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Ds(2317) AND RELATIVISTIC QUANTUM MECHANICS IN ONE DIMENSION*

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It is recalled that a ten year old calculation of all meson masses may explain the low value of the recently discovered Ds(2317) meson. This calculation was based on a fully relativistic quasiparticle theory, which has been applied to a large number of bound state problems and scattering processes. In this paper we want to show that also for one-dimensional systems the theory can be formulated in a compact way. After discussing the Lippmann–Schwinger equation for two nonrelativistic particles on a line, we show how to extend this momentum–space formulation to a Poincaré invariant theory. We then apply this theory to a simple example and compare the reflection and transmission coefficients, as well as the total diffusion and the bound state spectrum, to the results obtained from the nonrelativistic case. Also the relativistic corrections to the spectrum of two harmonically bound particles are calculated. It is found that especially the higher excited states become less massive.

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1. Motivation

After the discovery of the Ds(2317) meson [1], the question arises whether a new theory is required to explain the unexpectedly low value of its mass, or whether existing quasiparticle theories can explain this meson as a $c\overline{s}$ bound state. In the SLAC press release the former point of view was taken and the theorists ([2, 3]) were advised to return to their drawing boards.

However, it should be realised that the above mentioned theories

> ... did not carry out a relativization from first principles, but rather constructed a quark potential motivated by the expected relativistic properties [2].

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Another difficulty is the appearance of an ultraviolet divergence, which

> ...is due to the inconsistency of a static point-like source (the heavy quark) within a relativistic framework [3].

Therefore, before jumping to the conclusion that a new chapter of the physics book has been opened, it is advisable first to construct a relativistic two-particle theory in which these difficulties are solved in a natural way.

Many years ago such a manifestly Poincaré invariant theory has been developed by the author ([4–6] and references therein). It was applied to many physical systems, of which the article by Hersbach [7] on meson spectroscopy is the most relevant one for the present discussion.

He calculated the masses of all mesons and compared them with the experimental values known at that time. For the mass of the Ds(2317) seen as a $c\bar{s}$ bound state of type $0^+\ {}^3P_0$, he predicted 2436 MeV, 2366 MeV and 2349 MeV, depending on the choice of potential. These numbers should be compared to 2480 MeV from [2] and 2487 MeV from [3]. The Regge slope for his best solution was $\beta = 1.18 \pm 0.05$, compatible with the experimental value. For the same solution he found the running coupling constants $\alpha_{\rm s}(34\ {\rm GeV}) = 0.141$ and $\alpha_{\rm s}(M_{\rm Z} = 91\ {\rm GeV}) = 0.1164$, in good agreement with the experimental values of 0.14 ± 0.02 and 0.1134 ± 0.0035, respectively.

These encouraging results are reasons not yet to reject all quasiparticle theories when it comes to understanding the Ds(2317) meson. It is believed that a consistent incorporation of relativity is absolutely necessary. The author′s ideas, most fully explained in [4], provide such a theory. Whether potentials can be found that explain the data remains to be seen.

In the present paper we want to repeat the formulation of the theory, but now for the much simpler one-dimensional case. The application to the harmonic oscillator in Section 4 shows that the incorporation of relativity leads to a lowering of the mass of each of the bound states.

2. The nonrelativistic equations

2.1. The general formulation

Since the relativistic equations are written in the momentum representation, we will begin by also writing the nonrelativistic equations in this form. In particular it will be shown that also for the one-dimensional case the scattering problem can be cast in the form of the Lippmann–Schwinger equation [8].

As basis of our Hilbert space we take the states $|\,\alpha\rangle = |\,q_1, q_2\rangle$ of two free spinless particles with masses m_1 and m_2 and one-dimensional momenta

q_1 and q_2. In the coordinate representation these states are given by plane waves

$$\langle x_1, x_2 | \, q_1, q_2 \rangle = \frac{1}{2\pi} \, \mathrm{e}^{i(q_1 x_1 + q_2 x_2)} \, .$$

The basis states $|\,\alpha\rangle = |\,q_1, q_2\rangle$, $|\,\beta\rangle = |\,k_1, k_2\rangle$, *etc.* form a complete and orthonormal set *i.e.*

$$\langle \alpha \,|\, \beta \rangle = \delta(q_1 - k_1)\,\delta(q_2 - k_2) \equiv \delta(\alpha - \beta) \quad \text{and} \quad \int\limits_{\alpha} |\,\alpha\rangle\langle\alpha\,|\,\psi\rangle = |\,\psi\rangle \, ,$$

where $\int\limits_{\alpha} \cdots \equiv \int\limits_{-\infty}^{\infty} dq_1 dq_2 \cdots$. They are also eigenstates of the free particle Hamiltonian H_0

$$H_0 \,|\,\alpha\rangle = E_\alpha \,|\,\alpha\rangle \, , \quad \text{with} \quad E_\alpha = \varepsilon_1(q_1) + \varepsilon_2(q_2) = \frac{q_1^2}{2m_1} + \frac{q_2^2}{2m_2} \, .$$

Units are chosen in such a way that $\hbar = 1$. Supposing that in the coordinate representation the local and translational invariant potential W is given by

$$\langle x_1', x_2' \,|\, W \,|\, x_1, x_2 \rangle = V(x_1 - x_2)\,\delta(x_1' - x_1)\,\delta(x_2' - x_2) \, ,$$

then, in the momentum representation, this potential will be equal to

$$\begin{aligned} &\langle k_1, k_2 | \, W \,|\, q_1, q_2 \rangle = \widehat{V}(k_1 - q_1)\,\delta(k_1 + k_2 - q_1 - q_2) \, , \\ &\text{with} \quad \widehat{V}(k) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-ikx} V(x)\, dx. \end{aligned} \tag{1}$$

Since we will restrict ourselves to potentials $V(x)$, which are real and symmetric in x, it follows that also $\widehat{V}(k)$ will be real and symmetric in k.

As in the standard formulation we first have to find the stationary scattering states $|\,\alpha\rangle_\pm$, defined as the in- and outgoing eigenstates of the total Hamiltonian

$$H\,|\,\alpha\rangle_\pm = E_\alpha |\,\alpha\rangle_\pm \, , \quad \text{with} \quad H = H_0 + W \, . \tag{2}$$

Although they do not form a complete set when bound states exist, they can be taken as orthonormal ${}_\pm\langle \alpha \,|\, \beta \rangle_\pm = \delta(\alpha - \beta)$. Writing the scattering states in a standard way which exhibits the singularities explicitly,

$$|\,\alpha\rangle_\pm = |\,\alpha\rangle - \frac{1}{H_0 - E_\alpha \mp i0} \,|\,\chi(\alpha)\rangle_\pm \, , \tag{3}$$

we find on substitution into Eq. (2) that $|\,\chi(\alpha)\rangle_{\pm}$has to satisfy

$$|\,\chi(\alpha)\rangle_{\pm} = W\;|\,\alpha\rangle - W\frac{1}{H_0 - E_\alpha \mp i0}\,|\,\chi(\alpha)\rangle_{\pm} = W\,|\,\alpha\rangle_{\pm}\,. \tag{4}$$

With $i0$ we indicate an infinitesimally small positive imaginary number. Defining

$$W_{\beta\alpha} = \langle\beta\,|\,W\,|\,\alpha\rangle \quad \text{and} \quad T_{\beta\alpha}(\pm) = \langle\beta\,|\,\chi(\alpha)\rangle_{\pm}\,, \tag{5}$$

we obtain from Eq. (4), after left multiplication with $\langle\beta\,|$, the Lippmann–Schwinger equation for the T-matrix

$$T_{\beta\alpha}(\pm) = W_{\beta\alpha} - \int\limits_{\gamma} \frac{W_{\beta\gamma}T_{\gamma\alpha}(\pm)}{E_\gamma - E_\alpha \mp i0}\,. \tag{6}$$

Taking advantage of the special form of the potential as shown in Eq. (1), and putting $T_{\beta\alpha}(\pm) = M_{\beta\alpha}(\pm)\,\delta(P_\beta - P_\alpha)$, Eq. (6) becomes

$$M_{\beta\alpha}(\pm) = V_{\beta\alpha} - \int\limits_{\gamma} \frac{V_{\beta\gamma}M_{\gamma\alpha}(\pm)}{E_\gamma - E_\alpha \mp i0}\delta(P_\gamma - P_\alpha) \quad \text{for} \quad P_\alpha = P_\beta\,, \tag{7}$$

where the total momenta of the states $|\,\alpha\rangle = |\,q_1, q_2\rangle$, $|\,\beta\rangle = |\,k_1, k_2\rangle$ and $|\,\gamma\rangle = |\,p_1, p_2\rangle$ have been written as $P_\alpha = q_1 + q_2$, $P_\beta = k_1 + k_2$ and $P_\gamma = p_1 + p_2\,.$

In order to show the connection between the scattering amplitudes $M_{\beta\alpha}(\pm)$ and the S-matrix, we follow the presentation of Van Hove [9], without repeating the proofs.

1. The solution of the Schrödinger equation $i\frac{\partial\,|\phi(t)>}{\partial t} = (H_0 + W)|\,\phi(t)\rangle$, which for $t \to -\infty$ approaches the free particle solution $|\,\psi_+(t)\rangle = \int\limits_\alpha d_+(\alpha)\,\mathrm{e}^{-iE_\alpha t}|\,\alpha\rangle$, is given by $|\,\phi(t)\rangle = \int\limits_\alpha d_+(\alpha)\,\mathrm{e}^{-iE_\alpha t}|\,\alpha\rangle_+$.
2. The same solution can be expanded in terms of the stationary scattering states $|\,\alpha\rangle_-$, giving $|\,\phi(t)\rangle = \int\limits_\alpha d_-(\alpha)\,\mathrm{e}^{-iE_\alpha t}|\,\alpha\rangle_-$. For $t \to +\infty$ this approaches the free particle solution $|\,\psi_-(t)\rangle = \int\limits_\alpha d_-(\alpha)\,\mathrm{e}^{-iE_\alpha t}|\,\alpha\rangle$.
3. The S-matrix is defined by $d_-(\alpha) = \int\limits_\beta S_{\alpha\beta}\,d_+(\beta)$.
4. Using

$$\frac{\mathrm{e}^{i(E_\alpha - E_\beta)t}}{E_\alpha - E_\beta - i0} \Longrightarrow 2\pi i\,\delta(E_\alpha - E_\beta) \quad \text{for} \quad \mathrm{t} \to +\infty$$

it can be shown that

$$\lim_{t\to+\infty}\langle\psi_+(t)\,|\,\phi(t)\rangle = \int\limits_{\alpha} d_+^*(\alpha)d_-(\alpha)$$

$$= \int\limits_{\alpha\beta} d_+^*(\alpha)d_+(\beta)\left\{\delta(\alpha-\beta) - 2\pi i T_{\alpha\beta}(+)\delta(E_\alpha - E_\beta)\right\}.$$

5. Comparing the expressions under (3) and (4) it is seen that

$$S_{\alpha\beta} = \delta(\alpha-\beta) - 2\pi i M_{\alpha\beta}(+)\,\delta(P_\alpha - P_\beta)\,\delta(E_\alpha - E_\beta)\,.$$

6. It can be proved that this S-matrix is unitary $\int\limits_{\gamma} S_{\gamma\alpha}^* S_{\gamma\beta} = \delta(\alpha-\beta)$, from which one derives the unitarity relation

$$M_{\alpha\beta}^*(+) - M_{\beta\alpha}(+) = 2\pi i \int\limits_{\gamma} M_{\gamma\beta}^*(+)M_{\gamma\alpha}(+)\delta(P_\gamma - P_\beta)\delta(E_\gamma - E_\beta)\,. \quad (8)$$

For the one-dimensional case many of these formulas can be further simplified by introducing the total momentum and the relative momentum $K = k_1 + k_2$ and $k = (m_2 k_1 - m_1 k_2)/M$, and by writing the total energy as

$$E_\beta = \varepsilon_1(k_1) + \varepsilon_2(k_2) = \frac{k_1^2}{2m_1} + \frac{k_2^2}{2m_2} = \frac{K^2}{2M} + \frac{k^2}{2m}\,,$$

where $m = m_1 m_2/(m_1+m_2)$ is the reduced mass. Since the total momentum is conserved, the potential and also the scattering amplitude only depend on the relative momenta. The Lippmann–Schwinger equation (7) can, therefore, be simplified to

$$M_+(k|q) = \widehat{V}(k-q) - 2m \int\limits_{-\infty}^{\infty} \frac{\widehat{V}(k-p)\,M_+(p|q)}{p^2 - q^2 - i0}\,dp\,. \quad (9)$$

From the unitarity relation (8) one derives

$$\mathrm{Im}M_+(q|q) = -\pi\frac{m}{|q|}\left\{|M_+(q|q)|^2 + |M_+(-q|q)|^2\right\} \quad (10)$$

and

$$\mathrm{Im}M_+(q|-q) = -\pi\frac{m}{|q|}\left\{M_+^*(q|q)M_+(q|-q) + M_+(q|q)M_+^*(q|-q)\right\}. \quad (11)$$

The relation between the ingoing and outgoing amplitude becomes

$$d_-(q) = \left[1 - \frac{2\pi i m}{|q|} M_+(q|q)\right] d_+(q) - \frac{2\pi i m}{|q|} M_+(q|-q) d_+(-q)\,.$$

By writing the same relation for the opposite value of q we see that the S-matrix can be written in the form of a 2×2 matrix

$$\begin{aligned} S &= \begin{pmatrix} \sigma & \rho \\ \rho & \sigma \end{pmatrix}, \\ \sigma &= 1 - \frac{2\pi i m}{|q|} M_+(q|q) \quad \text{and} \quad \rho = -\frac{2\pi i m}{|q|} M_+(-q|q)\,, \end{aligned}$$

where we have used the obvious symmetry of the scattering amplitudes $M_+(q|q) = M_+(-q|-q)$ and $M_+(q|-q) = M_+(-q|q)$. The relations (10) and (11) imply the unitarity of this S-matrix. The coefficients for transmission, reflection and total diffusion are defined in the usual way

$$T = |\sigma|^2\,, \quad R = |\rho|^2 \quad \text{and} \quad I = |\sigma - 1|^2 + |\rho|^2\,.$$

From the unitarity of the S-matrix it is easy to derive

the conservation of probability $T + R = 1$,
the optical theorem $I = -2\mathrm{Re}(\sigma - 1)$.

Since the S-matrix is unitary its eigenvalues have unit magnitude. They can easily be expressed in terms of σ and ρ (see *e.g.* Galindo and Pascual [10] Section 4.5), with the result

$$\mathrm{e}^{2i\delta_0} = \sigma + \rho \quad \text{and} \quad \mathrm{e}^{2i\delta_1} = \sigma - \rho\,. \tag{12}$$

The above mentioned coefficients can be expressed in terms of these phase shifts and we find

$$T = \cos^2(\delta_0 - \delta_1)\,, \quad R = \sin^2(\delta_0 - \delta_1) \quad \text{and} \quad I = 2(\sin^2\delta_0 + \sin^2\delta_1)\,.$$

For the numerical calculation of these phase shifts one first has to solve the Lippmann–Schwinger equation (9). This task is simplified by introducing the K-matrix. This is done by first defining the functions

$$\begin{aligned} Z_\pm(k,q) &\equiv M_+(k\,|\,q) \pm M_+(-k\,|\,q)\,, \\ W_\pm(k,q) &\equiv \widehat{V}(k-q) \pm \widehat{V}(k+q)\,. \end{aligned}$$

From Eq. (9) one then easily derives the equations for $Z_\pm(k,q)$:

$$Z_\pm(k,q)=W_\pm(k,q)\left[1-\frac{i\pi m}{|q|}Z_\pm(q,q)\right]-m\int_{-\infty}^{\infty}\frac{W_\pm(k,p)\,Z_\pm(p,q)}{(p^2-q^2)_P}\,dp\,, \quad (13)$$

in which the principle value of the integral has to be taken. If we now define

$$K_\pm(k,q)\equiv-\frac{\pi m}{|\,q|}\frac{Z_\pm(k,q)}{1-\frac{i\pi m}{|\,q|}Z_\pm(q,q)} \quad \text{and} \quad U_\pm(k,q)\equiv-\frac{\pi m}{|\,k\,|}W_\pm(k,q)\,,$$

it follows from Eq. (13) that the K-matrix must satisfy

$$K_\pm(k,q)=\frac{|\,k\,|}{|\,q|}U_\pm(k,q)+\frac{|\,k\,|}{\pi}\int_{-\infty}^{\infty}\frac{U_\pm(k,p)\,K_\pm(p,q)}{(p^2-q^2)_P}\,dp\,. \quad (14)$$

This equation has the advantage that it relates real quantities and is therefore easier to solve numerically. Moreover, it can easily be checked that the values on the energy shell immediately give the phase shifts:

$$K_+(q,q)=\tan\delta_0\,, \qquad K_-(q,q)=\tan\delta_1\,.$$

For the bound state problem the separation of the centre of mass motion and the relative motion can be done in the standard way. In the momentum representation the eigenvalue problem then takes the form

$$(\varepsilon(q)-E_n)\,\psi_n(q)+\int_{-\infty}^{\infty}\widehat{V}(q-p)\,\psi_n(p)\,dp=0 \quad \text{with} \quad \varepsilon(q)=\frac{q^2}{2m}\,. \quad (15)$$

Other quantities which can also be expressed in terms of the phase shifts $\delta_0(q)$ and $\delta_1(q)$ are the time delays τ_T and τ_R of the transmitted and reflected parts of an incoming wavepacket, which is well localised in space and also in momentum q. After writing $\sigma=|\sigma|\,\mathrm{e}^{2i\delta_\mathrm{T}}$ and $\rho=|\rho|\,\mathrm{e}^{2i\delta_\mathrm{R}}$, it is shown in [10] (Vol. I, p. 154 and 155), that

$$\tau_\mathrm{T}=2\frac{d\,\delta_\mathrm{T}}{d\varepsilon}\,, \quad \tau_\mathrm{R}=2\frac{d\,\delta_\mathrm{R}}{d\varepsilon} \quad \text{with} \quad \varepsilon=\frac{q^2}{2m}\,.$$

In the formula for τ_R a constant term is omitted, which would be present if the potential were not centred around the origin of the coordinate system. Using the definitions Eq. (12) of the phase shifts $\delta_0(q)$ and $\delta_1(q)$, it simply follows that

$$2\,\delta_\mathrm{T}=\delta_0(q)+\delta_1(q)+\varphi_\mathrm{T}\,, \quad 2\,\delta_\mathrm{R}=\delta_0(q)+\delta_1(q)+\varphi_\mathrm{R}\,.$$

The phases φ_{T} and φ_{R} are given by

$$\varphi_{\mathrm{T}} = \begin{cases} \pi & \text{if} \quad \cos(\delta_0 - \delta_1) > 0 \\ 0 & \text{if} \quad \cos(\delta_0 - \delta_1) < 0 \end{cases} , \quad \varphi_{\mathrm{R}} = \begin{cases} \pi/2 & \text{if} \quad \sin(\delta_0 - \delta_1) > 0 \\ -\pi/2 & \text{if} \quad \sin(\delta_0 - \delta_1) < 0 \end{cases} .$$

The time delays become equal to

$$\tau \equiv \tau_{\mathrm{T}} = \tau_{\mathrm{R}} = \frac{m}{q} \frac{1}{1 + \tan^2(\delta_0 + \delta_1)} \frac{d}{dq} \tan(\delta_0 + \delta_1) \, .$$

2.2. Example

The simplest case for which the scattering and the bound state problem can be solved exactly is for the potential

$$V(x) = -\frac{\kappa}{m} \delta(x) \quad \text{for which} \quad \widehat{V}(k) = -\frac{\kappa}{2\pi m} \, . \tag{16}$$

As is well known there is one bound state if $\kappa > 0$

$$\psi(p) = \frac{\kappa^{3/2}}{\sqrt{2}(p^2 + \kappa^2)} \quad \text{with energy} \quad E = -\frac{\kappa^2}{2m} \, . \tag{17}$$

The solution of Eq. (14) is

$$K_+(k, q) = \frac{\kappa}{|\, q|} \quad \text{and} \quad K_-(k, q) = 0 \, .$$

For the phase shifts we obtain

$$\tan \delta_0 = \frac{\kappa}{|\, q|} \quad \text{and} \quad \delta_1 = 0 \, .$$

The transmission, reflection and diffusion coefficients become

$$T = \frac{q^2}{q^2 + \kappa^2} \, , \quad R = \frac{\kappa^2}{q^2 + \kappa^2} \, , \quad I = \frac{2\kappa^2}{q^2 + \kappa^2} \, , \quad \tau = \frac{-\kappa m}{|q| \, (q^2 + \kappa^2)} \, , \tag{18}$$

where τ is the time delay.

3. The relativistic equations

3.1. The general formulation

In the 3+1 dimensional case [4] the essence in changing the Lippmann–Schwinger equation into a relativistic equation was to replace the conservation of total three-momentum for intermediate states, by the conservation of the total three-velocity, which for a state of two particles with momenta p_1 and p_2 is defined as

$$v = \frac{p_1 + p_2}{p_1^0 + p_2^0} = \frac{P}{P^0} \quad \text{with} \quad p_i^0 = \sqrt{p_i^2 + m_i^2}. \tag{19}$$

Repeating the same procedure for the one-dimensional case leads us from Eq. (7) to

$$M_{\beta\alpha}(s \pm i0) = V_{\beta\alpha} - \int_\gamma V_{\beta\gamma} L_\gamma(v, s \pm i0)\, M_{\gamma\alpha}(s \pm i0) \quad \text{for} \quad v_\alpha = v_\beta \equiv v. \tag{20}$$

In this equation we have defined the relativistic propagator by

$$L_\gamma(v, s \pm i0) = \int_0^\infty \frac{ds'}{s' - s \mp i0}\, \delta_2\left(\boldsymbol{P}_\gamma - \frac{s'}{s}\boldsymbol{P}\right), \tag{21}$$

in which the two-dimensional Lorentz vectors $\boldsymbol{P}_\gamma$ and $\boldsymbol{P}$ are given in terms of the velocities v_γ and v by the expressions

$$\boldsymbol{P}_\gamma = \sqrt{\frac{s_\gamma}{1 - v_\gamma^2}}\,(1, v_\gamma) \quad \text{and} \quad \boldsymbol{P} = \sqrt{\frac{s}{1 - v^2}}\,(1, v)\,,$$

so that $s_\gamma = \boldsymbol{P}_\gamma^2 = (P_\gamma^0)^2 - (P_\gamma)^2$ and $s = \boldsymbol{P}^2 = (P^0)^2 - (P)^2$ coincide with the usual Mandelstam variables. From here on 2-vectors will be written in bold face: $\boldsymbol{P} = (P^0, P)$.

The propagator of Eq. (21) is manifestly Lorentz invariant. By performing the s'-integration it takes the form

$$L_\gamma(v, s \pm i0) = L_\gamma^0(s)\, L^1(v_\gamma, v) \tag{22}$$

in which

$$L_\gamma^0(s) = \frac{1}{\sqrt{s_\gamma}\,(\sqrt{s_\gamma} - \sqrt{s} \mp i0)}\,, \qquad L^1(v_\gamma, v) = (1 - v^2)\,\delta(v_\gamma - v)\,.$$

The latter formula clearly exhibits the velocity conservation. From Eq. (21) we easily derive

$$\mathrm{Im} L_\gamma(v, s + i0) = \pi \delta_2(\boldsymbol{P}_\gamma - \boldsymbol{P})\,,$$

which, together with the hermiticity of $V_{\alpha\beta}$, guarantees the unitarity of the S-matrix.

The integration over the states $\gamma = (\boldsymbol{p}_1, \boldsymbol{p}_2)$, occurring in Eq. (20), with $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ both being 2-vectors, is defined by

$$\int\limits_{\gamma} \cdots = \int d\boldsymbol{p}_1 d\boldsymbol{p}_2 \prod_{j=1}^{2} \delta(\boldsymbol{p}_j^2 - m_j^2)\, \theta(p_j^0) \cdots = \int\limits_{-\infty}^{\infty} \frac{dp_1}{2p_1^0} \frac{dp_2}{2p_2^0} \cdots .$$

If we take the velocities of the individual particles as integration variables, this integration element can be written as

$$\int\limits_{\gamma} \cdots = \frac{1}{4} \int\limits_{\gamma}^{*} \cdots = \frac{1}{4} \int\limits_{-\infty}^{\infty} \gamma^2(v_1)\, dv_1 \gamma^2(v_2)\, dv_2 \cdots , \quad \gamma^2(v) = \frac{1}{1-v^2} . \tag{23}$$

Using this notation and writing the propagator in its product form (22), Eq. (20) becomes

$$\begin{aligned} & M_{\beta\alpha}(s \pm i0) \\ & \quad = V_{\beta\alpha} - \frac{1}{4} \int\limits_{-\infty}^{\infty} \frac{V_{\beta\gamma}\, M_{\gamma\alpha}(s \pm i0)(1-v^2)\, \delta(v_\gamma - v)}{\sqrt{s_\gamma}(\sqrt{s_\gamma} - \sqrt{s} \mp i0)} \gamma^2(v_1)\, dv_1 \gamma^2(v_2)\, dv_2 , \\ & \quad \text{for} \quad v_\alpha = v_\beta \equiv v. \end{aligned} \tag{24}$$

The nonrelativistic limit of this equation is obtained by neglecting the kinetic energies of the particles making up the state γ, as compared to their total mass M. If their relative momentum is k, one can show, using the conservation of the total momentum, to which velocity conservation gives rise in this limit, that $\sqrt{s_\gamma} \simeq M + k^2/(2m)$, in which m is the reduced mass. If we now would replace $V_{\alpha\beta}$ by $4m_1m_2V_{\alpha\beta}^{\rm NR}$ and $M_{\alpha\beta}$ by $4m_1m_2M_{\alpha\beta}^{\rm NR}$, it is seen immediately that Eq. (24) becomes identical to the nonrelativistic Lippmann–Schwinger equation (7). So this minimal requirement is satisfied.

In addition, we want Eq. (20) to be invariant under space and time translations and under Lorentz transformations, *i.e.*, under boosts. The latter can be satisfied, because the integration element $\int_\gamma$ and the propagator $L_\gamma(v, s \pm i0)$ are both manifestly Lorentz invariant, so that we only have to choose $V_{\beta\alpha}$ as a function of the scalars that can be formed out of the momentum 2-vectors, which occur in the states α and β. Moreover, we want this potential to be local, so that it should be a function of the momentum transfer only. Strictly speaking this is impossible for the following reason. The relativistic form for the square of the momentum transfer is equal to t_1 or t_2(see figure 1 for the definition of the variables), where

$$t_1 = (\boldsymbol{p}_1' - \boldsymbol{p}_1)^2 , \qquad t_2 = (\boldsymbol{p}_2' - \boldsymbol{p}_2)^2 .$$

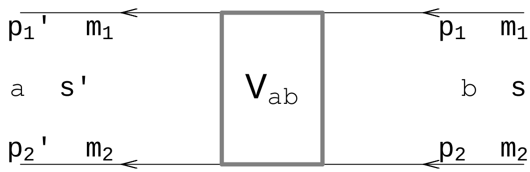

Fig. 1. Definition of momentum variables. (a) for α and (b) for β.

In the usual case where 2-momentum is conserved, $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_1' + \boldsymbol{p}_2'$, we see that the momentum transfers in the upper and in the lower vertex are equal, *i.e.*, $t_1 = t_2 \equiv t$. In the present case, however, we have replaced the momentum conservation by velocity conservation, so that it is not clear which of the t_1 and t_2 should be chosen as variable for the momentum transfer. This problem is solved as follows.

According to Eq. (19) the conservation of velocity can be written as

$$\frac{p_1 + p_2}{p_1^0 + p_2^0} = \frac{p_1' + p_2'}{p_1'^0 + p_2'^0}\,,$$

or, what amounts to the same, as

$$\frac{\boldsymbol{p}_1 + \boldsymbol{p}_2}{\sqrt{s}} = \frac{\boldsymbol{p}_1' + \boldsymbol{p}_2'}{\sqrt{s'}} \quad \text{with} \quad s = (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2 \quad \text{and} \quad s' = (\boldsymbol{p}_1' + \boldsymbol{p}_2')^2\,.$$

Only on the mass shell, *i.e.*, when $s = s'$, momentum is conserved and $t_1 = t_2 \equiv t$. In that case $\boldsymbol{q}_1 = \boldsymbol{p}_1' - \boldsymbol{p}_1$ and $\boldsymbol{q}_2 = \boldsymbol{p}_2 - \boldsymbol{p}_2'$ are equal. In order to be able, also for the off-shell case, to define a Lorentz invariant expression for the potential, we first define other Mandelstam like variables

$$\begin{aligned}
\overline{s} &= (\boldsymbol{p}_1' + \boldsymbol{p}_2') \cdot (\boldsymbol{p}_1 + \boldsymbol{p}_2)\,, \\
\overline{t} &= (\boldsymbol{p}_1' - \boldsymbol{p}_1) \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_2') = \boldsymbol{q}_1 \cdot \boldsymbol{q}_2\,, \\
\overline{u} &= (\boldsymbol{p}_1' - \boldsymbol{p}_2) \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_2')\,,
\end{aligned}$$

which satisfy the relations

$$\overline{s} = \sqrt{s's}\,, \qquad \overline{s} + \overline{t} + \overline{u} = 2\left(m_1^2 + m_2^2\right) - \left(\sqrt{s'} - \sqrt{s}\right)^2\,.$$

As general prescription for the definition of the relativistic potential we now take

$$V_{\alpha\beta} = 4 m_1 m_2 V_{\alpha\beta}^{\mathrm{NR}}(-\overline{t})\,. \tag{25}$$

Here we have replaced the square of the *relative* momentum transfer, by $-\overline{t}$. For any two states α and β we take this as our definition of the potential $V_{\alpha\beta}$.

In this way we now have guaranteed the Lorentz invariance of the theory. Although the strict locality of the interaction is sacrificed, we have shown in [6] that this has no measurable effect on the causality of the theory.

We still must prove that in the nonrelativistic limit, where the momenta are small as compared to the particle masses, $-\bar{t}$ indeed becomes equal to the square of the relative momentum transfer. In order to show this we define the relative and the total 2-momenta for α and β

$$\boldsymbol{k}_\alpha = \frac{1}{M}(m_2\, p_1' - m_1\, p_2')\,, \qquad \boldsymbol{k}_\beta = \frac{1}{M}(m_2\, p_1 - m_1\, p_2)\,,$$

and

$$\boldsymbol{K}_\alpha = \boldsymbol{p}_1' + \boldsymbol{p}_2'\,, \qquad \boldsymbol{K}_\beta = \boldsymbol{p}_1 + \boldsymbol{p}_2\,.$$

In terms of these momenta we obtain

$$\bar{t} = -\frac{m}{M}(\boldsymbol{K}_\alpha - \boldsymbol{K}_\beta)^2 + \frac{m_1 - m_2}{M}(\boldsymbol{K}_\alpha - \boldsymbol{K}_\beta)\cdot(\boldsymbol{k}_\alpha - \boldsymbol{k}_\beta) + (\boldsymbol{k}_\alpha - \boldsymbol{k}_\beta)^2\,,$$

or

$$\bar{t} = -\frac{m}{M}(\sqrt{s'} - \sqrt{s})^2 + \frac{m_1 - m_2}{M}\left(\sqrt{\frac{s'}{s}} - 1\right)\boldsymbol{K}_\beta\cdot(\boldsymbol{k}_\alpha - \boldsymbol{k}_\beta) + (\boldsymbol{k}_\alpha - \boldsymbol{k}_\beta)^2\,.$$

If *(particle momentum)/(particle mass)* is of order ε one easily sees that the first and second term on the right are of order ε^4. Also the time part of $(\boldsymbol{k}_\alpha - \boldsymbol{k}_\beta)^2$ is of order ε^4, so that we are left with the space part $\bar{t} \simeq -(k_\alpha - k_\beta)^2$, which is of order ε^2. This finally shows that for low energies the nonrelativistic theory is recovered if we make the substitution of Eq. (25) into Eq. (24).

In order to prove that the theory is also invariant under space and time translations, it is of some advantage to use stationary states.

3.2. Poincaré invariance

The discussion of Poincaré invariance will be simplified if we first introduce a Hilbert space, which is spanned by a set of free particle states $|\,\alpha\rangle = |v_1, v_2\rangle$ with given velocities. The normalisation we choose such that

$$\langle\alpha'|\,\alpha\rangle = \prod_{i=1}^{2} L^1(v_i', v_i) \equiv \delta(\alpha' - \alpha)\,, \tag{26}$$

in which $\delta(\alpha' - \alpha)$ has all the usual properties of a δ-function under the $\overset{*}{\underset{\gamma}{\int}}$ -integration as defined in Eq. (23). The completeness of these states is expressed by

$$\int_{\alpha}^{*} \langle \chi \,|\, \alpha \rangle \langle \alpha \,| \psi \rangle = \langle \chi \,|\, \psi \rangle \,.$$

By defining

$$W_{\beta\alpha} = \frac{V_{\beta\alpha}}{4(s_\beta s_\alpha)^{1/4}}\,, \quad T_{\beta\alpha} = \frac{M_{\beta\alpha}}{4(s_\beta s_\alpha)^{1/4}}\,, \quad G_0(s_\gamma, s) = \frac{1}{\sqrt{s_\gamma} - \sqrt{s}}\,, \quad (27)$$

Eq. (24) takes the more symmetrical form

$$T_{\beta\alpha}(s) = W_{\beta\alpha} - \int_{\gamma}^{*} W_{\beta\gamma}\, G_0(s_\gamma, s)\, L^1(v_\gamma, v)\, T_{\gamma\alpha}(s)\,, \quad \text{for} \quad v_\alpha = v_\beta = v\,, \quad (28)$$

in which s can now have any complex value. This equation can also be written in operator form when we first define the mass operator $\mathcal{M}$, the propagator $G_0(s)$, the interaction operator W and the scattering operator $T(s)$ by giving their matrix elements

$$\langle \gamma \,|\mathcal{M}|\, \gamma' \rangle = \sqrt{s_\gamma} \langle \gamma \,|\, \gamma' \rangle \quad \text{and} \quad G_0(s) = \frac{1}{\mathcal{M} - \sqrt{s}}$$

and

$$\langle \beta \,|W|\, \alpha \rangle = W_{\beta\alpha}\, L^1(v_\beta, v_\alpha) \quad \text{and} \quad \langle \beta \,|T(s)|\, \alpha \rangle = T_{\beta\alpha}\,(s)\, L^1(v_\beta, v_\alpha)\,. \quad (29)$$

In terms of these operators Eq. (28) becomes

$$T(s) = W - W\, G_0(s)\, T(s)\,.$$

The formal solution is

$$T(s) = W - W\, G(s)\, W \quad \text{with} \quad G(s) = \frac{1}{\mathcal{M} + W - \sqrt{s}}\,.$$

The discrete spectrum of invariant masses M_n is now defined by those values $s_n = M_n^2$ of s, for which the full Green function $G(s)$ becomes singular. This leads to the following eigenvalue equation for the states $|\, n, v\rangle$

$$(\mathcal{M} + W)\,|\, n, v\rangle = M_n \,|\, n, v\rangle\,, \quad (30)$$

in which the velocity has an arbitrary given value. This is not changed by the action of the operator W, because that is velocity conserving, due to the appearance of $L^1(v_\beta, v_\alpha)$ in Eq. (29). If in the velocity representation the wave functions ψ^n_γ are defined by

$$\langle \gamma \,|\, n, v\rangle = \psi^n_\gamma \, L^1(v_\gamma, v)\,,$$

the eigenstates can be expanded in free-particle states

$$|\, n, v\rangle = \int\limits_\gamma^* \psi^n_\gamma \, L^1(v_\gamma, v)\,|\,\gamma\rangle\,.$$

From Eq. (30) we then obtain the eigenvalue equation for the wave functions

$$\left(\sqrt{\boldsymbol{P}^2_\beta} - M_n\right)\,\psi^n_\beta + \int\limits_\gamma^* W_{\beta\gamma} L^1(v_\gamma, v)\,\psi^n_\gamma = 0\,. \tag{31}$$

The hermiticity of $\mathcal{M}$ and W in Eq. (30) guarantees the orthogonality of the eigenfunctions. If we also require the states $|\, n, v\rangle$ to be normalised in the same way as the single free-particle states, compare Eq. (26), then we should have

$$\langle n', v' |\, n, v\rangle = \delta_{n'n}\, L^1(v', v)\,.$$

For the wave functions this implies the following normalisation

$$\int\limits_\gamma^* \psi^{n'*}_\gamma \psi^n_\gamma \, L^1(v_\gamma, v) = \delta_{n'n}\,. \tag{32}$$

Eq. (31) is the basic manifestly Lorentz invariant equation, from which the mass spectrum can be calculated.

If, in analogy with Eqs. (3)–(5), we define the stationary scattering states in the relativistic case by

$$|\,\beta\rangle_+ = |\,\beta\rangle - G_0(s + i0)T(s_\beta + i0)\,|\,\beta\rangle = |\,\beta\rangle - G(s+i0)W\ |\,\beta\rangle\,,$$

for a prescribed, but not shown velocity v, it can simply be proved that these states are also eigenstates of the operator $\mathcal{M} + W$:

$$(\mathcal{M} + W)\ |\,\beta\rangle_+ = \sqrt{s_\beta}|\,\beta\rangle_+\,.$$

Again the velocities of $|\,\beta\rangle$ and $|\,\beta\rangle_+$ are equal, because W is velocity conserving. We now define the operator for the total 2-momentum as

$$\boldsymbol{P}_\mu = (\mathcal{M} + W)\ \boldsymbol{u}_\mu\,, \quad \text{with} \quad \boldsymbol{u} = \frac{(1, v)}{\sqrt{1 - v^2}} \quad \text{and} \quad \mu = 0, 1\,.$$

Both the bound states and the stationary scattering states are eigenstates of this operator with eigenvalues $M_n \boldsymbol{u}_\mu$ and $\sqrt{s_\beta}\boldsymbol{u}_\mu$ for energy and momentum.

We consider P_0 and P_1 as the generators for infinitesimal time and space translations, which can now be used to calculate the effect of a translation over a time t, or over a distance a of an arbitrary state $|\,v\rangle$ with a given velocity:

$$|\,v,t\rangle = \mathrm{e}^{iP_0 t}\,|\,v\rangle \quad \text{and} \quad |\,v,a\rangle = \mathrm{e}^{-iP_1 a}\,|\,v\rangle\,.$$

The operators P_0 and P_1 commute

$$[\boldsymbol{P}_\mu, \boldsymbol{P}_\nu] = 0\,,$$

which is the first requirement for Poincaré invariance. For the full Poncaré invariance to hold we must still define the generator J for infinitesimal boosts, which must satisfy the commutation rules

$$[J, P_0] = iP_1 \quad \text{and} \quad [J, P_1] = iP_0\,. \tag{33}$$

This is simply done by taking for J the same operator as for free particles. In that case P_0 and P_1 do not contain the interaction W, so that Eq. (33) is satisfied. Since, however, W (and also $\mathcal{M}$) is a scalar under Lorentz transformations, it commutes with J. Therefore, we only have to show that

$$[J, u_0] = i\,u_1 \quad \text{and} \quad [J, u_1] = i\,u_0\,. \tag{34}$$

But this follows from the fact that $\boldsymbol{u} = (u_0, u_1)$ transforms like a 2-vector under boosts. It can also be shown explicitly by taking the momentum representation of J

$$J = iP_0 \frac{\partial}{\partial P_1} \quad \text{with} \quad P_0 = \sqrt{P_1^2 + m_2}\,,$$

which in the velocity representation becomes

$$J = i\,(1 - v^2)\frac{\partial}{\partial v}\,.$$

Then Eqs. (33) and (34) follow easily.

When J is applied to a state $|\boldsymbol{u}\rangle = |u_0, u_1\rangle$ the result is $J|\boldsymbol{u}\rangle = i|u_1, u_0\rangle$. A simple calculation shows that a finite boost results in

$$\mathrm{e}^{-i\xi J}\,|\,\boldsymbol{u}\rangle = |\,\boldsymbol{u}'\rangle\,,$$

where

$$u_0' = \frac{1}{\sqrt{1-w^2}}(u_0 + w\,u_1)\,, \quad u_1' = \frac{1}{\sqrt{1-w^2}}(w\,u_0 + u_1)\,, \quad w = \tanh\xi\,.$$

This new 2-vector can also be written as

$$\boldsymbol{u}'=\frac{(1,v')}{\sqrt{1-v'^2}} \quad \text{with} \quad v'=\frac{v+w}{1+v\,w}\,,$$

which gives the usual velocity transformation of the boosted system.

We close this section with the observation that our choice of the infinitesimal generators of the Poincaré group is an example of the "point-form" of Dirac [11]. In this form of a relativistic *classical* theory the interaction was introduced by adding terms to the four components of the momentum, while the generators for rotations and boosts remained unchanged. For this classical case nobody, so far, has succeeded in constructing a potential such that the Poincaré brackets were all satisfied. The reason probably is that, in efforts to build such a construction, different kinds of tacit assumptions were made, like *e.g.*, the Lorentz invariance of world lines, which may be incompatible with a point like interaction. A discussion of these aspects can be found in [12].

3.3. Example

In order to see the relativistic theory at work we consider the same example as in Section 2.2, where in Eq. (16) the interaction in the momentum representation was given by $\widehat{V}(k)=-\kappa/2\pi m$. Since this is independent of the momentum transfer, the prescription Eq. (25) for the construction of the relativistic potential is very simple and leads via Eq. (27) to

$$W_{\beta\alpha}=-\frac{\kappa M}{2\pi(s_\beta s_\alpha)^{1/4}}\,.$$

The eigenvalue equation (31) takes its simplest form in the centre of momentum system. The wave function will then depend only on the relative momentum p. Defining

$$\chi^n(p)=s^{1/4}(p)\,\psi^n(p)\,, \quad \text{with} \quad s^{1/2}(p)=\sqrt{p^2+m_1^2}+\sqrt{p^2+m_2^2}\,,$$

the eigenvalue problem for M_n becomes

$$\left[\sqrt{q^2+m_1^2}+\sqrt{q^2+m_2^2}-M_n\right]\chi^n(q)=\frac{\kappa M}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\chi^n(p)}{\sqrt{(p^2+m_1^2)\,(p^2+m_2^2)}}\,dp\,. \tag{35}$$

By taking $s(p)$ as independent variable, instead of p, this equation reads

$$\left[\sqrt{s}-M_n\right]\chi^n(s)=\frac{\kappa M}{\pi}\int\limits_{M^2}^{\infty}\frac{\chi^n(s')}{\sqrt{s'\lambda(s',m_1^2,m_2^2)}}\,ds'\equiv C_n\,, \tag{36}$$

with the triangle function

$$\lambda(s, m_1^2, m_2^2) = (s - s_+)(s - s_-) \quad \text{and} \quad s_\pm = (m_1 \pm m_2)^2 \,.$$

Since the rhs of Eq. (36) is a constant independent of s, the wave function takes the form

$$\chi^n(s) = \frac{C_n}{[\sqrt{s} - M_n]} \,.$$

When this is substituted into the integral expression for C_n, we get

$$\int\limits_{M^2}^{\infty} \frac{ds}{\sqrt{s\,\lambda(s, m_1^2, m_2^2)}(\sqrt{s} - M_n)} = \frac{\pi}{\kappa M} \,, \tag{37}$$

which is the equation from which the mass M_n of the only bound state must be solved. The normalisation constant C_n follows from Eq. (32), which in terms of $\chi^n(s)$ is

$$\int\limits_{M^2}^{\infty} \frac{|\chi^n(s)|^2 \; ds}{\sqrt{s\,\lambda(s, m_1^2, m_2^2)}} = \frac{1}{2} \,.$$

In the nonrelativistic limit, when $\sqrt{q^2 + m_1^2} \simeq m_1 + q^2/(2m_1)$ Eq. (35) becomes identical to Eq. (15), if we write $M_n = M + E_n$.

For two equal masses, $m_1 = m_2 = 2\,m = \frac{1}{2}M$, Eq. (37) simplifies considerably. The integral can be calculated explicitly and the equation for M_n becomes

$$\frac{4M}{\pi\sqrt{M^2 - M_n^2}} \arctan\sqrt{\frac{M + M_n}{M - M_n}} = 1 + \frac{M_n}{\kappa} \,. \tag{38}$$

In order to find the mass of deeply bound states we expand the lhs in powers of M_n/M and then obtain the solution

$$\frac{M_n}{M} = -\frac{4}{\pi} + \frac{2M}{\kappa} + \mathcal{O}\left(\left(\frac{M}{\kappa}\right)^2\right) \,.$$

This shows that for values of κ larger than a critical value $\kappa_{\rm cr} = \frac{1}{2}\pi M$, the total mass becomes negative. Although in the non-relativistic theory this phenomenon already occurs for values of κ larger than $\frac{1}{2}\sqrt{2}M$, it is not altogether removed from the relativistic theory.

For small values of κ/M we expand the lhs of Eq. (38) in powers of $x = (M - M_n)/M = \varepsilon(\kappa)/M$. Up to the next to lowest order we then find for the binding energy

$$\varepsilon(\kappa) = \frac{\kappa^2}{2\,m}\left[1 - \left(\frac{1}{2} + \frac{1}{\pi}\right)\frac{\kappa}{m} + \ldots\right] \,,$$

which is slightly less then the value in Eq. (17) from the non-relativistic theory. The general dependence of the bound state mass M_n on the strength κ of the potential is shown in Fig. 2. The full line results from the present relativistic theory, the dashed line from the Schrödinger equation.

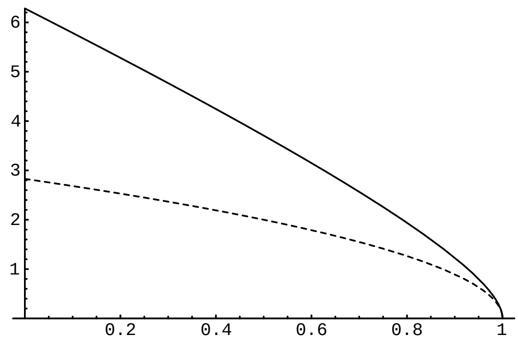

Fig. 2. M_n/M (horizontal axis) *versus* κ/m (vertical axis).

Also the scattering problem, described by Eq. (24), can easily be solved for the potential $V_{\beta\alpha} = -2\kappa M \,/\, \pi$. The scattering amplitude turns out only to depend on the total energy $\sqrt{s_0} = \sqrt{q^2+m_1^2} + \sqrt{q^2+m_2^2}$, in which q is the relative momentum of the two particles in the initial state:

$$\begin{aligned} M(s_0) &= -\frac{2\,\kappa M}{\pi(1-\frac{\kappa M}{\pi}C(s_0))}\,, \\ C(s_0) &= \int\limits_{M^2}^{\infty} \frac{ds}{\sqrt{s\lambda(s,m_1^2,m_2^2)}[\sqrt{s}-\sqrt{s_0}-i0]}\,. \end{aligned} \tag{39}$$

By comparing with Eq. (37) we see that for the bound state energy $\sqrt{s_0} = M_n$ this scattering function indeed becomes singular. From the unitarity condition

$$\mathrm{Im}M(s_0) = -\frac{\pi\,|M(s_0)|^2}{\sqrt{\lambda(s_0,m_1^2,m_2^2)}}\,,$$

which one easily proves from Eq. (39), it follows that $M(s_0)$ can be written in the form

$$M(s_0) = -\frac{\sqrt{\lambda(s_0,m_1^2,m_2^2)}}{\pi}\,\mathrm{e}^{i\,\delta(s_0)}\sin\delta(s_0)\,.$$

Eq. (39) now enables us to derive the following equation for the phase shift

$$\tan\delta(s_0) = \frac{2\,\kappa M}{\sqrt{\lambda(s_0,m_1^2,m_2^2)}(1-\frac{\kappa M}{\pi}Q(s_0))}\,,$$

in which $Q(s_0)$ is the principal value integral

$$Q(s_0) = \mathrm{Re}C(s_0) = \int\limits_{M^2}^{\infty} \frac{ds}{\sqrt{s\,\lambda(s, m_1^2, m_2^2)}[\sqrt{s} - \sqrt{s_0}]_P}\,.$$

For a given strength κ of the potential the reflection coefficient $R(s_0) = \sin^2\delta(s_0) = (\tan^2\delta)/(1+\tan^2\delta)$ can now be calculated for any value of the energy $\sqrt{s_0}$. In the case of two equal masses and with $\kappa = m$, the result of such a calculation is shown in Fig. 3. For comparison we also plotted the reflection coefficient from the nonrelativistic theory Eq. (18) for the same value of κ. We see that for high energies the reflection is considerably suppressed.

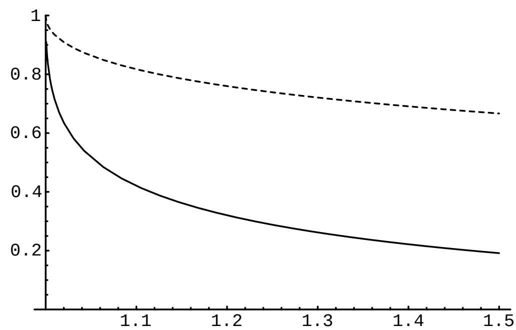

Fig. 3. Reflection coefficients for the relativistic theory (full curve) and the nonrelativistic theory (dashed line) for $\kappa = m$, as functions of the energy in units of M.

With this value of κ we also calculated the delay time $\tau = d\,\delta/(d\,E)$. In Fig. 4 this delay time is shown as a function of the energy (in units of M), together with the value obtained from Eq. (18) from the nonrelativistic theory. The relativistic effect is largest for low energies.

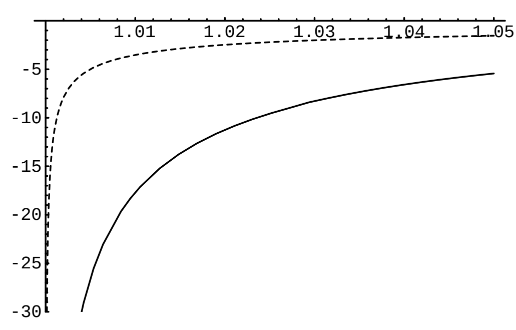

Fig. 4. Delay times (in units of M^{-1}) for the relativistic theory (full curve) and the nonrelativistic theory (dashed line) for $\kappa = m$, as functions of energy in units of M.

4. The relativistic harmonic oscillator

The nonrelativistic potential for the interaction between two particles with masses $m_1$and m_2 is $V^{\rm NR}(x_1,x_2)=\frac{1}{2}m\omega^2(x_1-x_2)^2$, in which m is the reduced mass. In order to write the relativistic eigenvalue equation Eq. (31) in a more explicit form, first the Fourier transform Eq. (1) of this harmonic potential must be calculated. Since this does not exist, a cutoff potential is introduced

$$V_{\rm R}^{\rm NR}(x)=m\omega^2R^2\left[1-{\rm e}^{-\frac{x^2}{2R^2}}\right],$$

which has the correct limit when $R\to\infty$ and which *does* have a Fourier transform

$$V_{\rm R}^{\rm NR}(k,k') \;=\; m\omega^2R^2\left[\delta(q)-\frac{\rm R}{\sqrt{2\pi}}\,{\rm e}^{-\frac{1}{2}q^2R^2}\right], \tag{40}$$

where $q=k'-k$.

With Eq. (25) the potential $V(k,k')$ is determined, which figures in the eigenvalue equation obtained from Eq. (31). In the centre of momentum system this equation takes the form

$$\left[\sqrt{k^2+m_1^2}+\sqrt{k^2+m_2^2}-M_n\right]\chi_n(k)+\frac{1}{4}\int\limits_{-\infty}^{\infty}\frac{V(k,k')}{\sqrt{(k'^2+m_1^2)(k'^2+m_2^2)}}\chi_n(k')\,dk'=0\,,$$

where the following substitution was made

$$\chi_n(k)=\left[\sqrt{k^2+m_1^2}+\sqrt{k^2+m_2^2}\right]^{1/2}\psi_n(k)\,.$$

The ortho-normality of the eigenfunctions Eq. (32) is now expressed by

$$\frac{1}{4}\int\limits_{-\infty}^{\infty}\frac{\chi_l^*(k)\chi_m(k)}{\sqrt{(k^2+m_1^2)(k^2+m_2^2)}}{\rm d}k=\delta_{lm}\,.$$

It must be kept in mind that in the expression for the potential $V(k,k')$, in which $V_{\rm R}^{\rm NR}(k,k')$ of Eq. (40) occurs, the replacement $q^2\to-\bar{t}$ must be made. The details of this replacement were explained in [5], Eqs. (111)–(115). Since for large R the potential $V(k,k')$ is very much peaked around $k'\approx k$, the integrand, including the function $\chi_n(k')$, can be expanded in powers of $k'-k$. This then gives rise to a second order differential equation for $\chi_n(k)$ with only finite terms, because those which tend to infinity with $R\to\infty$, cancel.

In the static case, in which m_1 is kept finite, but $M = m_1 + m_2 \to \infty$, this equation takes the form

$$\frac{d^2\chi_n(x)}{dx^2} - xP(x)\frac{d\chi_n(x)}{dx} + Q_n(x)\chi_n(x) = 0\,, \tag{41}$$

with

$$\begin{aligned} P(x) &= \frac{2g}{1+gx^2}\,, \\ Q_n(x) &= \frac{g(1-2gx^2)}{(1+gx^2)^2} - \frac{2}{g}\sqrt{1+gx^2}\left\{\sqrt{1+gx^2} - \sqrt{1+gx_n^2}\right\}. \end{aligned}$$

The following abbreviations have been introduced

$$x = \frac{k}{\sqrt{m\omega}}\,, \quad g = \frac{\omega}{m}\,, \quad M_n = M - m + \varepsilon_n\,, \quad \varepsilon_n = m\sqrt{1+gx_n^2}\,.$$

The nonrelativistic approximation is obtained by letting $g \to 0$. In that case Eq. (41) becomes

$$-\frac{d^2\widetilde{\chi}_n(x)}{dx^2} + x^2\,\widetilde{\chi}_n(x) = \widetilde{x}_n^2\,\widetilde{\chi}_n(x)\,,$$

which is the Schrödinger equation for the harmonic oscillator. The eigenvalues are $\widetilde{x}_n^2 = 2n+1$ and correspondingly

$$\widetilde{\varepsilon}_n = m\left(1 + \frac{1}{2}\,g\,\widetilde{x}_n^2\right) = m + \left(n + \frac{1}{2}\right)\omega\,.$$

For a range of g-values the ground state and the first five excited states were calculated by numerically solving Eq. (41). The difference in mass as compared to the nonrelativistic case

$$\widetilde{\varepsilon}_n - \varepsilon_n = m\left[1 + \left(n + \frac{1}{2}\right)g - \sqrt{1+gx_n^2}\right]$$

is plotted in Fig. 5.

The picture shows that relativistic effects are important in explaining the mass of bound states. Especially the higher excited states experience a tighter binding and are therefore lighter than expected from a nonrelativistic theory.

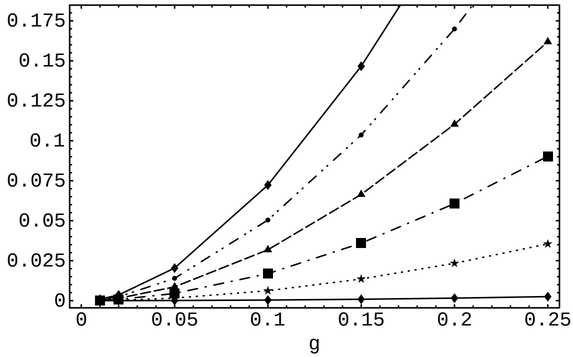

Fig. 5. Decrease of mass in units of m.

Concluding it can be said that, as suggested by the results of the present paper, the mass of the Ds(2317) meson can perhaps still be understood on the basis of a quasipotential theory, provided relativity is included in a consistent way.

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