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LECTURES ON SUPERSYMMETRIC GAUGE THEORIES*

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To the memory of Ian Kogan and Jan Kwieciński

Last year or so has seen a revival of interest in the dynamics of supersymmetric gauge theories. In this review we give *(i)* an introduction and a review of the earlier results in the field; *(ii)* discuss a more recent work of my own and of my collaborators on non-Abelian monopoles, vortices and confinement; and in the last lecture, we discuss *(iii)* the latest development in the dynamics of $\mathcal{N}=1$ gauge theories.

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1. Introduction and review of earlier results

1.1. Basics of SUSY gauge theories

Supersymmetric gauge theories continue to surprise us for deep insights they give us about the dynamics of non-Abelian gauge theories. We start with the basics of the supersymmetric theory [1].

The basic SUSY algebra contains

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha,\dot\alpha} P_\mu \,.$$

In order to construct supersymmetric theories it is convenient to introduce superfields

$$\begin{aligned} F(x,\theta,\bar\theta) &= f(x) + \theta\psi(x) + \ldots \,, \\ Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha,\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \,, \quad \bar{Q}_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha,\dot\alpha}\partial_\mu \,. \end{aligned}$$

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In general they are reducible with respect to supersymmetry transformations. We construct smaller irreducible multiplets. Chiral superfields are defined by the constraint $\bar{D}\Phi = 0$ ($D\Phi^\dagger = 0$) so that

$$\Phi(x,\theta,\bar\theta) = \phi(y) + \sqrt{2}\theta\,\psi(y) + \theta\theta\,F(y)\,, \quad y = x + i\theta\sigma\bar\theta\,,$$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha,\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\,, \qquad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha,\dot\alpha}\partial_\mu\,.$$

Vector superfields are defined to be real $V^\dagger = V$. They are conveniently expressed in terms of a chiral (fermionic) superfield

$$W_\alpha = -\frac{1}{4}\bar{D}^2 \mathrm{e}^{-V} D_\alpha \mathrm{e}^V = -i\lambda + \frac{\mu,}{2}\,(\sigma^\mu\,\bar\sigma^\nu)_\alpha^\beta\,F_{\mu\nu}\,\theta_\beta + \ldots\,.$$

Supersymmetric Lagrangian ($\int d\theta_1\,\theta_1 = 1$, *etc.*) can then be written simply as

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\,\tau_{\mathrm{cl}}\left[\int d^4\theta\;\Phi^\dagger \mathrm{e}^V\,\Phi + \int d^2\theta\,\frac{1}{2}WW\right] + \int d^2\theta\,\mathcal{W}(\Phi)\,, \qquad (1)$$

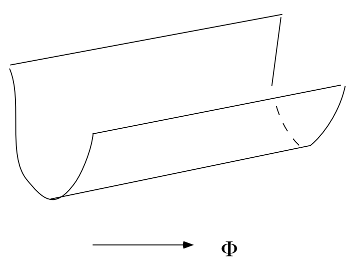
where $\mathcal{W}(\Phi)$ is the superpotential and $\tau_{\mathrm{cl}} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. The scalar potential is the sum of the F-term and D-term:

$$V_{\mathrm{sc}} = \sum_{\mathrm{mat}}\left|\frac{\partial\mathcal{W}}{\partial\phi}\right|^2 + \frac{1}{2}\sum_a\left|\sum_{\mathrm{mat}}\phi^* t^a\phi\right|^2\,.$$

For SQCD, $\{\Phi\} \to Q \sim \underline{N}$, $\tilde{Q} \sim \underline{N}^*$ of $\mathrm{SU}(N)$

$$G_F = \mathrm{SU}(n_f)\times\mathrm{SU}(n_f)\times\mathrm{U}_V(1)\times\mathrm{U}_A(1)\times\mathrm{U}_\lambda(1)\,.$$

Flat directions (CMS) *e.g.*, for $n_f < n_c$,



$$Q = \tilde{Q}^\dagger = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & \ddots & & \\ 0 & \ldots & & a_{n_f} \\ 0 & 0 & \ldots & 0 \\ \ldots & & & \ldots \end{pmatrix}$$

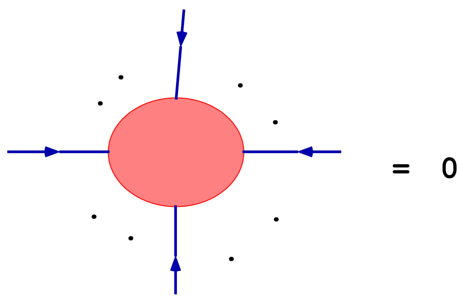
The problem is: is superpotential generated? Is CMS modified?

1.2. Nonrenormalization theorem

A crucial ingredient of the analysis is a set of so-called nonrenormalization theorems. Perturbative nonrenormalization theorem follows from the supergraph technique [1]

$$\begin{aligned}\mathcal{L} &= \int d^2\theta d^2\bar\theta \left(\bar\Phi\Phi + \tfrac{1}{2}\Phi^2\delta^2(\bar\theta) + \text{h.c.}\right),\\ &\langle T\,\Phi(x,\theta,\bar\theta)\,\Phi(x',\theta',\bar\theta')\rangle \\ &= -m\,\delta^2(\theta-\theta')\,\mathrm{e}^{-i(\theta\sigma^\mu\bar\theta-\theta'\sigma^\mu\bar\theta')\partial_\mu}\,\Delta_c(x-x'),\end{aligned}$$

etc. By using a particular property of Grassmannian delta functions (which encodes the known cancellation mechanism between bosonic and fermionic loops) one easily finds the standard result that only D-terms (of the form of $\int d^2\theta d^2\bar\theta(\ldots)$) get generated. No F-terms arise from radiative corrections.



If there exists an exact non-anomalous symmetry G then no terms violating G can be generated even non-perturbatively.

There are subtleties, however. Certain perturbative anomaly has been discovered in [2] which however all turns out [3] to be a fake F-term of the type

$$\Delta L = \int d^2\theta\, d^2\bar\theta\;\; \Phi^2\frac{D^2}{\Box}\Phi \sim \int d^2\theta\; \Phi^3\,.$$

No such nonlocal term simulating F-term arise in the Wilsonian action S_{W}.

Terms protected only by anomalous (*e.g.* $\mathrm{U}_A(1)$) symmetries can be generated by instantons.

Generalized non-renormalization theorem [3] tells that the gauge kinetic term

$$\int d^2\theta\, W_\alpha W^\alpha = \int d^2\theta\, d^2\bar\theta \left[\left(\mathrm{e}^{-V}D_\alpha\mathrm{e}^V\right)W^\alpha\right]\,,$$

being a kind of D-term, can be generated, *but by 1 loop corrections only.* This leads immediately to the so-called NSVZ exact β functions [4]. For instance, for SU(N) SQCD:

$$
\begin{aligned}
L \;=\; & \frac{1}{4}\int d^2\theta \left(\frac{1}{g^2(M)} - \frac{b_0}{8\pi^2}\log\frac{M}{\mu}\right) W^a W^a \\
& + \int d^4\theta \sum_i Z_i(\mu, M)\, \Phi_i^\dagger \mathrm{e}^{2V_i}\, \Phi_i \,,
\end{aligned}
$$

where $b_0 = -3N_c + \sum_i T_{Fi}$. A re-normalization of the matter fields Φ $\Phi_i \to Z_i^{-1/2}\,\Phi_i$ leads to the NSVZ β function

$$
\beta_h(g) \equiv \mu\frac{d}{d\mu}\, g(\mu) = -\frac{g^3}{16\pi^2}\left(3N_c - \sum_i T_{Fi}(1-\gamma_i)\right)\,,
$$

where $\gamma_i(g(\mu)) = -\mu\frac{\partial}{\partial\mu}\log Z_i(\mu, M)|_{M,g(M)}$. Actually by rescaling the fields further [5] (see [6] for discussion) $A_\mu = g_{\mathrm{c}} A_{c\mu}$, $\lambda = g_{\mathrm{c}}\lambda_c$, one gets a more frequently cited form of the NSVZ beta function

$$
\beta(g_{\mathrm{c}}) = -\frac{g_{\mathrm{c}}^3}{16\pi^2}\frac{3N_c - \sum_i T_{Fi}(1-\gamma_i)}{1 - N_c g_{\mathrm{c}}^2/8\pi^2}
$$

where g_{c} is the "canonical" coupling constant, related to the more natural "holomorphic" coupling constant defined above by

$$
\frac{1}{g^2} = \frac{1}{g_{\mathrm{c}}^2} + \frac{N_c}{8\pi^2}\log g_{\mathrm{c}}^2\,.
$$

1.3. SUSY Ward–Takahashi identities

SUSY transformation of components of the chiral superfield $\Phi(x,\theta,\bar\theta) = \phi(y) + \sqrt{2}\theta\psi + \theta\theta F(y)$,

$$
[\bar Q^{\dot\alpha}, \phi] = 0\,, \qquad \{\bar Q^{\dot\alpha}, \psi_\alpha\} = -\sqrt{2}\bar\sigma^\mu\partial_\mu\phi\,,
$$

implies that the chiral n-point function

$$
G = \langle T\phi_1(x_1)\phi_2(x_2)\ldots\phi_k(x_k)\rangle
$$

is independent of the spacetime argument [4]

$$
\bar\sigma^\mu\partial_\mu^{x_1}\, G = \langle T[\,\bar Q^{\dot\alpha}, (\psi_1(x_1)\phi_2(x_2)\ldots)]\rangle = 0\,,
$$

so that G is equal to $\prod_i\langle\phi_i\rangle$. Also they depend analytically on g_i, m_i *etc.* $(\mathcal{W}(\Phi) = m\Phi^2 + g_2\Phi^3 + \ldots)$

$$
\frac{\partial G}{\partial m^*} = \langle T[\,\bar Q^{\dot\alpha}, (\bar{\mathcal{W}}|_{\bar\theta}\,\phi_1(x_1)\phi_2(x_2)\ldots)]\rangle = 0\,,
$$

Fields	Δ	q_V	q_λ	q_X
$Q, \tilde{Q}$	1	$1, -1$	1	$n_c - n_f$
$\psi_Q, \psi_{\tilde{Q}}$	$3/2$	$1, -1$	0	n_c
λ_α	$\frac{3}{2}$	0	1	$-n_f$
g_l	$2-l$	$-(l+1)$	$1-l$	2
Λ^{2N}	$2N$	$2N$	$\frac{4N}{3}$	0

1.4. Anomalies and instanton

The anomaly plays an important role in many aspects of physics. The classical example is the $\mathrm{U}_A(1)$ anomaly

$$\partial_\mu J_5^\mu = \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

which, applied for QCD, yields the reasonable $\pi_0 \to 2\gamma$ decay rate. The $\mathrm{U}_A(1)$ anomaly due to QCD interactions is

$$\partial_\mu J_{\mathrm{L}}^\mu = \frac{g^2}{32\pi^2} G_{\mu\nu^a}\tilde{G}^{a,\mu\nu}$$

which leads to a chirality non conservation

$$\Delta Q_5 = 2\,n_f \int d^4x\,\frac{g^2}{32\pi^2} G_{\mu\nu^a}\tilde{G}^{a,\mu\nu} \neq 0\,.$$

This $\mathrm{U}_A(1)$ breaking leads to the solution of the "U(1)" problem ($m_\eta \gg m_\pi$? Why NO $\mathrm{U}_A(1)$ Goldstone boson) [7].

Actually, the fact that $\frac{g^2}{32\pi^2} G_{\mu\nu^a}\tilde{G}^{a,\mu\nu} = \partial_\mu K^\mu$ means that its integral is a topological invariant. In fact, a finite energy configuration must be asymptotically of the form

$$A_\mu \sim \mathrm{U}^{-1}(x)\partial_\mu \mathrm{U}(x), \qquad x \to \infty$$

representing the homotopy class $\Pi_3(\mathrm{SU}(2)) = Z$, with the Pontryagin number,

$$\int d^4x\,\frac{g^2}{32\pi^2} G_{\mu\nu^a}\tilde{G}^{a,\mu\nu} = n, \qquad n = 0, \pm 1, \pm 2, \dots\,.$$

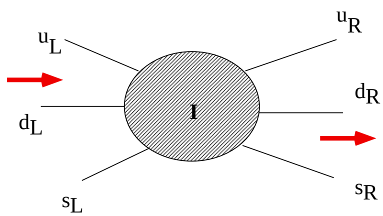
The configuration of minimum action with $n = 1$ is known as the instanton

$$A_\mu = -\frac{2i}{g^2}\frac{\tau_{\mu\nu}(x-x_0)_\nu}{(x-x_0)^2+\rho^2}\,, \qquad \tau_{\mu\nu} = \frac{\tau_\mu\bar{\tau}_\nu - \tau_\nu\bar{\tau}_\mu}{4}\,.$$

The instanton ('t Hooft) yields in QCD the effective Lagrangian

$$\mathcal{L}_{\rm eff} \sim \epsilon^{i_1\ldots i_{n_f}} \epsilon_{j_1\ldots j_{n_f}} \bar{\psi}_{\rm L}^{j_1}(x)\ldots\bar{\psi}_{\rm L}^{j_{n_f}}(x)\psi_{{\rm R},{\rm i}_1}(x)\ldots\psi_{{\rm R},{\rm i}_{\rm n_f}}(x),$$

leaving $\mathrm{U}_A(1)$ broken to Z_{2n_f} while $\mathrm{SU}_{\rm L}(n_f)\times\mathrm{SU}_{\rm R}(n_f)$ remains unbroken. Equivalently, the $2\,n_f$ point function



$$\langle\epsilon^{i_1\ldots i_{n_f}} \epsilon_{j_1\ldots j_{n_f}} \bar{\psi}_{\rm L}^{j_1}(x_1)\ldots\bar{\psi}_{\rm L}^{j_{n_f}}(x_{n_f})\psi_{{\rm R},{\rm i}_1}(y_1)\ldots\psi_{{\rm R},{\rm i}_{\rm n_f}}(y_{n_f})\rangle \neq 0$$

is nonvanishing (while zero to all orders in perturbation theory). The presence of the instanton effects means that the θ term is possible in QCD:

$$\mathcal{L} = \theta\frac{g^2}{32\pi^2}G_{\mu\nu^a}\tilde{G}^{a,\mu\nu}$$

which is renormalizable. The experimental limit ($d_n < 10^{-28}$ e cm) means that

$$|\theta| < 10^{-9}$$

which is the "Strong CP Problem" (why is $|\theta|$ so small?) A possible solution is the Peccei–Quinn symmetry (axions); another possibility is that $m_u = 0$. Another problem, possibly related to the instanton effects is the so-called $\Delta I = \frac{1}{2}$ problem (why is the ratio $\frac{A(K\to\pi\pi)^{\Delta I=1/2}}{A(K\to\pi\pi)^{\Delta I=3/2}} \sim 25$ so large?)

1.5. Instanton calculation in SUSY QCD

In the so-called strong coupling (standard) instanton method [9] one calculates

$$\langle\lambda\lambda(x_1)\lambda\lambda(x_2)\ldots\lambda\lambda(x_{n_c})\rangle = \text{const.}\,\Lambda^{3n_c}$$

which gives

$$\text{L.H.S.} = \text{const.} = \prod\langle\lambda\lambda\rangle = \langle\lambda\lambda\rangle^{n_c}.$$

The fact that the Z_{2n_c} discrete symmetry is not broken by the instanton effect means that one must disentangle the vacuum sum to get the vacuum expectation value (VEV) of the gluino condensate.

In the weak coupling instanton method [10] one considers:

(i) SQCD with massless $(Q, \tilde{Q})$'s with flat direction, and computes;

(ii) the instanton corrections at large $\langle Q \rangle \gg \Lambda$, which yields the Affleck–Dine–Seiberg effective superpotential;

$$\Delta \mathcal{W}^{(\mathrm{ADS})} = (n_c - n_f) \frac{\Lambda^{(3n_c - n_f)/(n_c - n_f)}}{(\det Q\tilde{Q})^{1/(n_c - n_f)}}, \tag{2}$$

(iii) adds then the mass term $\mathcal{W}_{\mathrm{mass}} = mQ\tilde{Q}$ and computes the minimum of the potential;

(iv) then decouple the quarks $m \to \infty$, $\Lambda^*_{\mathrm{YM}} = m\Lambda^*$, to get $\langle \lambda\lambda \rangle = \Lambda^3$.

There is a numerical discrepancy ("4/5 puzzle") between the results obtained in the two methods, which led to various speculations. Other methods, all involving computation of the gluino condensate *in a single vacuum*, give WCI results. For SU$(r+1)$, SO$(2r+1)$, USp$(2r)$, SO$(2r)$ SYM, the result is [11, 12]

$$\left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{\mathrm{SU}(r+1)} = \Lambda^3 , \qquad \left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{\mathrm{SO}(2r+1)} = 2^{\frac{4}{2r-1}-1} \Lambda^3 ,$$
$$\left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{\mathrm{USp}(2r)} = 2^{1-\frac{2}{r+1}} \Lambda^3 , \qquad \left\langle \frac{\mathrm{Tr}\lambda^2}{16\pi^2} \right\rangle_{\mathrm{SO}(2r)} = 2^{\frac{2}{r-1}-1} \Lambda^3 .$$

1.6. U(1)-related (Konishi) anomaly

Another useful ingredient is the anomalous identities related to the axial (or chiral) U(1) anomaly.

- For SQCD it reads, in the superfield form,

$$-\frac{1}{4} \bar{D}^2 (\, Q^\dagger \mathrm{e}^V Q \,) = m \, \tilde{Q} \, Q + \frac{g^2}{16\pi^2} \mathrm{Tr} W_\alpha W^\alpha \, .$$

  The imaginary part of the F-component of the both sides corresponds to the $\mathrm{U}_A(1)$ anomaly;

- the lowest component instead gives

$$\{ \bar{Q}_{\dot{\alpha}}, \bar{\psi}^{\dot{\alpha}} Q \} = m \, \tilde{Q} \, Q - \frac{g^2}{16\pi^2} \mathrm{Tr} \lambda_\alpha \lambda^\alpha$$

which leads to an exact relation

$$\langle m_i \tilde{Q}_i Q_i \rangle = \left\langle \frac{g^2}{16\pi^2} \mathrm{Tr}\lambda_\alpha \lambda^\alpha \right\rangle \qquad \text{(no sum)} \quad i = 1, \dots n_f ,$$

(*cf.* $\langle \bar{\psi}_i \psi_i \rangle = -\Lambda^2$ ($i = u, d, s$) in QCD).

- In a general chiral gauge theory with superpotential $\mathcal{W}(\Phi_i)$ the corresponding identity is

$$-\frac{1}{4}\bar{D}^2 (\Phi_i^\dagger \mathrm{e}^V \Phi_i) = \Phi_i \frac{\partial \mathcal{W}}{\partial \Phi_i} + C(\Phi_i) \frac{g^2}{16\pi^2} \mathrm{Tr} W_\alpha W^\alpha .$$

- These relations can be used as a check of dynamical calculation (instantons), of various approximations or of general arguments.

- To obtain the Konishi anomaly, one considers the functional change of variables, $\delta\Phi_i = iA(z)\Phi_i$ ($A(z)$ arbitrary chiral superfields): the Jacobian is

$$J = \det \left(\frac{\delta \Phi'_{z'}}{\delta \Phi_z} \right) = \det \langle z' | iA(z) \left(-\frac{\bar{D}^2}{4} \right) | z \rangle = \mathrm{e}^{\mathrm{Tr} iA(z) \frac{-\bar{D}^2}{4}} .$$

Regularizing the high frequency modes by ($L \equiv \bar{D}^2 \mathrm{e}^{-V} D^2 \mathrm{e}^V / 16$) gives

$$\mathrm{Tr} \left[iA(z) \frac{-\bar{D}^2}{4} \right] \to \lim_{M \to \infty} \mathrm{Tr} \left[iA(z) \mathrm{e}^{L/M^2} \frac{-\bar{D}^2}{4} \right] .$$

- To compute it, note that L can be written, acting on $\frac{-\bar{D}^2}{4}$, as

$$L = P^2 - \frac{1}{2} W^\alpha D_\alpha + C^\mu P_\mu + F ,$$

where

$$\begin{aligned} W^\alpha &= -\frac{1}{4} (\bar{D}^2 \mathrm{e}^{-V} D^\alpha \mathrm{e}^V) , \\ C^\mu &= -\frac{1}{2} \sigma^\mu_{\alpha\dot{\alpha}} (\bar{D}^{\dot{\alpha}} \mathrm{e}^{-V} D^\alpha \mathrm{e}^V) , \end{aligned}$$

and

$$F = \frac{(\bar{D}^2 \mathrm{e}^{-V} D^2 \mathrm{e}^V)}{16} .$$

- Each power of terms in L is accompanied by a factor M^{-1} as $M \to \infty$; on the other hand one needs at least two powers of D's:

$$\langle \theta\bar{\theta}|DD\bar{D}^2|\theta\bar{\theta}\rangle \neq 0\,.$$

  The net result is that only terms quadratic in $\frac{1}{2}W^{\alpha}D_{\alpha}$ contribute. Computing that term by going to a plane wave basis yields the anomaly.

- Although the Konishi anomaly has been obtained in almost all known methods, such as Pauli–Villars regularization, explicit 1-loop calculation, point-splitting, BPHZ, both in component and superfield formalisms, the functional integral method seems to be particularly adequate for generalization (see Section 3).

1.7. Phases of SQCD; Seiberg's duality

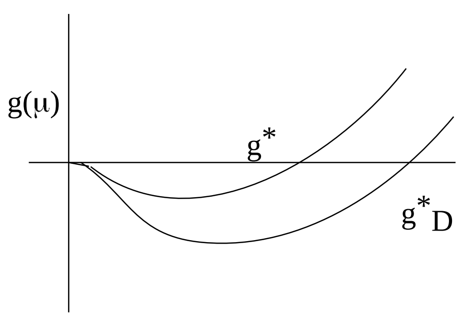
These analyses and Seiberg's duality [14] have established the following picture of the vacuum in the massless SQCD.

- The dynamically generated superpotential (2) implies the vacuum runaway ($n_f < n_c$); while no superpotential is generated for $n_f > n_c$.

- For $n_f = n_c$ the moduli space (space of vacua) is quantum mechanically modified as

$$\det M - B\,\tilde{B} = \Lambda^{2n_f}\,.$$

- $\frac{3n_c}{2} < n_f < 3n_c$ (conformal window), the system is in an infrared fixed point (SCFT): the low energy physics is described either as the original SQCD or as the dual $\mathrm{SU}(\tilde{n}_c) = \mathrm{SU}(n_f - n_c)$ theory with dual quarks.

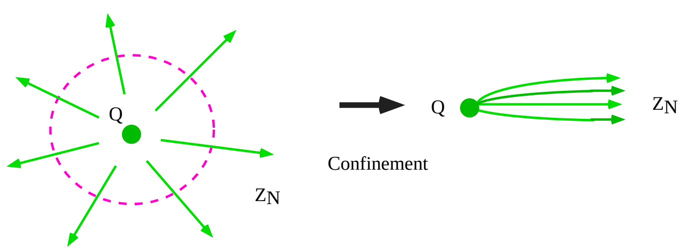


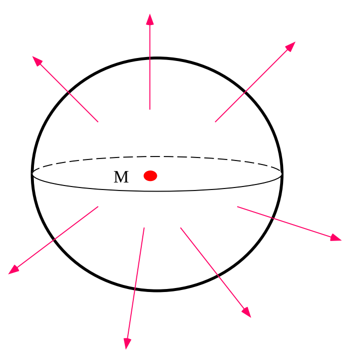
N_f	Deg. freedom	Eff. gauge group	Phase	Symmetry
0 (SYM)	—	—	Confinement	—
$1 \leq N_f < N_c$	—	—	no vacua	—
N_c	$M, B, \tilde{B}$	—	Confinement	$\mathrm{U}(N_f)$
$N_c + 1$	$M, B, \tilde{B}$	—	Confinement	Unbroken
$N_c + 1 < N_f < \frac{3N_c}{2}$	$q, \tilde{q}, M$	$\mathrm{SU}(\tilde{N}_c)$	Free-magnetic	Unbroken
$\frac{3N_c}{2} < N_f < 3N_c$	$q, \tilde{q}, M$ or $Q, \tilde{Q}$	$\mathrm{SU}(\tilde{N}_c)$ or $\mathrm{SU}(N_c)$	SCFT	Unbroken
$N_f = 3N_c$	$Q, \tilde{Q}$	$\mathrm{SU}(N_c)$	SCFT (finite)	Unbroken
$N_f > 3N_c$	$Q, \tilde{Q}$	$\mathrm{SU}(N_c)$	Free Electric	Unbroken

2. Non-Abelian monopoles, vortices and confinement

2.1. SU(N) YM

The test charges in $\mathrm{SU}(N)$ YM theory take values in $(Z_N^{(M)}, Z_N^{(E)})$ where Z_N is the center of $\mathrm{SU}(N)$ and $Z_N^{(M)}, Z_N^{(E)}$ refer to the magnetic and electric center charges. $(Z_N^{(M)}, Z_N^{(E)})$ classification of phases ('t Hooft) follows (see figure).





$\Pi_1(SU(N)/Z_N) \sim Z_N$

2.2. SU(N) YM and solvable cousins

- If field with $x = (a, b)$ condense, particles $X = (A, B)$ with

$$\langle x, X \rangle \equiv a\,B - b\,A \neq 0 \quad (\mathrm{mod}\ N)$$

  are confined (*e.g.* $\langle \phi_{(0,1)} \rangle \neq 0 \to$ Higgs phase).

- Quarks are confined if some field χ exist, such that

$$\langle \chi_{(1,0)} \rangle \neq 0\,.$$

- In the softly broken $N = 4$ (to $N = 1$) theory (often referred to as $N = 1^*$) all different types of massive vacua, related by $\mathrm{SL}(2, Z)$, appear; the chiral condensates in each vacua are known.

- In softly broken $N = 2$ Gauge Theories, dynamics turns out to be particularly transparent.

The questions we wish to address are: What is χ in QCD? How do they interact? Is XSB related to confinement? θ vacua?; $\frac{\epsilon'}{\epsilon}$; $\Delta I = \frac{1}{2}$?

2.3. QCD as dual superconductor

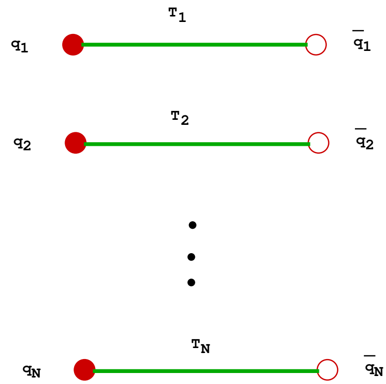
A familiar idea is that the ground state of QCD is a dual superconductor [15]. There exist no elementary nor soliton monopoles in QCD; however, monopoles can appear as topological singularities (lines in 4D) of Abelian gauge fixing, $\mathrm{SU}(3) \to \mathrm{U}(1)^2$. Alternatively, one can assume that certain configuration similar to Wu–Yang monopole (SU(2))

$$A_\mu^a = \tilde{\sigma}(x)(\partial_\mu n \times n)^a + \ldots, \qquad n(r) = \frac{r}{r} \quad \Rightarrow \quad A_i^a = \epsilon_{aij}\frac{r^j}{r^3}$$

dominate [16]. Although there are some evidence in lattice QCD [17] for "Abelian dominance", there are several questions to be answered. Do Abelian monopoles carry flavor? ($\mathcal{L}_{\mathrm{eff}}$?) What about the gauge dependence? Most significantly, does dynamical $\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1}$ breaking occur? That would imply a richer spectrum of mesons ($T_1 \neq T_2$, *etc.*) not seen in Nature and not expected in QCD. Both in Nature and presumably in QCD there is only one "meson" state

$$\text{meson} \sim \sum_{i=1}^{N} |\, q_i\, \bar{q}_i \,\rangle$$

i.e., 1 state *vs* $\left[\frac{N}{2}\right]$ states ($\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1} \times$ Weyl not enough).



2.4. Dirac's monopoles

As is well known QED admits pointlike magnetic monopoles if Dirac's quantization condition

$$g\,e = \frac{n}{2}\,, \qquad n \in \ Z\,, \tag{3}$$

is satisfied. In the presence of a magnetic monopole, there cannot be a gauge vector potential which is everywhere regular. A possible singularity (Dirac string) along $(0,0,0) \to (0,0,-\infty)$ is invisible if (3) is satisfied. A proper formulation is to cover S^2 by two regions $\mathtt{a}$ $(0 \le \theta < \frac{\pi}{2} + \epsilon)$ and $\mathtt{b}$ $(\frac{\pi}{2} - \epsilon < \theta \le \pi)$ [18]

$$(A_\phi)^{\mathtt{a}} = \frac{g}{r\sin\theta}(1-\cos\theta)\,, \qquad (A_\phi)^{\mathtt{b}} = -\frac{g}{r\sin\theta}(1+\cos\theta)\,,$$

so that in each neighborhood the vector potential is regular. The two descriptions are related along the equator by a gauge transformation

$$A_i^{\mathtt{a}} = A_i^{\mathtt{b}} - U^\dagger \frac{i}{e}\partial_i U\,, \qquad U = \mathrm{e}^{2ige\phi}.$$

The gauge transformation is well-defined if the condition (3) is met. More generally, for dyons (e_1, g_1), $(e_2 g_2)$, the quantization condition reads

$$e_1\,g_2 - e_2\,g_1 = \frac{n}{2}\,, \qquad n \in \ Z\,. \tag{4}$$

The topology involved is: $\Pi_1(\mathrm{U}(1)) = \ Z$.

2.5. Non-Abelian gauge theories

In the case of a non-Abelian gauge group, one might embed Dirac's monopole in a U(1) subgroup. However, the homotopy group properties such as

$$\mathrm{SU}(2) \sim S^3\,, \qquad \Pi_1(\mathrm{SU}(2)) = 1\,, \tag{5}$$

$$\mathrm{SO}(3) \sim \frac{S^3}{Z_2}\,, \qquad \Pi_1(\mathrm{SO}(3)) = Z_2\,, \tag{6}$$

show that there are no monopoles in SU(2), SU(N); there is only one type of monopole in SO(3), and so on [18].

In spontaneously broken gauge theories, instead, there are ('t Hooft–Polyakov) monopoles [19]

$$\mathrm{SU}(2) \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} \mathrm{U}(1)$$

$$\begin{aligned} \mathcal{D}\phi \stackrel{r\to\infty}{\longrightarrow} 0, \quad &\Rightarrow \quad \phi \sim U\cdot\langle\phi\rangle\cdot U^{-1}\,; \\ A_i \sim U\cdot\partial_i U^\dagger \quad &\Rightarrow \quad F_{ij} = \epsilon_{ijk}\frac{r_k}{r^3}m\frac{\tau_3}{2} \end{aligned}$$

which are regular, finite energy soliton-like solutions, with topology $\Pi_2(\mathrm{SU}(2)/\mathrm{U}(1)) = \Pi_1(\mathrm{U}(1)) = Z$. The static energy can be written as (Bogomolny)

$$\begin{aligned} H &= \int d^3x\left[\frac{1}{4}(F^a_{ij})^2 + \frac{1}{2}(D_i\phi^a)^2 + \frac{\lambda}{2}(\phi^2 - v^2)^2\right] \\ &= \int d^3x\left[\frac{1}{4}(F^a_{ij} - \epsilon_{ijk}D_k\phi^a)^2 + \frac{1}{2}F^a_{ij}\epsilon_{ijk}D_k\phi^a + \mathrm{pot.}\right]\,, \end{aligned}$$

where $\frac{1}{2}F^a_{ij}\epsilon_{ijk}D_k\phi^a = \partial_i S_i$; $S_i = \frac{1}{2}\epsilon_{ijk}F^a_{ij}\phi^a$: the second term in the square bracket is a topological invariant. It follows that in a given sector

$$H \geq \int d^3x \nabla\cdot S = \frac{4\pi v}{g}m\,, \qquad m = 1, 2\,, \ldots\,.$$

If $\lambda = 0$ the configuration of the minimum energy is given by the solution of the linear (Bogomolny) equations

$$F^a_{ij} - \epsilon_{ijk}D_k\phi^a = 0\,; \qquad B^a_i = D_i\phi^a$$

whose solutions are known in analytic form.

A more general situation is the spontaneously broken gauge theory with

$$G \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} H\,,$$

where H is non-Abelian [20, 21]. The asymptotic behavior is

$$\begin{aligned} \mathcal{D}\phi \stackrel{r\to\infty}{\longrightarrow} 0, \quad \Rightarrow \quad \phi &\sim U\cdot\langle\phi\rangle\cdot U^{-1} \sim \Pi_2(G/H) = \Pi_1(H)\,; \\ A_i &\sim U\cdot\partial_i U^{\dagger} \to \quad F_{ij} = \epsilon_{ijk}\frac{r_k}{r^3}\beta_\ell T_\ell\,, \end{aligned}$$

where $T_i \in$ Cartan S.A. of H. Topological quantization leads to $2\,\alpha\cdot\beta \subset Z$, where $\beta_i =$ weight vectors of $\tilde{H} =$ dual of H. Examples of the dual of several groups are given in the Table.

$$\tilde{H} \Leftrightarrow H$$

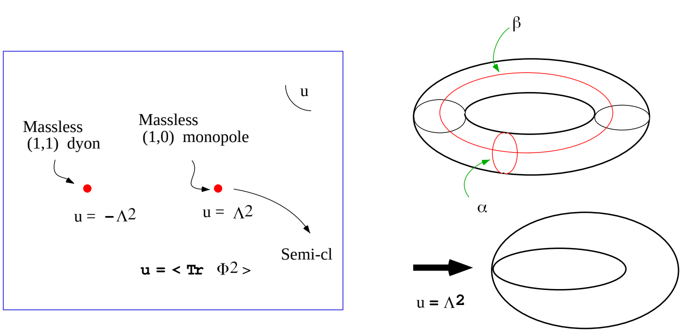
$\mathrm{SU}(N)/Z_N$	$\Leftrightarrow$	$\mathrm{SU}(N)$
$\mathrm{SO}(2N)$	$\Leftrightarrow$	$\mathrm{SO}(2N)$
$\mathrm{SO}(2N+1)$	$\Leftrightarrow$	$\mathrm{USp}(2N)$

They reduce to singular Dirac-like monopoles (Wu–Yang) for $|\phi| \to \infty$; 't Hooft–Polyakov monopoles for $G = \mathrm{SU}(2)$, $H = \mathrm{U}(1)$.

2.6. Seiberg–Witten, $\mathcal{N} = 2$ gauge theory

The Lagrangian of a $\mathcal{N} = 2$ YM theory is Eq. (1) with $\mathcal{W}(\Phi) = 0$. For SU(2) the vacuum degeneracy (moduli space) is parametrized as

$$\langle\Phi\rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix},$$



$a \neq 0$ breaks SU(2) $\to$ U(1): at IR,

$$\mathcal{L}_{\rm eff} = {\rm Im}\left[\int d^4\theta\, \bar{A}\ \frac{\partial \mathcal{F}_p(A)}{\partial A} + \int \frac{1}{2}\frac{\partial^2 \mathcal{F}_p(A)}{\partial A^2} W_\alpha W^\alpha\right],$$

where $W_\alpha,\ A$ describe $\mathcal{N}=2$ U(1) theory, $\mathcal{F}_p(A)$ is the prepotential. Define the dual of A, $A_{\rm D} \equiv \frac{\partial \mathcal{F}_p(A)}{\partial A}$: then

$$\frac{d\,A_{\rm D}}{d\,u} = \oint_\alpha \frac{dx}{y}\,, \qquad \frac{d\,A}{d\,u} = \oint_\beta \frac{dx}{y}\,,$$

where the curve is ($u \equiv {\rm Tr}\,\langle \Phi^2 \rangle$ describes the quantum moduli space — QMS)

$$y^2 = (x-u)(x+\Lambda^2)(x-\Lambda^2)\,.$$

The exact mass formula (BPS) following from the $\mathcal{N}=2$ SUSY algebra is

$$m_{n_m,n_e} = \sqrt{2}\,|\,n_m A_{\rm D} + n_e A\,|\,.$$

The above four formulae constitute the Seiberg–Witten solution [22–24].

The adjoint scalar mass ($\mu\,\Phi^2$ perturbation) leads to the low-energy effective superpotential near the singularity, $u \simeq \Lambda^2$:

$$\mathcal{W}_{\rm eff} = \sqrt{2}\,A_{\rm D}\,M\tilde{M} + \mu\,U(A_{\rm D})\,.$$

Minimization of the potential leads to the condensation of the monopole $\langle M \rangle \sim \sqrt{\mu\,\Lambda}$ (confinement).

It is interesting to note that at the singularities $u = \pm\Lambda^2$, instanton sum diverges

$$\langle {\rm Tr}\,\Phi^2 \rangle = \frac{a^2}{2} + \frac{\Lambda^4}{a^2} + \ldots = \ldots + 1 + 1 + 1 + \ldots\,.$$

The discussion can be generalized to $\mathcal{N}=2$ pure YM theories with other gauge groups. In general, dynamical abelianization occurs near the monopole singularities, for instance, SU(N) gauge group gets dynamically broken as SU(N) $\to$ U(1)$^{N-1}$ (*cf.* QCD).

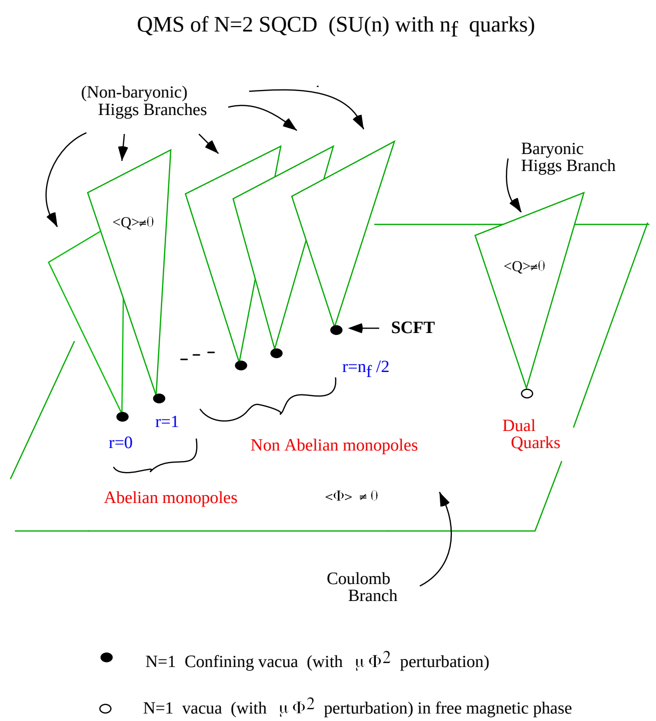
It is important to realize that these light "monopoles" are indeed 't Hooft–Polyakov monopoles becoming light by quantum corrections. This can be proven by studying the charge fractionalization [25]. For instance, the electric charge of the monopole is known to behave as

$$\frac{2}{g}Q_e = n_e + \left[-\frac{4}{\pi}{\rm Arg}\,a + \frac{1}{2\pi}\sum_{f=1}^{N_f}{\rm Arg}\,(m_f^2 - 2a^2)\right]\,n_m + \ldots\,.$$

in the semi-classical region. The Seiberg–Witten exact solution, when extrapolated back to the semi-classical domain, reproduces exactly the previously known one-loop results. An analogous check has been done for the quark number fractionalization. There is an interesting phenomenon of quantum quenching of quark numbers of massless, condensing monopole. Also, the non-Abelian flavor quantum number of the quantum monopoles as encoded in the Seiberg–Witten solution is consistent with the well-known Jackiw–Rebbi mechanism.

2.7. More general $\mathcal{N} = 2$ models

The study of the more general class of $\mathcal{N} = 2$ theories [27, 28] has shown that there are variety of confining vacua (see figure):



1. There are vacua ($r = 0, 1$) in which the low-energy effective action is an Abelian (dual) gauge theory. Upon the adjoint scalar mass perturbation $\mu\, \Phi^2$, the magnetic monopole condenses (confinement); the system displays dynamical abelianization, a feature not shared by the real world QCD.

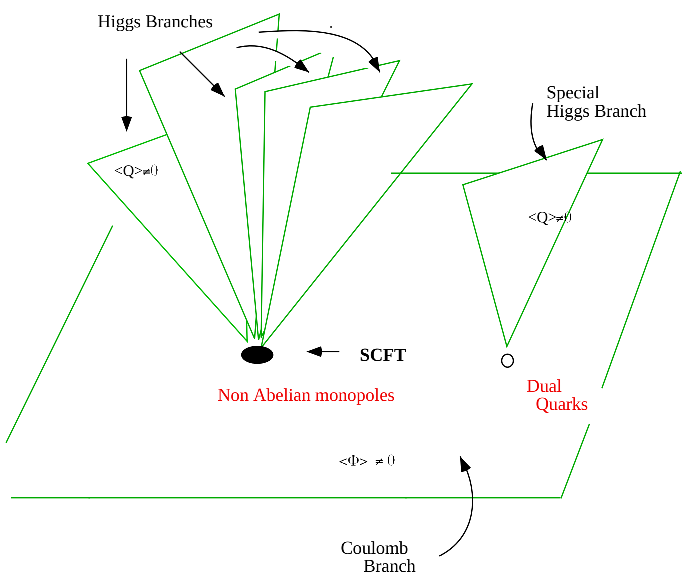
2. In a series of r-vacua, the effective action is $G_{\rm eff} \sim \mathrm{SU}(r) \times \mathrm{U}(1)^{n_c - r - 1}$; with n_f dual quarks in $\underline{r}$ of the low-energy $\mathrm{SU}(r)$ group. The "dual quarks" can be identified with the standard non-Abelian monopoles.

3. These r-vacua exist for $r \leq \frac{n_f}{2}$.

4. Superconformal theory (SCFT) occurs at $r = \frac{n_f}{2}$. Here the question is what (mutually nonlocal) degrees of freedom describe the SCFT (which confines upon the perturbation $\mu\, \Phi^2$).

Physics of $\mathrm{USp}(2\, n_c)$ ($\mathrm{SO}(n_c)$ similar) theory is even more interesting. All r vacua (at finite m) collapse into a single SCFT at $m \to 0$; in other words, all confining vacua (with $\mu\, \Phi^2$) are of this type; also, the global $\mathrm{SO}(2\, n_f) \to \mathrm{U}(n_f)$ symmetry breaking pattern (in the case of $\mathrm{USp}(2\, n_c)$ theory) is very reminiscent of what happens in QCD (*cf.* $\langle \bar{\psi}\, \psi \rangle^{(\mathrm{QCD})} \neq 0$).

QMS of N=2 USp(2n) Theory with n_f Quarks



● N=1 Confining vacua (with $\mu\,\Phi^2$ perturbation)

○ N=1 vacua (with $\mu\,\Phi^2$ perturbation) in free magnetic phase

2.8. More about non-Abelian monopoles

Consider the system with gauge symmetry breaking

$$\mathrm{SU}(3) \xrightarrow{\langle\phi\rangle} \mathrm{SU}(2) \times \mathrm{U}(1), \qquad \langle\phi\rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}.$$

By making use of the 't Hooft–Polyakov solutions in $\mathrm{SU}_U(2), \mathrm{US}_V(2) \subset \mathrm{SU}(3)$ one finds two degenerate SU(3) solutions. Analogously, for the system

monopoles	$\tilde{\mathrm{SU}}(2)$	$\tilde{\mathrm{U}}(1)$
$\tilde{q}$	$\underline{2}$	1

with symmetry breaking

$$\mathrm{SU}(n) \xrightarrow{\langle\phi\rangle} \mathrm{SU}(r) \times \mathrm{U}^{n-r}(1), \qquad \langle\phi\rangle = \begin{pmatrix} v_1 1_{r\times r} & 0 & \dots & 0 \\ 0 & v_2 & 0 & \dots \\ 0 & 0 & \ddots & \dots \\ 0 & 0 & \dots & v_{n-r+1} \end{pmatrix}$$

one finds the following set of the minimal monopoles:

monopoles	$\tilde{\mathrm{SU}}(r)$	$\tilde{\mathrm{U}}_0(1)$	$\tilde{\mathrm{U}}_1(1)$	$\tilde{\mathrm{U}}_2(1)$	$\dots$	$\tilde{\mathrm{U}}_{n-r-1}(1)$
q	$\underline{r}$	1	0	0	$\dots$	0
e_1	$\underline{1}$	0	1	0	$\dots$	0
e_2	$\underline{1}$	0	0	1	0	0
$\vdots$	$\underline{1}$	0	$\dots$			0
e_{n-r-1}	$\underline{1}$	0	0	$\dots$	$\dots$	1

We note [26] that

- they represent a degenerate r-plet of monopole solutions (q);
- they have *the same charge structure as that of the "dual quarks" appearing in the r-vacua of $\mathcal{N} = 2$ SQCD;*
- they have the correct flavor quantum numbers due to the Jackiw–Rebbi mechanism.

2.9. Subtleties

There are however certain subtleties about the non-Abelian monopoles.

- No "colored dyons" are known to exist [29]. In other words, no charge fractionalization is possible for non-Abelian quantum numbers. This being so, it does not preclude the non-Abelian monopoles of our interest: magnetic particles having Abelian and non-Abelian charges, *both* magnetic, can perfectly well exist, and do appear in the r-vacua of the softly broken $\mathcal{N}=2$ SQCD [26]!

- It is not justified to study the system $G \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} H$ as a limit of maximally broken cases as sometimes done.

- Non Abelian monopoles are never really semi-classical, even if

$$\langle\phi\rangle \gg \Lambda_H .$$

  For H were broken it would produce an *approximately* degenerate set of monopoles as, for instance, in the pure $\mathcal{N}=2$, SU(3) theory. Only if H remains unbroken do non-Abelian monopoles in irreps of $\tilde{H}$ appear.

- This option *is* realized in the r-vacua of $\mathcal{N}=2$ SQCD with $\mathrm{SU}(r)\times \mathrm{U}(1)^{n_c-r+1}$ gauge group, where $r<\frac{N_f}{2}$. This last constraint can be understood from a renormalization-group consideration: for $r<\frac{N_f}{2}$ there is a sign flip in the beta functions of the dual magnetic gauge group, with respect to that in the underlying theory:

$$b_0^{(\mathrm{dual})} \propto -2\,r+n_f>0, \qquad b_0 \propto -2\,n_c+n_f<0\,.$$

- In fact, when such a sign flip not possible (*e.g.*, pure $\mathcal{N}=2$ YM) dynamical abelianization occurs!

- *The quantum behavior of non-Abelian monopoles thus depend critically on the presence of massless fermions in the underlying theory.*

- $r=\frac{n_f}{2}$ is a boundary case: the corresponding vacua are SCFT (non-trivial IR fixed point). Non Abelian monopoles and dyons still show up as low-energy degrees of freedom, but their interactions are nonlocal and strong. The possible mechanism of confinement in these vacua has been recently studied [31].

2.10. $\boldsymbol{Z}_N$ vortices

In the Abelian dual superconductor picture of confinement in a SU(N) YM theory, the quarks would be confined by Abelian Abrikosov–Nielsen–Olesen vortices of $\mathrm{U}(1)^{N-1}$. However, this leads to the difficulty mentioned at the end of the Subsection 2.3. The quarks must be confined by some sort of non-Abelian chromoelectric vortices.

The simplest type of vortices involving a non-Abelian gauge group is the $\boldsymbol{Z}_N$ vortex, which occurs in a system with gauge symmetry breaking as

$$\mathrm{SU}(N) \Rightarrow \boldsymbol{Z}_N\,.$$

An analogous vortex appear in a system with a general symmetry breaking pattern, $H \Rightarrow \mathcal{C}$, a discrete center. Vortices represent nontrivial elements of $\Pi_1(H/\mathcal{C})$, *e.g.* $\Pi_1(\mathrm{SU}(N)/\boldsymbol{Z}_N) = \boldsymbol{Z}_N$. The asymptotic behavior of the fields is

$$A_i \sim \frac{i}{g}\,\mathrm{U}(\phi)\partial_i \mathrm{U}^\dagger(\phi)\,; \quad \phi_A \sim \mathrm{U}\phi_A^{(0)}\mathrm{U}^\dagger\,, \qquad \mathrm{U}(\phi) = \exp i \sum_j^r \beta_j T_j \phi\,,$$

where T_j are the generators of the Cartan subalgebra of H. The quantization condition is (α = root vectors of H)

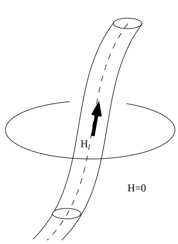
$$\mathrm{U}(2\pi) \in \boldsymbol{Z}_N\,, \qquad \alpha \cdot \beta \in \boldsymbol{Z}\,:$$

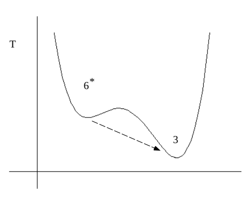
the vortices are characterized by the *weight vectors* of the group $\tilde{H}$, dual of H [30]. It seems as though the vortex solutions thus appear in the irreps of $\tilde{H} = \mathrm{SU}(N)$. Actually, the fact that the topology involved is $\Pi_1(\mathrm{SU}(N)/\boldsymbol{Z}_N) = \boldsymbol{Z}_N$ means that the stable vortices are characterized by $\boldsymbol{Z}_N$ charge (N-ality) only.

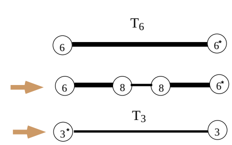
These $\boldsymbol{Z}_N$ vortices are non BPS and this makes the analysis of these objects so far relatively little explored. However there are interesting quantities which characterize these systems such as the tension ratios for which certain intriguing proposal (sine formula) [32, 33]

$$T_k \propto \sin\frac{\pi\, k}{N}$$

and which can be measured on the lattice.







2.11. Non-Abelian BPS vortices; non-Abelian superconductors

Systems with BPS vortices with a non-Abelian flux — non-Abelian superconductors — have been recently proven to exist [34, 35]. Consider a gauge theory in which the gauge group is broken at two different scales

$$G \stackrel{\langle\phi\rangle\neq 0}{\longrightarrow} H \stackrel{\langle\phi'\rangle\neq 0}{\longrightarrow} \emptyset\,, \qquad \langle\phi\rangle \gg \langle\phi'\rangle\,,$$

where the unbroken (non-Abelian) group H gets broken at a much lower scale, $\langle\phi'\rangle$. We are interested in the physics at scales between the two scales $\langle\phi\rangle$ and $\langle\phi'\rangle$. When $\Pi_1(H) \neq \emptyset$ the system develops vortices. If the theory contains an exact continuous symmetry G_F, respected both by the interactions and by the vacuum (not spontaneously broken), but broken by a vortex solution, then there is a nontrivial degeneracy of vortex solutions (zero modes).

An example [35] is the SU(3) $\mathcal{N}=2$ theory with $n_f=4,5$ quark flavors with large common (bare) mass m, with the $\mathcal{N}=2$ symmetry broken softly to $\mathcal{N}=1$ by the adjoint mass term, $\mu\,\mathrm{Tr}\,\Phi^2$. We consider a particular vacuum, the "$r=2$" vacuum of this system, which is characterized by the VEVs

$$\Phi = -\frac{1}{\sqrt{2}}\begin{pmatrix} m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & -2m\end{pmatrix}, \quad \left\langle q^{kA}\right\rangle = \left\langle \bar{\tilde{q}}^{kA}\right\rangle = \sqrt{\frac{\xi}{2}}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}.$$

At the mass scales below m (but above $\xi=\sqrt{\mu\, m}$) the system has an exact $\mathrm{SU}(2)\times\mathrm{U}(1)/Z_2$ gauge symmetry as well as an $\mathrm{SU}(n_f)$ global symmetry. The action has the form, after the Ansatz $\Phi=\langle\Phi\rangle$; $q=\tilde{q}^\dagger$; and $q\to\frac{1}{2}q$:

$$\begin{aligned} S = \int d^4x \Bigg[&\frac{1}{4g_2^2}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{4g_1^2}\left(F^8_{\mu\nu}\right)^2 + \left|\nabla_\mu q^A\right|^2 \\ &+\frac{g_2^2}{8}\left(\bar{q}_A\tau^a q^A\right)^2 + \frac{g_1^2}{24}\left(\bar{q}_A q^A - 2\xi\right)^2\Bigg]. \end{aligned} \tag{7}$$

The tension can be rewritten à la Bogomolny:

$$\begin{aligned} T \;=\; \int d^2x \Bigg(&\sum_{a=1}^{3}\left[\frac{1}{2g_2}F^{(a)}_{ij} \pm \frac{g_2}{4}\left(\bar{q}_A\tau^a q^A\right)\epsilon_{ij}\right]^2 \\ &+\left[\frac{1}{2g_1}F^{(8)}_{ij} \pm \frac{g_1}{4\sqrt{3}}\left(|q^A|^2 - 2\xi\right)\epsilon_{ij}\right]^2 \\ &+\frac{1}{2}\left|\nabla_i\, q^A + i\,\varepsilon\,\epsilon_{ij}\nabla_j\, q^A\right|^2 \pm \frac{\xi}{2\sqrt{3}}\tilde{F}^{(8)}\Bigg), \end{aligned}$$

where the first three terms are positive definite and the fourth term is a topological invariant, U(1) flux. The non-Abelian Bogomolny equations

$$
\begin{aligned}
\frac{1}{2g_2} F_{ij}^{(a)} \pm \frac{g_2}{4}\Big(\bar{q}_A \tau^a q^A\big)\epsilon_{ij} &= 0\,, \qquad (a=1,2,3);\\
\frac{1}{2g_1} F_{ij}^{(8)} \pm \frac{g_1}{4\sqrt{3}}\left(|q^A|^2 - 2\xi\right)\epsilon_{ij} &= 0\,,\\
\nabla_i\, q^A + i\varepsilon\epsilon_{ij}\nabla_j\, q^A &= 0\,, \qquad A=1,2
\end{aligned}
\tag{8}
$$

follow from the last formula. The equations (8) have Abelian (n,k) solutions of the type (where n,k are integers) studied in [36]

$$
q^{kA} = \begin{pmatrix} \mathrm{e}^{i\,n\,\varphi}\phi_1(r) & 0 \\ 0 & \mathrm{e}^{i\,k\,\varphi}\phi_2(r) \end{pmatrix},
$$

$$
\begin{aligned}
A_i^3(x) &= -\varepsilon\epsilon_{ij}\,\frac{x_j}{r^2}\,\left((n-k) - f_3(r)\right)\,,\\
A_i^8(x) &= -\sqrt{3}\,\varepsilon\epsilon_{ij}\,\frac{x_j}{r^2}\,\left((n+k) - f_8(r)\right)
\end{aligned}
\tag{9}
$$

can be seen to exist, where $\phi_1(r)$, $\phi_2(r)$, $f_3(r)$, $f_8(r)$ are profile functions with appropriate boundary conditions.

The crucial observation is that the system (7) has an exact $\mathrm{SU}(2)_{C+F}$ symmetry, which is neither broken by the interactions nor by the squark VEVS. However, an individual vortex configuration breaks it as $\mathrm{SU}(2)_{C+F} \to \mathrm{U}(1)$ therefore the vortex acquires zero modes parametrizing

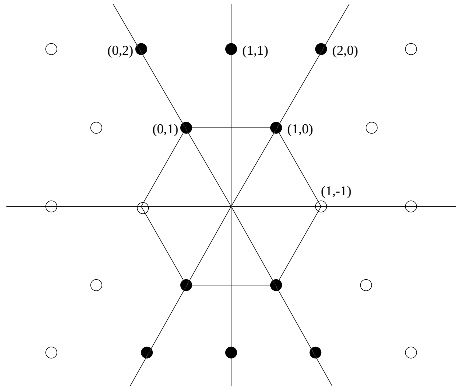
$$
\frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \sim \mathrm{CP}^1 \sim S^2\,.
$$

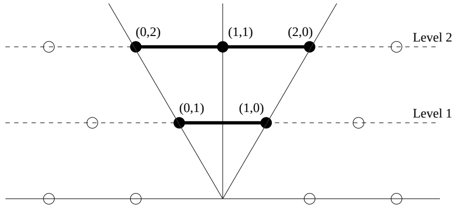
For instance, a minimum vortex of generic orientation can be explicitly constructed as

$$
\begin{aligned}
q^{kA} &= U \begin{pmatrix} \mathrm{e}^{i\varphi}\phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1} = \mathrm{e}^{\frac{i}{2}\varphi(1+n^a\tau^a)}\, U \begin{pmatrix} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1},\\
A_i(x) &= U\left[-\frac{\tau^3}{2}\,\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_3(r)]\right]U^{-1} = -\frac{1}{2}\,n^a\tau^a\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_3(r)]\,,\\
A_i^8(x) &= -\sqrt{3}\,\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_8(r)]\,,
\end{aligned}
\tag{10}
$$

where U is an SU(2) matrix.

The (massive) non-Abelian monopoles resulting from the gauge symmetry breaking $\mathrm{SU}(3) \to \mathrm{SU}(2)\times \mathrm{U}(1)/Z_2$ by the adjoint Φ VEV, are confined by these non-Abelian vortices.

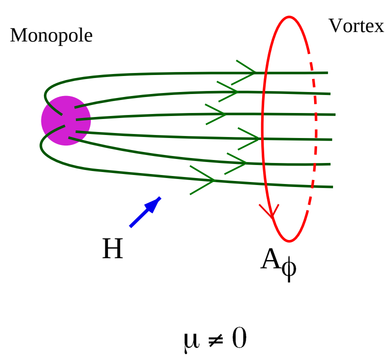




Remarks [35]:

- The reduction of the vortex spectrum (meson spectrum): (figures) is due to the topology change
$$\Pi_1\left(\frac{\mathrm{U}(1)\times\mathrm{U}(1)}{\boldsymbol{Z}_2}\right)=\boldsymbol{Z}^2 \quad\rightarrow\quad \Pi_1\left(\frac{\mathrm{SU}(2)\times\mathrm{U}(1)}{\boldsymbol{Z}_2}\right)=\boldsymbol{Z};$$
- The transition from the Abelian ($m_i \neq m_j$) to the non-Abelian ($m_i = m$) superconductivity is here **reliably** and **quantum mechanically** analyzed (it is important to have $n_f = 4$ or $n_f = 5$ for this);
- Our findings provide an (indirect) solution to the “existence problem” of non-Abelian **monopoles**, as the latter act as the sources of the non-Abelian vortices (see figure below);
- The dynamics of vortex zero modes can be shown to be equivalent to the two-dimensional $\mathrm{O}(3) = \mathbf{CP}^1$ sigma model ($\boldsymbol{n} \to \boldsymbol{n}(z,t)$):
$$S_\sigma^{(1+1)} = \beta \int dz\, dt\, \frac{1}{2}\,(\partial\, n^a)^2 + \text{fermions}.$$
It is dual [37,38] to a chiral theory with two vacua. The exact $\mathrm{SU}(2)_{C+F}$ symmetry is not spontaneously broken. The dual ($\mathcal{N}=1$) SU(2) theory is in confinement phase and has, correctly, two vacua (Witten index).

- The whole picture generalizes naturally to the case of $\mathrm{SU}(N) \to \frac{\mathrm{SU}(N-1)\times \mathrm{U}(1)}{\boldsymbol{Z}_{N-1}} \to \emptyset$ system with $2N > N_f \geq 2(N-1)$ flavors. It has vortices with $2(N-2)$-parameter family of zero modes representing
$$\frac{\mathrm{SU}(N-1)}{\mathrm{SU}(N-2)\times \mathrm{U}(1)} \sim \mathbf{CP}^{N-2}\,.$$
- The analysis was made at large m (large $\langle\phi\rangle$) where the system is semi-classical. Though more difficult to analyze, the situation at small m where the non-Abelian monopoles condense and the quarks are confined by non-Abelian chromoelectric vortices, is related smoothly to the non-Abelian superconductor studied here, via holomorphic dependence of the physics on m and though the isomonodromy (in which quarks become monopoles and *vice versa*).



2.12. *Lessons from $\mathcal{N}=2$ SQCD*

Softly broken $\mathcal{N}=2$, $\mathrm{SU}(n_c)$ gauge theories with n_f quarks $\Rightarrow$ confining vacua with: physics quite different for

(i) $r = 0, 1 \Rightarrow$ Weakly coupled Abelian monopoles;

(ii) $r < \frac{n_f}{2} \Rightarrow$ Weakly coupled non-Abelian monopoles;

(iii) $r = \frac{n_f}{2} \Rightarrow$ Strongly coupled non-Abelian monopoles.

Both at generic r-vacua and at the SCFT ($r = n_f/2$) vacua,
$$\langle \mathcal{M}^i_\alpha \rangle = \delta^i_\alpha\, v \neq 0, \qquad (\alpha = 1,2,\ldots,r; \quad i = 1,2,\ldots,n_f)$$
("Color–Flavor-Locked Phase"). Gauge invariant condensates are
$$\epsilon^{\alpha_1\alpha_2\ldots\alpha_r}\mathcal{M}^{i_1}_{\alpha_1}\mathcal{M}^{i_2}_{\alpha_2}\ldots\mathcal{M}^{i_r}_{\alpha_r} \sim \mathrm{U}(1) \text{ monopole?}$$

2.13. Hint for QCD

- Dynamical abelianization neither is observed in the real world nor is believed to occur in QCD.
- QCD with n_f flavor and its possible dual have the beta function coefficients ($\tilde{n}_c = 2, 3$, $n_f = 2, 3$)

$$b_0 = 11\, n_c - 2\, n_f \quad vs \quad \tilde{b}_0 = 11\, \tilde{n}_c - n_f$$

so no sign flip (no weakly-coupled non-Abelian monopoles) is possible.

- It leaves the possibility of a strongly-interacting non-Abelian superconductor.
- Taking a hint from supersymmetric models one might assume that non-Abelian magnetic monopoles condense in a color-flavor-locked form

$$\langle \mathcal{M}^i_{\mathrm{L},\alpha} \rangle = \delta^i_\alpha\, v_{\mathrm{R}} \neq 0\,, \quad \langle \mathcal{M}^\alpha_{\mathrm{R,i}} \rangle = \delta^\alpha_i\, v_{\mathrm{L}} \neq 0\,,$$

$(\alpha = 1, 2, \ldots \tilde{n}_c; i = 1, 2, \ldots n_f)$.

- A better picture might be

$$\langle \mathcal{M}^i_{\mathrm{L},\alpha}\, \mathcal{M}^\alpha_{\mathrm{R,j}} \rangle = \mathrm{const.}\, \delta^i_j \neq 0\,;$$

which yields for $\tilde{n}_c = 2$, $n_f = 2$ the correct symmetry breaking pattern

$$G_F = \mathrm{SU_L}(2) \times \mathrm{SU_R}(2) \Rightarrow \mathrm{SU}_V(2)\,.$$

3. Recent developments in $\mathcal{N} = 1$ gauge dynamics

An exciting new development in the analysis of the dynamics of supersymmetric gauge theories has been initiated by the work by Dijkgraaf–Vafa [39], Cachazo–Douglas–Seiberg–Witten [40] and others. Here we shall review briefly the field-theoretic approach of the latter.

3.1. Veneziano–Yankielovicz effective action

The Veneziano–Yankielovicz in the $\mathcal{N} = 1$ SUSY SU(N) Yang–Mills ($W_\alpha = -i\lambda + \frac{i}{2}\,(\sigma^\mu\, \bar{\sigma}^\nu)^\beta_\alpha\, F_{\mu\nu}\, \theta_\beta + \ \ldots$)

$$\mathcal{L}^{\mathrm{bare}} = \int d^2\theta\, \frac{1}{g_0^2} W W = \int d^2\theta\, \frac{1}{g_0^2}\, S\,,$$

where

$$S \equiv W^\alpha W_\alpha = -\lambda\lambda + \ldots - \frac{1}{2} F^2_{\mu\nu} - i\lambda\sigma^\nu \mathcal{D}_\nu \bar{\lambda} + \ldots\,,$$

has the form

$$\mathcal{L}^{\rm VY} = \text{kin. term} - \int d^2\theta\, S \left[\log \frac{S^N}{\Lambda^{3N}} - N \right] + \text{h.c.}$$

It basically represents the exact, one-loop renormalization effect

$$\left[\frac{1}{g_0^2} + b_0 \,\log \frac{M}{S^{1/3}} \right] S = \frac{1}{g(S)^2}\, S = b_0\, S \log \frac{S^{1/3}}{\Lambda}, \qquad b_0 = 3N.$$

The minimization of the potential leads to the well known N vacua, $\langle S\rangle = \Lambda^3 \exp 2\pi i k/N$, with $k = 1, 2, \ldots N$ ($Z_{2N} \subset {\rm U}_A(1)$ broken to Z_2); It was constructed in order to reproduce correctly the anomaly under $\lambda \to {\rm e}^{i\alpha}\lambda$

$$\Delta \mathcal{L}^{\rm VY} = 2N\, \alpha\, F_{\mu\nu} \tilde{F}^{\mu\nu}.$$

Actually, $\int {\rm e}^{i\,S}$ invariant under Z_{2N}.

3.2. Chiral rings in theory with adjoint field Φ

Consider now $\mathcal{N} = 1$ SUSY U(N) gauge theory

$$\begin{aligned} \mathcal{L} &= \frac{1}{8\pi} {\rm Im}\, \tau_{\rm cl} \left[\int d^4\theta\, \Phi^\dagger {\rm e}^V \Phi + \int d^2\theta\, \frac{1}{2} W W \right] + \int d^2\theta\, \mathcal{W}(\Phi) + \text{h.c.}\,, \\ \tau_{\rm cl} &\equiv \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2} \end{aligned}$$

($\mathcal{N} = 1$) multiplets $\Phi = \phi + \sqrt{2}\theta\psi + \ldots$; $W_\alpha = -i\lambda + \frac{i}{2}(\sigma^\mu\bar\sigma^\nu)_\alpha^\beta F_{\mu\nu}\theta_\beta + \ldots$ and the superpotential is taken to be of the form,

$$\mathcal{W}(\Phi) = \sum_{k=0}^{n} \frac{g_k}{k+1} {\rm Tr}\; \Phi^{k+1}. \tag{11}$$

Gauge invariant chiral composite can be taken, modulo $\{\bar Q, \ldots\}$ to be the set

$$\{ {\rm Tr}\; \Phi^k, \qquad {\rm Tr}\, W_\alpha\, \Phi^k\,, \qquad {\rm Tr}\, W^\alpha W_\alpha\, \Phi^k \}\,.$$

Perturbatively (for $k > N$), there are relations such as,

$$\begin{aligned} {\rm Tr}\; \Phi^k &= \mathcal{P}(\{u_i\})\,, \qquad u_i = {\rm Tr}\; \Phi^j\,, \qquad j \le N\,, \\ \frac{\partial}{\partial \Phi} \mathcal{W}(\Phi) &= \bar D^2(\ldots) = 0\,, \qquad S^N = 0\,. \end{aligned}$$

We wish to know how these get modified by the quantum effects.

3.3. Problem

Classically if $\{a_i\}$ = eigenvalues of Φ,

$$\mathcal{W}'(z) = g_n \prod_i^n (z - a_i)$$

the gauge group is broken as $\mathrm{U}(N) \Rightarrow \prod_i \mathrm{U}(N_i)$. The low-energy effective degrees of freedom are

$$\begin{aligned} S_i &= \frac{1}{16\pi^2} \mathrm{Tr}\, W_i^{\alpha} W_{\alpha i}\,, \qquad w_{\alpha i} = \frac{1}{4\pi} \mathrm{Tr}\, W_{\alpha i} \;: \\ \int \mathcal{D}\Phi\, \mathrm{e}^{i\,S} &= \mathrm{e}^{i \int \mathcal{L}_{\mathrm{eff}}}\,, \quad \mathcal{L}_{\mathrm{eff}} = \int d^2\theta\, \mathcal{W}_{\mathrm{eff}}(S_i, w_{\alpha i}, g_k) + \ldots\,. \end{aligned}$$

The problem is to compute $\mathcal{W}_{\mathrm{eff}}(S_i, w_{\alpha i}, g_k)$.

The idea of the solution is to observe that

$$\frac{\partial}{\partial g_k} \mathcal{W}_{\mathrm{eff}}(S_i, w_{\alpha i}, g_k) = \left\langle \mathrm{Tr}\, \frac{\Phi^{k+1}}{k+1} \right\rangle\,, \tag{12}$$

namely the problem is reduced to that of determining all the chiral condensates as functions of $S_i, w_{\alpha i}, g_k$.

3.4. Symmetries

The first ingredient is the use of the symmetries (Table) which implies that, apart from the anomalous one-loop contribution, the effective action must depend on the field variables and on the coupling constants as

Fields	Δ	Q_Φ	Q_{R}	Q_θ
Φ	1	1	$\frac{2}{3}$	0
W_α	$\frac{3}{2}$	0	1	1
g_l	$2-l$	$-(l+1)$	$\frac{2}{3}(2-l)$	2
Λ^{2N}	$2N$	$2N$	$\frac{4N}{3}$	0

$$\mathcal{W}_{\mathrm{eff}} = W_\alpha^2\, F\left(\frac{g_k\, W_\alpha^{k-1}}{g_1^{(k+1)/2}} \right)\,,$$

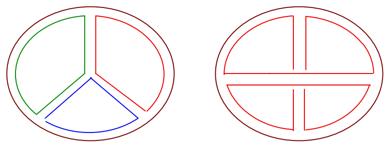
or

$$\left[\sum_k (2-k)\, g_k \frac{\partial}{\partial g_k} + \frac{3}{2}\, W_\alpha \frac{\partial}{W_\alpha} \right] \mathcal{W}_{\mathrm{eff}} = 3\, \mathcal{W}_{\mathrm{eff}}\,.$$

On the other hand, the number of the index loops (L), of the vertices (k_i), and the genus (g) of the two-dimensional surface (upon which the Feynman diagram can be embedded without crossing the index lines) are related as

$$L = 2 - 2g + \frac{1}{2}\sum_i (k_i - 1) :$$

the same relation used by 't Hooft in deriving his $1/N$ expansion. It follows



that only planar diagrams contribute to $\mathcal{W}_{\rm eff}$ (proof of the conjecture by Dijkgraaf–Vafa), which is remarkable as it involves no large N expansion.

A second ingredient is the observation that $\mathrm{U}(1) \subset \mathrm{U}(N)$ is free — $\mathcal{W}_{\rm eff}(S_i, w_{\alpha i}, g_k)$ is invariant under

$$W_\alpha \to W_\alpha - 4\pi\psi_\alpha \,.$$

The general form of $\mathcal{W}_{\rm eff}$ is then determined to be

$$\begin{aligned}\mathcal{W}_{\rm eff} &= \sum N_i \frac{\partial \mathcal{F}_p(S_k, g_k)}{\partial S_i} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 \mathcal{F}_p(S_k, g_k)}{\partial S_i\, \partial S_j} w_{\alpha i} w_j^\alpha \\ &= \int d^2\psi\ \mathcal{F}_p(\mathcal{S}_i, g_k)\,,\end{aligned}$$

where

$$\mathcal{S}_i = -\frac{1}{2}\mathrm{Tr}\left(\frac{1}{4\pi}W_{\alpha i} - \psi_\alpha\right)\left(\frac{1}{4\pi}W_i^\alpha - \psi^\alpha\right) = S_i + \psi_\alpha w_i^\alpha - N_i\psi_\alpha\psi^\alpha \,. \quad (13)$$

Note that the idea is similar to the way supersymmetric Lagrangians are constructed as the highest components of products of various superfields. It follows that

$$\begin{aligned}\frac{\partial \mathcal{W}_{\rm eff}}{\partial g_k} &= \int d^2\psi\ \frac{\partial \mathcal{F}_p(\mathcal{S}_i, g_k)}{\partial g_k} = \left\langle \frac{\Phi^{k+1}}{k+1} \right\rangle_\Phi \\ &= -\frac{1}{2(k+1)} \int d^2\psi \left\langle \mathrm{Tr}\left(\frac{1}{4\pi}W_\alpha - \psi_\alpha\right)^2 \Phi^{k+1} \right\rangle_\Phi \,.\end{aligned}$$

So

$$\frac{\partial \mathcal{F}_p(\mathcal{S}_i, g_k)}{\partial g_k} = -\frac{1}{2(k+1)} \int d^2\psi \left\langle \mathrm{Tr}\left(\frac{1}{4\pi}W_\alpha - \psi_\alpha\right)^2 \Phi^{k+1} \right\rangle_\Phi$$

and the problem is to find the right-hand side.

3.5. Generalized Konishi Anomaly

The last ingredient is the generalized Konishi Anomaly. The anomaly in its original form reads

$$\begin{aligned} \bar{D}^2 \mathrm{Tr}\{\bar{\Phi} \mathrm{e}^V \Phi\} &= \mathrm{Tr}\, \Phi \frac{\partial \mathcal{W}}{\partial \Phi} + \frac{1}{32\pi^2} \mathrm{Tr}\,(ad\, W_\alpha\, ad\, W^\alpha)\,, \\ \bar{D}^2 \mathrm{Tr}\{\bar{\Phi} \mathrm{e}^V \Phi\} &= \mathrm{Tr}\, \Phi \frac{\partial \mathcal{W}}{\partial \Phi} + \frac{N}{16\pi^2} \mathrm{Tr}\,(W_\alpha W^\alpha) - \frac{1}{16\pi^2} \mathrm{Tr}\, W_\alpha \mathrm{Tr}\, W^\alpha \end{aligned} \tag{14}$$

which are just the supersymmetrized form of the chiral $\mathrm{U}_\Phi(1)$ anomaly. Taking the VEVS of the both sides

$$\left\langle \mathrm{Tr}\, \Phi \frac{\partial \mathcal{W}}{\partial \Phi} \right\rangle = -\frac{N}{16\pi^2} \left\langle \mathrm{Tr}\,(W_\alpha W^\alpha) \right\rangle \,.$$

But the left-hand side is equal to

$$\left\langle \mathrm{Tr} \sum_k g_k \Phi^{k+1} \right\rangle = \sum_k (k+1) g_k \frac{\partial}{\partial g_k} \mathcal{W}_{\mathrm{eff}}$$

which are precisely a kind of quantity one wants to know.

The original Konishi anomaly follows à la Fujikawa (Shizuya–Konishi) by a functional change of variables, $\delta\Phi = \alpha\Phi$. As noted Cachazo, Douglas, Seiberg and Witten, the Konishi–Shizuya derivation straightforwardly generalizes to the anomalous Ward identities for more general current $J_f = \mathrm{Tr}\{\bar{\Phi}\mathrm{e}^V f(\Phi, W_\alpha)\}$:

$$\delta\Phi = f(\Phi, W_\alpha); \tag{15}$$

which leads to the identities

$$\bar{D}^2 J_f = \mathrm{Tr}\, f(\Phi, W_\alpha) \frac{\partial \mathcal{W}}{\partial \Phi} + \frac{1}{32\pi^2} \sum_{ij} \left[W_\alpha, \left[W^\alpha, \frac{\partial f}{\partial \Phi_{ij}} \right] \right]_{ij} . \tag{16}$$

The $\langle$R.H.S.$\rangle = 0$ leads basically to the solution of the problem.

Define

$$\begin{aligned} \mathcal{R}(z,\phi) &= -\frac{1}{2} \mathrm{Tr} \left(\frac{1}{4\pi} W_\alpha - \psi_\alpha \right)^2 \frac{1}{z - \Phi}, \\ &= R(z) + \psi_\alpha w^\alpha(z) - \psi^1 \psi^2 T(z)\,, \end{aligned} \tag{17}$$

where generating functions are

$$\begin{aligned} T(z) &= \mathrm{Tr}\, \frac{1}{z - \Phi}\,, \quad w^\alpha = \frac{1}{4\pi} \mathrm{Tr}\, W_\alpha \frac{1}{z - \Phi}\,, \\ R(z) &= -\frac{1}{32\pi^2} \mathrm{Tr}\, W_\alpha W^\alpha \frac{1}{z - \Phi}\,. \end{aligned}$$

By choosing $f(\Phi) = W_\alpha W^\alpha \frac{1}{z-\Phi}$ in (15):

$$\left\langle -\frac{1}{32\pi^2} \sum_{ij} \left[W_\alpha, \left[W^\alpha, \frac{\partial}{\partial \Phi_{ij}} \left(W_\beta W^\beta \frac{1}{z-\Phi} \right) \right] \right]_{ij} \right\rangle = \left\langle \mathrm{Tr} \left[\frac{\partial \mathcal{W}}{\partial \Phi} W_\alpha W^\alpha \frac{1}{z-\Phi} \right] \right\rangle .$$

By identity

$$\sum_{ij} \left[\chi_1, \left[\chi_2, \frac{\partial}{\partial \Phi_{ij}} \frac{\chi_1 \chi_2}{z-\Phi} \right] \right]_{ij} = \left(\mathrm{Tr} \frac{\chi_1 \chi_2}{z-\Phi} \right)^2$$

(valid if $\chi_1^2 = \chi_2^2 = 0$, $[\chi_i, \Phi] = 0$) one gets

$$R(z,\psi)^2 = \mathrm{Tr}\left(\mathcal{W}'(\Phi) R(z,\psi) \right) .$$

Analogously, with $f(\Phi) = \mathcal{R}$ (r.h.s of (17) without trace)

$$\mathcal{R}(z,\psi)^2 = \mathrm{Tr}\left(\mathcal{W}'(\Phi) \mathcal{R}(z,\psi) \right) \tag{18}$$

which can be rewritten as

$$\mathcal{R}(z,\psi)^2 = \mathrm{Tr}\left(\mathcal{W}'(z) \mathcal{R}(z,\psi) \right) + \frac{1}{4} f(z,\psi), \tag{19}$$

or

$$\begin{aligned} R^2(z) &= \mathcal{W}'(z) R(z) + \tfrac{1}{4} f(z); \\ 2\, R(z)\, w^\alpha(z) &= \mathcal{W}'(z) w^\alpha(z) + \tfrac{1}{4} \rho^\alpha; \\ 2\, R(z)\, T(z) + w_\alpha(z) w^\alpha(z) &= \mathcal{W}'(z) T(z) + \tfrac{1}{4} c(z). \end{aligned}$$

with

$$\begin{aligned} f(z,\psi) &= \frac{1}{8\pi^2} \mathrm{Tr} \, \frac{(\mathcal{W}'(z) - \mathcal{W}'(\Phi))(W_\alpha - 4\pi\psi_\alpha)\,(W^\alpha - 4\pi\psi^\alpha)}{z-\Phi} \\ &= f(z) + \psi_\alpha\, \rho^\alpha(z) - \psi_1 \psi_2\, c(z)\,, \end{aligned}$$

where $f(z)$ is an nth order polynomial in z.

Solving the quadratic equation

$$2\, \mathcal{R}(z,\psi) = \mathcal{W}'(z) - \sqrt{\mathcal{W}'(z)^2 + f(z,\psi)}$$

or

$$2\, R(z) = \mathcal{W}'(z) - \sqrt{\mathcal{W}'(z)^2 + f(z)}$$

etc. Namely, $R(z) = -\frac{1}{32\pi^2}\mathrm{Tr}\, W_\alpha W^\alpha \frac{1}{z-\Phi}$ has been determined in terms of f_i where

$$f(z) = \sum_{i=0}^{n-1} f_i\, z^i .$$

The relations among $\{f_i\}$ and (S_i, w_i^α) are given by

$$\mathcal{S}_i = S_i + \psi_\alpha w_i^\alpha - N_i \psi_\alpha \psi^\alpha = \frac{1}{2\pi i}\oint_{C_i} dz\, \mathcal{R}(z,\psi)\,.$$

Finally,

$$\begin{aligned}\frac{\partial \mathcal{F}_p(\mathcal{S}_i, g_k)}{\partial g_k} &= -\frac{1}{2(k+1)}\int d^2\psi \left\langle \mathrm{Tr}\left(\frac{1}{4\pi}W_\alpha - \psi_\alpha\right)^2 \Phi^{k+1}\right\rangle_\Phi \\ &= -\frac{1}{2(k+1)}\oint dz\, z^{k+1}\mathcal{R}(z,\phi)\,.\end{aligned}$$

By integrating over g_k and adding integration constant — g_k independent, 1-loop, contribution (VY), we get $\mathcal{W}_{\rm eff}$ in terms of $(S_i, w_i^\alpha, \Lambda_i)$.

3.6. Matrix model (Dijkgraaf–Vafa)

In the matrix approach one considers an integral over $\hat{N}\times\hat{N}$ Hermitian matrices M

$$\exp -\frac{\hat{N}^2}{g_m^2} F_{\rm m.m.} = \int d^{\hat{N}^2}M\ \exp -\frac{\hat{N}}{g_m}\,\mathrm{Tr}\,\mathcal{W}(M)\,.$$

By considering the matrix change of variable $\delta M = \epsilon\, M^{n+1}$ one gets

$$0 = \int d^{\hat{N}^2}M\ \mathrm{e}^{-\frac{\hat{N}}{g_m}\mathrm{Tr}\,\mathcal{W}(M)}\left[\mathrm{Tr}\frac{\partial}{\partial M}M^n - \frac{\hat{N}}{g_m}\mathrm{Tr}\mathcal{W}' M^n\right]$$

from which a relation

$$\langle R_m(z)^2\rangle = \langle W'(z) R_m(z)\rangle + \tfrac{1}{4} f_m(z)\,, \quad \text{where} \quad R_m(z) = \frac{g_m}{\hat{N}}\left\langle \mathrm{Tr}\frac{1}{z-M}\right\rangle$$

follows. Take now $\hat{N}\to\infty$ and use the known factorization property to obtain

$$\langle R_m(z)\rangle^2 = \langle W'\rangle\langle R_m(z)\rangle + \tfrac{1}{4} f_m(z)$$

but this is identical to Eq. (19)! By

$$S_i = \frac{1}{2\pi i}\oint_{C_i} R_m(z)\, dz\,, \qquad \frac{\partial F_{\rm m.m.}}{\partial g_k} = \left\langle \frac{\mathrm{Tr}\, M^{k+1}}{k+1}\right\rangle$$

and by identification of $F_{\rm m.m.}(S_i, g_k)$ with $\mathcal{F}_p(\mathcal{S}_i, g_k)$ one gets $\mathcal{W}_{\rm eff}(S_i, w_i^\alpha, \Lambda_i)$.

3.7. Further development

The class of the models being studied

$$\mathcal{L}^{\mathrm{U}(N)} = \frac{1}{8\pi}\mathrm{Im}\,\tau_{\mathrm{cl}}\left[\int d^4\theta\,\Phi^\dagger \mathrm{e}^V\,\Phi + \int d^2\theta\,\frac{1}{2}WW\right] + \int d^2\theta\,\mathcal{W}(\Phi)\,,$$

where $\mathcal{W}(\Phi) = \sum_{r=0}^{k} \frac{g_r}{r+1}\mathrm{Tr}\,\Phi^{r+1}$ has in its basis the structure of the $\mathcal{N} = 2$ theory ($\mathcal{W}(\Phi) = 0$) which was not obvious in the result so far discussed. A much deeper analysis becomes possible by fully taking into account of such a structure (Cachazo, Seiberg, Witten [40]).

When $\mathcal{W}(\Phi) = 0$ the theory has $\mathcal{N} = 2$ and as is well known the gauge group is (dynamically) broken as $G \sim \mathrm{U}(1)^{N-1}$ on a generic point of QMS. At special points where some $N - n$ monopoles become massless (condensation and Higgs mechanism for $N - n$ dual gauge bosons), however, the low-energy gauge group is smaller: $G \sim \mathrm{U}(1)^n$. At such points the curve factorizes (cond. on QMS)

$$y^2 = P_N^2(x) - 4\Lambda^2 = F_{2n}\,H_{N-n}^2(x)\,. \tag{20}$$

Classically, if the VEV of Φ has the form,

$$\mathrm{diag}\,\Phi = \{a_1,\ldots a_1, a_2,\ldots,\ldots,a_n,\ldots,a_n\}\,,\quad \mathcal{W}'(z) = g_k\prod_i^k(z-a_i)\,,$$

the low-energy gauge group is

$$\mathrm{U}(N) \Rightarrow \prod_i^n \mathrm{U}(N_i) \Rightarrow \mathrm{U}(1)^n \qquad n \le k \tag{21}$$

as the $\mathrm{SU}(N_i)$ parts become strong. The problem is to find, quantum mechanically, the relation

$$\text{vacua (21)} \Longleftrightarrow \mathcal{W}(\Phi)\,.$$

The answer is the factorization condition (20) with (for $k = n$)

$$F_{2n}(x) = \frac{1}{g_n^2}\mathcal{W}'(x)^2 + f_n(x)$$

$f_n(x) = O(x^{n-1})$ with n unknown coefficients. For more general choice of k see [40].

It is now possible to obtain the generalized Konishi anomaly relations (which are written in terms of the electric variables) starting from $\mathcal{N} = 2$ curves (whose singularities can be characterized by magnetic variables). This is a kind of link which was earlier not fully exploited[1].

[1] Such a link was used in $\mathcal{N} = 2$, $n_f = 1$, SU(2) theory by Gorsky–Vainshtein–Yung [41].

3.8. Confinement index

A quantity which characterize a particular confining vacuum is the *confinement index* which is equal to the smallest possible $r \in Z_N^{(E)}$ for which Wilson loop displays no area law. For instance, for SU(N) YM, $r = N$ in the vacuum with complete confinement; $r = 1$ in the totally Higgs vacuum. Another example is that $r = 1$ in a theory with

$$\mathrm{SU}(N) \to \mathrm{SU}(N-1) \times \mathrm{U}(1).$$

A third example is the $\mathcal{N} = 1$ SUSY SU(N) theory broken (by adjoint VEV) as

$$\mathrm{SU}(N) \to \mathrm{SU}(N_1) \times \mathrm{SU}(N_2) \times \mathrm{U}(1).$$

Then

$$r = \text{l.c.d.}\, \{N_1, N_2, r_1 - r_2\}\,,$$

where

$$r_1 = 0, 1, 2, \ldots N_1 - 1\,, \qquad r_2 = 0, 1, 2\,, \ldots N_2 - 1$$

label the vacua in which $(n_m, n_e) = (1, r_1)$ and $(n_m, n_e) = (1, r_2)$ are condensed.

3.9. Multiplication map

Such a concept can be used to define highly nontrivial maps between the vacua of a pair of gauge theories such as U(N)⇔U(tN) with the same superpotential $\mathcal{W}(\Phi)$. In particular it can be shown that

- the vacua with the low-energy symmetry $\prod_{i=1}^{n} \mathrm{U}(N_i)$ of the U(N) theory are mapped to the vacua with $\prod_{i=1}^{n} \mathrm{U}(tN_i)$ of the theory U(tN);
- the confinement index r gets simply multiplied by t in such a correspondence;
- *all* confining vacua with $r = t$ in the U(tN) theory, arise from the Coulomb vacua of U(N) theory;
- the chiral condensates are related as

$$\left\langle \mathrm{Tr} \frac{1}{x - \Phi} \right\rangle = t \left\langle \mathrm{Tr} \frac{1}{x - \Phi_0} \right\rangle .$$

These and other relations lead to deep understanding of the vacua in supersymmetric gauge theories, their unexpected structures and new types of dualities, *etc.* A remarkable new example of this kind is the $\mathrm{USp}(N) \Leftrightarrow \mathrm{U}(N+2n)$ map found by Cachazo [42], with the order $n+1$ (common) superpotential $\mathcal{W}(\Phi)$

$$\prod_{i=1}^{n} \mathrm{USp}(N_i) \Leftrightarrow \prod_{i=1}^{n} \mathrm{U}(N_i+2):$$

$$\left\langle \mathrm{Tr}\frac{1}{z-\Phi} \right\rangle = \left\langle \mathrm{Tr}\frac{1}{z-\Phi_{\mathrm{U}}} \right\rangle - \frac{d}{dz}\log(W'(z)^2 + f(z)).$$

4. Summary

Many more consequences of the ideas discussed in the last section are being explored these days, and it is not easy to foresee where these efforts take us. It is hoped that one day the improved understanding in the non-Abelian dynamics in the context of supersymmetric gauge theories such as these would, perhaps combined with the insights obtained in the approach discussed in Section 2, shed light in the mechanism of confinement and dynamical symmetry breaking in Quantum Chromodynamics.

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