by modes at small $r \to 0$. A concrete illustration of this so called IR/UV correspondence is provided by the scale breaking instanton solution in Yang–Mills located at $x^\mu$ with size $\rho$. This corresponds exactly to a $D0$-brane located at five dimensional co-ordinate $x^\mu, r = 1/\rho$ in the AdS$^5$ manifold.

Ironically Maldacena's first example of Yang–Mills/String duality does not confine because the $\mathcal{N} = \infty$ Yang–Mills field theory is exactly conformal. Wilson loops have pure Coulomb (rather than area law) behavior. When a rectangular Wilson test loop, $L \times T$ is introduced on the boundary of AdS, the red shift factor $r^2/R^2$ of the metric allows the minimal surface area spanning the loop to remain finite by curving into the interior nearer and nearer to $r = 0$ as we increase the area: $L \times T \to \infty$ .

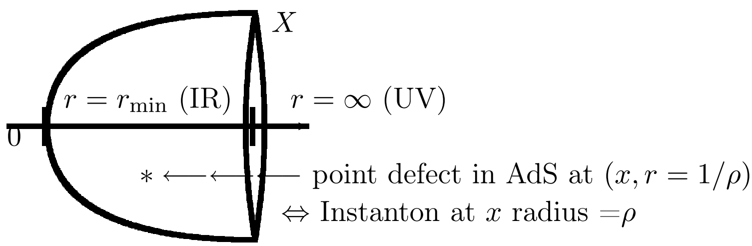


Fig. 2. Picture of an AdS black hole with its co-ordinate singularity at the Euclidean horizon at the origin of $(r, \tau)$ plane and a $D0$-brane instanton located at $x^\mu, 1/\rho$.

To look for string models closer to QCD, we must break conformal and supersymmetries. These models typically modify the metric in the IR cutting it off at a finite value $r = r_{\min}$. Two simple examples were suggested by Witten by introducing a Euclidean AdS black hole background with a compact dimension (called $\tau$) with radius set by the Hawking temperature:

- AdS$^5 \times$ S$^5$ Black Hole for 10-d IIB string theory;
- AdS$^7 \times$ S$^4$ Black Hole for 11-d M-theory.

The metric has the general form (see Fig. 2),

$$ds^2 = \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2[1-(r_{\min}/r)^d]}\,dr^2 + \frac{r^2}{R^2}[1-(r_{\min}/r)^d]\,d\tau^2 + ds_X^2\,. \tag{3.6}$$

This introduces a scale breaking cut-off which we can identify roughly as $\Lambda_{\rm QCD} = 1/r_{\min}$ or as we will see the scale of the glueball mass.

Now the area of the minimal surface as the Wilson loop increases in size eventually must grow when it approaches $r = r_{\min}$. For example in the AdS$^5$ black hole the proper areas grows like $r^2_{\min}/R^2$ giving a QCD tension

$T_{\rm QCD} = 1/2\pi\alpha'_{\rm QCD}$ or

$$\alpha_{\rm QCD} = \alpha'_s R^2/r^2_{\rm min} \sim \sqrt{g^2_{\rm YM} N_c} \Lambda^2_{\rm QCD} \,. \tag{3.7}$$

### *3.4. Hard scattering at wide angles*

One of the most baffling features of string theory (in flat space) at odds with QCD is the lack of hard scattering. As a theory of gravity, the softening of the short distance physics leads to a finite quantum theory, so it would seem to be intrinsic to superstrings. However we know that QCD, even in leading order of large $N_c$, exhibit asymptotic freedom and hard parton scattering properties. The fundamental "Rutherford experiment" for hadrons — scattering them at wide angles — has power law fall off precisely due to hard processes.

$$A_{\rm QCD}(s,t) \sim \left( \frac{1}{\sqrt{\alpha'_{\rm QCD} s}} \right)^{n-4} , \tag{3.8}$$

where $n = \sum_i \tau_i = \sum_i (d_i - s_i)$ with twist $\tau_i$ and conformal dimension $d_i$. In stark contrast the fundamental superstrings (in flat space) have exponential damped wide angle scattering

$$A_{\rm closed}(s,t) \to \exp\left[ -\frac{1}{2}\alpha'(s\ln s + t\ln t + u \ln u) \right] . \tag{3.9}$$

Polchinski and Strassler made the essential observation on how string scattering in a confining AdS background might avoid this conflict with QCD.

Suppose you have a background that is cut-off for small $r < r_{\rm min}$ and approximated by $\mathrm{AdS}^5 \times X_5$ for large $r$,

$$ds^2 = \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2 + ds^2_X \,. \tag{3.10}$$

A plane wave external glueball, $\phi(r)\exp[ixp]$, scatters locally in $r$ through a string amplitude with a red shifted proper distance or equivalently an effective momenta,

$$\hat{p}_{\rm s}(r) = \frac{R}{r} p \,.$$

Relative to the string scale, $l_{\rm s} = \sqrt{\alpha'_{\rm s}}$, the exponential cut-off at high momenta ($l_{\rm s} p_{\rm s} > 1$), suppresses string scattering in the IR region ($r < r_{\rm scatt}$), leaving a residual amplitude in a decreasing small window in the UV ($l_{\rm s}\, p_{\rm s} < 1$),

$$r > r_{\rm scatt} \equiv \sqrt{\alpha'_{\rm s}} R p \,.$$

Since the tail of the glueball wavefunction, $\phi_i(r) \sim (r/r_{\min})^{-\Delta_4^{(i)}}$, is entirely determined in the String/Gauge dictionary by the conformal weight $\Delta_4^{(i)}$ of the gauge operator dual to the string state, one is led back to the standard parton or naive dimensional analysis result used in the wide angle power counting,

$$\phi_i(r_{\rm scatt}) \sim \left(\frac{r_{\rm scatt}}{r_{\min}}\right)^{-\Delta_4^{(i)}} \sim (\sqrt{\alpha'_{\rm QCD}}\, p)^{-\Delta_4^{(i)}}\,, \tag{3.11}$$

where we have converted to the hadronic scale, using Eq. (3.7).

In the corresponding M-theory construction, all of this appears to be upset because the scaling of the wave function in AdS$^7$ changes. For example the scalar glueball with interpolating field $\mathrm{Tr}[F^{\mu\nu}F_{\mu\nu}]$ in AdS$^5$ has $\Delta_4 = 4$ as expected but in AdS$^7$ the wavefunction scales with $\Delta_6 = 6$ at large $r$. This apparent conflict with partonic expectations is avoided when one realizes that from an M-theory perspective, strings are a consequence of membranes wrapping the "11th" dimension and that in AdS$^7$ this 11th dimension is warped just like another spatial coordinate ($x^\mu$) with the proper size: $\hat{R}_{11}(r) = (r/R)R_{11}$. To account for this effect, one must also introduce local values for the effective string length and coupling constant:

$$\hat{l}_{\rm s}^2(r) = \frac{R}{r}(l_p^3/R_{11})\,, \quad \text{and} \quad \hat{g}_{\rm s}^2(r) = \frac{r^3}{R^3}(R_{11}^3/l_p^3)\,.$$

This additional deformation is precisely what is required. The new definition of the scattering region at wide angles,

$$r > r_{\rm scatt} = \hat{l}_{\rm s}(r_{\rm scatt})R\,p = \sqrt{\alpha'_{\rm s}}\,R^{\frac{2}{3}}\,r_{\rm scatt}^{-\frac{1}{2}}\,p\,,$$

leads to

$$\phi_i(r_{\rm scatt}) \sim \left(\frac{r_{\rm scatt}}{r_{\min}}\right)^{-\Delta_6^{(i)}} \sim \left(\sqrt{\alpha'_{\rm QCD}}\,p\right)^{-\frac{2}{3}\Delta_6^{(i)}}\,, \tag{3.12}$$

for each external line. For example, for the $0^{++}$ scalar glueball corresponding to the interpolating YM operator $\mathrm{Tr}[F^2]$, the factor of 2/3 exactly compensates for the shift in the conformal dimension from $\Delta_4 = 4$ for AdS$^5$ to $\Delta_6 = 6$ for AdS$^7$ to give the parton results, $n_i = 2\Delta_6^{(i)}/3$. This time, in converting to the hadronic scale in Eq. (3.12), we must realize the relationship of $\alpha'_{\rm QCD}$ to the string scale is $\alpha'_{\rm QCD} \sim (R/r_{\min})^3\alpha'_{\rm s}$. The 3rd power is a consequence of the fact that in M-theory the area law for the Wilson loop really comes from a minimal volume for a wrapped membrane world volume stabilized at $r \simeq r_{\min}$ rather than a minimal world surface area for a string which gave quadratic behavior in Eq. (3.7).

Putting all factors together, we can summarizing the results on hard scattering:

- Strong coupling AdS$^5$: $\Delta\sigma_{2\to m} \simeq \frac{1}{s}\, f\big(\frac{p_i\cdot p_j}{s}\big)\frac{(\sqrt{g^2 N_c})^m}{N_c^{2m}}\prod_i\Big(\frac{\sqrt{g^2 N_c}\Lambda^2_{\rm QCD}}{s}\Big)^{n_i-1}$
- Strong coupling AdS$^7$: $\Delta\sigma_{2\to m} \simeq \frac{1}{s}\, f\big(\frac{p_i\cdot p_j}{s}\big)\frac{1}{N_c^{2m}}\prod_i\Big(\frac{1}{\alpha'_{\rm QCD}s}\Big)^{n_i-1}$
- Weak coupling QCD: $\Delta\sigma_{2\to m} \simeq \frac{1}{s}\, f\big(\frac{p_i\cdot p_j}{s}\big)\frac{(g^2 N)^m}{N^{2m}}\prod_i\Big(\frac{g^2 N\Lambda^2_{\rm QCD}}{s}\Big)^{n_i-1}$

### *3.5. Near-forward scattering and Regge behavior*

The importance of scattering at large $r$ also implies the presence of a hard component in the near-Regge limit, $t/s \to 0$ as $s \to \infty$. The assumption of a single local scattering leads to $T(s,t) = \int_{r_h}^{\infty} dr\, \mathcal{K}(r)\, A(s,t,r)$, where $A$ is a local four-point amplitude, $\mathcal{K}(r) \sim r^5\phi_1(r)\phi_2(r)\phi_3(r)\phi_4(r)$, up to a constant, and $r_h$ is a cut-off, $r_h \gg r_{\rm min}$. After converting to local string parameters as discussed above, the amplitude $A(s,t,r)$ depends only on $\alpha'_{\rm s}\hat{s}$ and $\alpha'_{\rm s}\hat{t}$, where $\hat{s} = (R/r)^3 s$ and $\hat{t} = (R/r)^3 t$. In the Regge limit the amplitude becomes

$$T(s,t) = \int_{r_h}^{\infty} dr\, \mathcal{K}(r)\, \beta(\hat{t})(\alpha'_{\rm s}\hat{s})^{\alpha_0+\alpha'_{\rm s}\hat{t}}\,. \tag{3.13}$$

For small $t \simeq 0$, this corresponds to an exchange of a BFKL-like Pomeron, with a small effective Regge slope,

$$\alpha'_{\rm BFKL}(0) \sim (r_{\rm min}/r_h)^3\alpha'_{\rm QCD} \ll \alpha'_{\rm QCD}. \tag{3.14}$$

Such an exchange naturally leads to an elastic diffraction peak with little shrinkage. In the coordinate space, one finds, for a hard process, the transverse size is given by

$$\langle \vec{X}^2\rangle \sim (r_{\rm min}/r_h)^3\alpha'_{\rm QCD}\log s + \text{constant}\,. \tag{3.15}$$

There will be no transverse spread, if the cut-off, $r_h \sim \log s$, which characterizes a hard process, increases mildly with $s$. In the language of a recent study by Polchinski and Susskind, this corresponds to "thin" string fluctuations.

In spite of this progress in seeing some hard effects in the string picture, there is much more to understand. For instance, we note that, consistent with the known spectrum of glueballs at strong coupling, the IR-region must in addition give a factorizable Regge pole contribution,

$$T(s,t) \sim A(s,t,r_{\rm min}) \sim (\alpha'_{\rm QCD}s)^{\alpha_P(0)+\alpha'_{\rm QCD}t}\,. \tag{3.16}$$

The "soft" Pomeron must mix with the corresponding hard component, leading to a single Pomeron singularity in the large $N$ limit. However, addressing this issue requires a more refined treatment for the partonic structure within a hadron. As emphasized by Polchinski and Strassler, this is also what is required for treating deep inelastic scattering in a String/Gauge approach.

## 4. Lecture three: String *vs* lattice spectra

Based on the conformally broken backgrounds using Maldacena Gauge/String duality, we can begin to do some calculation in QCD like theories, at least in the strong coupling limit. We are in the position somewhat similar to a lattice cut-off theory. The strong coupling limit brings along non-universal cut-off dependent effects. For AdS models, the "cut-off" does not break Lorenz invariance but it introduces new charged Kaluza–Klein modes. Moreover unlike the lattice, we have (as yet) no algorithm (theoretical or numerical) in principle to send the cut-off to infinity. It is a coupled problem. The world sheet sigma model emits gravitons that perturb the background which in turn has a back reaction on the sigma model. Even finding the beta function perturbatively to the next order in $1/\alpha'$ is difficult. Still it is worth while to see if there is a reasonable spectrum in the strong coupling limit.

On the lattice side, where one can numerical take the weak coupling (continuum) limit, the spectra for glueballs and the quantum states of a stretch string are becoming quiet accurately determined including some studies of the extrapolation to $N_c = \infty$. In short the lattice has given and is capable of giving accurate spectral data for the quantum QCD string. If it exists, there can be only one answer. This is a unique opportunity: A concrete string theory problem with copious "experimental" data to constrain its construction.

### *4.1. Glueball spectra*

The first such comparison was the computation of the strong coupling glueball spectrum in the AdS$^7$ M-theory black hole. The correspondence for the quantum numbers for the gravity modes in terms of the Yang–Mills fields are read off the effective Born–Infeld action on the brane,

$$S = \int d^5x \det \left[G_{\mu\nu} + e^{-\phi/2}(B_{\mu\nu} + F_{\mu\nu})\right] + \int d^4x (C_1 F \wedge F + C_3 \wedge F + C_5)\,. \tag{4.1}$$

The entire spectrum for all states in the QCD super selection sector are now known and can be compared with lattice data for SU(3). The comparison is rather encouraging as a first approximation (see Fig. 3). All the states are

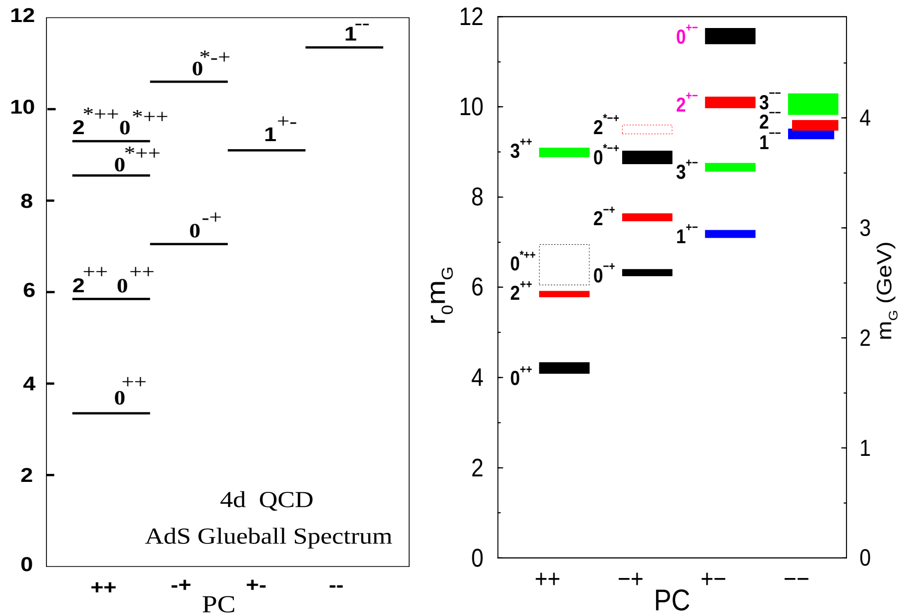


Fig. 3. The AdS glueball spectrum for $\mathrm{QCD}_4$ in strong coupling (left) compared with the lattice spectrum [8] for pure SU(3) QCD (right). The AdS Glueball mass scale, $1/r_{\min} = \Lambda_{\mathrm{QCD}}$ is adjusted to the lattice scale $1/r_0 = 410$ to fit the lowest $2^{++}$ tensor state.

in the correct relative order and the missing states at higher $J$ are a direct consequence of strong coupling which pushes the string tension to infinity. It appears plausible that the $\mathrm{AdS}^7$ black hole phase at strong coupling is rather smoothly connected to the weak coupling (confined) fixed point of QCD.

### *4.2. Stretched string spectra*

An even more direct observation of the string spectrum in lattice gauge data is the quantum modes of a fixed stretch string between infinitely heavy quarks (see Fig. 4). This is an open QCD string with Dirichlet boundary conditions. From the AdS/CFT view point starting with the string ends separated by a small distance $L$, we are able to see first the short distance Coulomb regime. Then as we increase $L$, the minimal surface moves into the interior probing more and more IR physics. Finally at very large $L$ we see only the lowest mass transverse "Goldstone modes" of the string leading to the universal spectrum of Lüscher,

$$E_n = T_0 L - \frac{\pi(D-2)}{12L} + \frac{2\pi a_n^\dagger a_n}{L}\,. \tag{4.2}$$

Indeed at large separation $L$ the lattice data for the stretched string spectrum appears to be approaching this form with $D-2=2$ transverse oscillators (see Fig. 4). A clever lattice algorithm developed by Lüscher confirmed

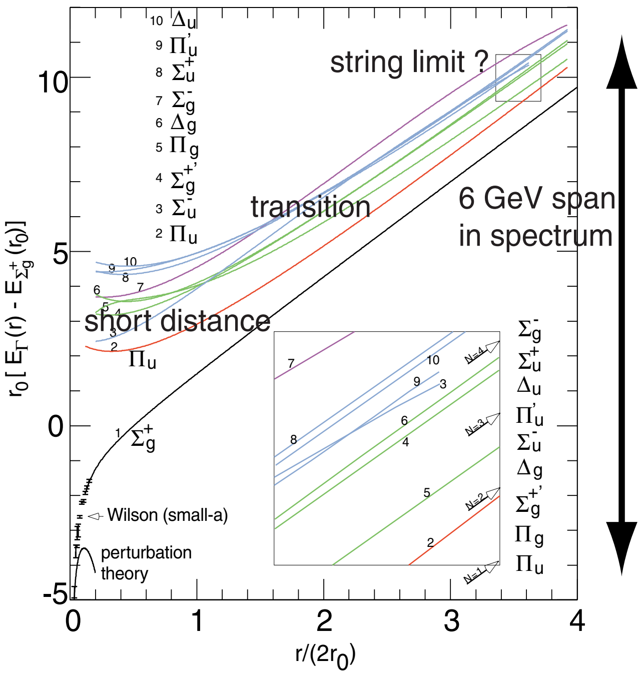


Fig. 4. The SU(3) QCD stretched string spectrum by Juge, Kuti and Morningstar [9] plotted relative to the Summer scale, $r_0 \simeq 0.5\,\mathrm{fm}$.

the universal "Lüscher term"

$$E_0(L) = \frac{\pi}{12}(1 + \frac{0.12\,\mathrm{fm}}{L})\,, \tag{4.3}$$

from a fit to the ground state (*i.e.* static potential) in the range $L$ from 0.5 to 1.0 fm.

The challenge to the AdS/CFT approach to the QCD string is to understand the interpolation between large $L$ and small $L$. As a first step one can consider an string inspired model for a "warped" metric

$$ds^2 = V(y)dx^\mu dx_\mu + dy^2 + U(y)d^2\tau + \cdots \tag{4.4}$$

suggested by an AdS$^{d+2}$ black hole with $V(y) = \frac{r^2}{r^2_{\mathrm{min}}} = \left[\cosh(\frac{d+1}{2R}y)\right]^{4/(d+1)}$.

The minimal surface for the classical potential obeys the following limits,

$$E_0 \to \frac{r_{\min}^2}{2\pi R^2\alpha'}\, L + O(Le^{-cL}) \quad \text{and} \quad E_0 \to -\frac{8\pi^2\sqrt{2}}{2\pi\alpha' \Gamma(1/4)^4}\, \frac{R^2}{L}\,, \tag{4.5}$$

which fits the lattice data almost perfectly after adjusting the mass $1/r_{\min} = \Lambda_{\rm QCD}$ and Regge slope $\alpha'_{\rm QCD} = \alpha'_{\rm s} R^2/r^2_{\min}$. This is reassuring but also highlights the limitation of our present situation. In the continuum limit QCD will fix the ratio of scales, $\alpha'_{\rm QCD}\Lambda^2_{\rm QCD}$, so there is only one free parameter. But at strong coupling in the AdS/CFT (or on the lattice for that matter) the fundamental string scale, $\alpha'_{\rm s} = R^2 f(g^2_{\rm YM} N_c)$, and the cut-off, $r^{-1}_{\min}$, provides two parameters that can be fit arbitrarily.

One can also investigate the quantized fluctuations in this model for the "warped" background. Choosing the string stretch symmetrically in the interval $z \in [-L/2, L/2]$ in a gauge with $z = X^3 = \sigma$, the transverse fluctuation obey the equation,

$$-\rho_0(z)\partial_t^2 X_\perp + \partial_z^2 X_\perp = 0\,, \tag{4.6}$$

and the radial mode,

$$-\rho_0(z)\partial_t^2 \xi + \partial_z^2 \xi'' = M^2(z)\xi\,, \tag{4.7}$$

with $\rho_0(z) = V^2(z)/V^2(0)$ and $M^2(z) = V''(z) - \frac{3}{2}\frac{V'(z)}{V(z)} \sim \Lambda^2_{\rm QCD}$. It is clear that at large $L$, this will reproduce the Lüscher's result for a $D-2=2$ string and as expected the mass of the extra radial mode is set by the mass scale of the glueballs. Indeed this is essentially just the open string analog of the close string glueball with amplitude concentrated near $r = r_{\min}$. However at best this is just a qualitative model of how a QCD string in warped space might behave. It is hoped that some insight can be gained by this comparison with lattice spectral data and that more fully self consistent string models can be solved in this limited context of low energy spectrum of fluctuations between infinitely heavy quarks.

## 5. No conclusions yet

The construction of the QCD string theory remains a tantalizing but unrealized goal. Recent progress has certainly begun to show how such a String/Gauge duality may arise. Indeed the intimate relations between Yang–Mills theory and string theory is a dramatic change in our understanding, which might be thought of as the "First String Counter Revolution" bring the subject back to its earliest roots. In this short lecture notes, it has not been possible to describe many important issues concerning the introduction of dynamical quarks and spontaneous chiral symmetry breaking, non-perturbative terms beyond the $g_{\rm s} \sim 1/N_c$ expansion such as the giant graviton baryon connection and attempts to identify short distance QCD physics. There is still much confusion on each of these topics with new ideas streaming forth. The most definitive progress based on String/Gauge duality relies on more tractable "toy models" of QCD with some residual Supersymmetry or special limits where semi-classical methods can be applied. This technical progress is of course the fundamental work that is needed to make real progress.

However it must be admitted that formidable challenges remain. First, even in the simplest case of pure $\mathrm{AdS}^5 \times \mathrm{S}^5$ it is not yet possible to quantize the superstring analytically. So hard evidence for the AdS/CFT duality is often somewhat indirect. When you break conformal and SUSY symmetries analysis becomes harder. Second, a basic difficult seems to remain in finding a way to really lift the mass scale for all charged Kaluza–Klein state outside the QCD sector above the physical states that should survive at a QCD fixed point. Perhaps the framework of starting from a critical string is flawed. Finally, it must be acknowledged that a direct constructive method for **the** QCD string (however difficult the mathematics may prove to be) is lacking. Still the conjecture that QCD is in fact a version of string theory has become more plausible and we are finding more and more about how such dualities arise. Let us hope that a young "Veneziano" in the "String Millennium" will come (quickly) to the rescue.

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[1] WARNING: This is not intended as a history of early string theory. A fair or even moderately complete treatment of history does not lend itself to a short pedagogical set of lectures.

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