SYMMETRIES OF NUCLEAR HAMILTONIANS WITH REDUNDANT VARIABLES*

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The problem of symmetries within the redundant-variable method is revisited. Using physical conditions similar to those suggested by Bohr and Mottelson an appropriate symmetry subspace of the full space of the redundant-variable Hamiltonian has been constructed. Such a construction is a prerequisite condition for entering into any realistic calculation scheme.

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1. Introduction

It is well known that no general method exists which allows to solve the nuclear many-body problem exactly. A class of approximate methods can be constructed by dividing the whole set of nuclear degrees of freedom into two sub-sets. The first one usually describes the single-particle, and the second one — the collective properties of the motion. In principle one can try to construct directly the 'collective' variables out of those that characterize the nucleonic degrees of freedom, however, the general solution of this difficult problem has never been obtained, *cf. e.g.* [1].

An alternative approach employs the so-called redundant variables. Such an approach has been used by many authors and an extensive presentation of the nuclear physics applications can be found in [2] and [1]. In this case the configuration space of the system is enlarged by introducing, in addition to the $6A$ single-nucleonic degrees of freedom, a number of 'collective' ones. Thus the ensemble of the variables used in this case must contain some redundant ones since the number of all the degrees of freedom must

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not exceed $6A$. The presence of redundant variables usually introduces some spurious solutions. To remove them, one needs to bring-in some extra constraints and this complicates the mathematical formulation of the problem that can be treated explicitly only in very simple cases like *e.g.* that of the translational motion. Even for rotational excitations one can formulate this procedure only approximately. The problem is still opened but despite that, using methods of redundant variables in one of their approximate realizations remains in practice a source of numerous successful model-approaches.

Another problem with the redundant variable method is to relate true symmetries of the system to those that can be described in a mathematically correct manner in an enlarged space. A solution to this problem has been proposed in [2], but only for some simple Abelian symmetry groups. Recently, a possible importance of higher, non-Abelian, symmetry groups in nuclei has been pointed to [3]. Here we briefly describe how to handle discrete symmetries within a redundant-variable approach for the case of non-Abelian symmetry groups expected to play an important role in the advanced nuclear physics applications.

2. Symmetry groups and enlarged configuration spaces

The space of all the states appearing in the redundant-variable approach can be represented as a tensor product of the intrinsic, $\mathcal{K}_{\rm intr}$, and collective, $\mathcal{K}_{\rm coll}$, spaces: $\mathcal{K}_{\rm intr}\otimes\mathcal{K}_{\rm coll}$. In the following it will be of advantage to shorten the notation. We will write $\mathcal{K}_{\rm intr}\to\mathcal{K}$, $\mathcal{K}_{\rm coll}\to\tilde{\mathcal{K}}$ and $\mathcal{K}_{\rm intr}\otimes\mathcal{K}_{\rm coll}\to\mathcal{K}\otimes\tilde{\mathcal{K}}$.

Suppose that the original Hamiltonian, the one acting in the single-nucleon space, is invariant under transformations of a certain group $\mathbf{G}$. The corresponding symmetry group of the redundant-variable Hamiltonian will be denoted $\mathbf{G}\times\tilde{\mathbf{G}}$; it can be realized by the product operators $\mathcal{T}(g,\tilde{h}) = T(g)\tilde{T}(\tilde{h})$, where T and $\tilde{T}$ act in $\mathcal{K}$ and $\tilde{\mathcal{K}}$, respectively. Needless to say, symbols $\mathbf{G}$ and $\tilde{\mathbf{G}}$ represent the same symmetry group in two different physical realizations; in $\mathcal{T}(g,\tilde{h})$ the arguments g and $\tilde{h}$ may take all possible values independently of one another.

In Ref. [2] there is a series of examples showing how one can construct relations between $T(g)$ and $\tilde{T}(\tilde{g})$ operators to represent the same symmetry of the physical system in the corresponding subspaces. Here, g and $\tilde{g}$ represent the same symmetry operation but belonging to $\mathbf{G}$ and $\tilde{\mathbf{G}}$, respectively. However, all the examples in [2] concern the Abelian groups where a correspondence condition written in the form

$$g=\tilde{g} \quad \text{and} \quad T=\tilde{T} \tag{1}$$

plays a central role. Although the above relation can be intuitively understood, strictly speaking it is incorrect because the operators in question

act in different spaces. In the following we show some implications of the condition (1) but rewritten in a formally correct way.

The symmetry group $\mathbf{G} \times \tilde{\mathbf{G}}$ may contain several subgroups. In what follows we restrict our considerations to operators $\mathcal{T}$ that generate such a 'physical' symmetry-subgroup $\mathcal{G} \subset \mathbf{G} \times \tilde{\mathbf{G}}$. The intuitive relation in Eq. (1) can be made more precise by requiring that

$$\mathcal{G} \ni \mathcal{T}(g, \tilde{g}) = T(g)\,\tilde{T}(\tilde{g}) = \mathbb{I}_{[\text{in } \mathcal{G}]}. \tag{2}$$

This condition can be realized mathematically if the operators $\tilde{T}(\tilde{g})$ are taken to be *'conjugate'*, as opposed to 'usual', representations (*cf.* Ref. [4] and also below). In other words: Eq. (2) expresses the fact that representations $\mathcal{T}(g, \tilde{g})$ acting in the redundant variable space $\mathcal{K} \otimes \tilde{\mathcal{K}}$ are scalar. The above relation may be given the following geometrical interpretation: an action of an intrinsic-space operator in $\mathcal{K}$ represents the same as the action of the corresponding collective-space operator in $\tilde{\mathcal{K}}$. The symbol 'tilde' in $\tilde{g}$ stresses the fact that the second argument of $\mathcal{T}(g, \tilde{g})$ corresponds to the same symmetry operation as g but acting in the collective space. For one dimensional irreducible representations such a condition may lead to unique relations among symmetry quantum numbers of the intrinsic and collective spaces. Some examples of such relations are shown in Eqs (20)–(21) below, some others in [2]. However, the multidimensional representations require some special care and will be discussed below.

To proceed we will need to specify the action of the symmetry operators $T(g)$ and $\tilde{T}(\tilde{g})$ in the corresponding vector spaces more precisely. We will denote the intrinsic wave functions by Φ. The basis of irreducible representations in the intrinsic space will be denoted by $|\Phi(\Gamma; \gamma; a)\rangle$, where Γ labels the irreducible representations, γ is a set of quantum numbers distinguishing among physically different but mathematically equivalent representations (therefore γ stands for all the quantum numbers that distinguish physical properties not related to the group structure) and a denotes all the quantum numbers required for unique labeling of the vectors that span the irreducible representation space. Similarly, the corresponding basis in the collective space will be denoted by $|\Psi(\Gamma; \gamma; a)\rangle$.

Let us introduce the matrices of the irreducible representations, $\Delta^{(\Gamma)}(g)$, and their transpose-inverse matrices $\bar{\Delta}^{(\Gamma)}(g)$:

$$\bar{\Delta}^{(\Gamma)}(g) = \left\{ [\Delta^{(\Gamma)}(g)]^T \right\}^{-1} = [\Delta^{(\Gamma)}(g^{-1})]^T . \tag{3}$$

The matrices $\bar{\Delta}^{(\Gamma)}(g)$ themselves provide irreducible representations of the group $\mathbf{G}$ called adjoint or sometimes, what is a better idea from the physics context point of view, inverse representation. Using this notation it can be

shown that the required action of the intrinsic symmetry-operator can be written as

$$T(g)\,|\,\Phi(\Gamma;\gamma;a)\rangle \;=\; \sum_{a'}\left\{[\Delta^{(\Gamma)}]^{-1}\right\}_{aa'}(g)\,|\,\Phi(\Gamma;\gamma;a')\rangle$$
$$=\; \sum_{a'}\bar{\Delta}^{(\Gamma)}_{a'a}(g)\,|\,\Phi(\Gamma;\gamma;a')\rangle\,, \tag{4}$$

and the action of the collective symmetry-operator $\tilde{T}(\tilde{g})$ as

$$\tilde{T}(\tilde{g})\,|\,\Psi(\Gamma;\gamma;a)\rangle = \sum_{a'}\Delta^{(\Gamma)}_{a'a}(\tilde{g})\,|\,\Psi(\Gamma;\gamma;a')\rangle. \tag{5}$$

As stated already, condition (2) expresses the fact that the physically acceptable irreducible representations of $\mathcal{G}$ are scalar. One can demonstrate, Ref. [5], that the corresponding representations can be obtained by projection using projection operator:

$$\hat{P}_{\,\Gamma} = \frac{\dim(\Gamma)}{\mathrm{card}(\mathbf{G})}\sum_{g\in\mathbf{G}}\;[\chi^{(\Gamma)}(g)]^{\star}\;T(g,\tilde{g})\,, \tag{6}$$

where $\dim(\Gamma)$ denotes the dimension of the irreducible representation Γ, $\mathrm{card}(\mathbf{G})$ denotes the number of group elements in the considered point group and $\chi^{(\Gamma)}(g)$ is the character of the element g corresponding to the irreducible representation Γ. For the case of a scalar representation one obtains the following result:

$$\hat{P}_0\,|\,\Phi(\Gamma_1;\gamma_1;a_1)\,\Psi(\Gamma_2;\gamma_2;a_2)\rangle = \frac{\delta_{\Gamma_1\,\Gamma_2}\delta_{a_1\,a_2}}{\dim[\Gamma_1]}\sum_{a'}|\,\Phi(\Gamma_1;\gamma_1;a')\,\Psi(\Gamma_1;\gamma_2;a')\rangle. \tag{7}$$

Let us observe that any non zero state-vector (7) is independent of any particular choice of quantum numbers a_1 and a_2, since no term on the right-hand side of this equation depends on them. We can further simplify the notation and rewrite Eq. (7) as

$$|\Gamma;\gamma_1;\gamma_2\rangle = \frac{1}{\sqrt{\dim[\Gamma]}}\sum_{a}|\,\Phi(\Gamma;\gamma_1;a)\,\Psi(\Gamma;\gamma_2;a)\rangle. \tag{8}$$

This is an orthonormal basis in the subspace in which the required relation (2) is automatically fulfilled; it spans a subspace of $\mathcal{K}\otimes\tilde{\mathcal{K}}$, adapted to the future use of property (2). All vectors $|\,\Psi\rangle$ that are associated with $|\,\Phi\rangle$ within the redundant-variable method, in order to fulfill (2), must belong to the constructed subspace and consequently they can be decomposed using

the basis in Eq. (8); each of the Φ and Ψ wave-functions (states) belongs to an irreducible representation Γ of the symmetry group.

Comparing to the phenomenological construction of Ref. [2] where the wave functions take always the form of $[\Phi_K D^I_{M,K} \pm \Phi_{-K} D^I_{M,-K}]$ one observes a structural difference: in general none of the terms in the square brackets belongs to any irreducible representation basis of the symmetry group.

Within the subspace of the considered direct-product space we have again only non-degenerate spectra owing to the construction of the scalar irreducible representations, despite the fact that they were constructed out of double degenerate intrinsic and collective energies.

3. Energy spectra and symmetry considerations

Let us consider a nuclear Hamiltonian that can be written down as

$$H = H_{\rm intr} + H_{\rm rot}\,. \tag{9}$$

Hamiltonians of this form have been widely used in the literature; at the same time they have the mathematical structure that is already appropriate to present our symmetry considerations.

Let $\mathbf{G}$ be the symmetry group of $H_{\rm intr}$ (similarly $\tilde{\mathbf{G}}$ the symmetry group of $H_{\rm rot}$). Let Γ denote the related s-dimensional irreducible representations, *i.e.* vectors that span the corresponding irreducible representation spaces can be enumerated with $a = 1, 2, \ldots\ s$, for both $\mathbf{G}$ and $\tilde{\mathbf{G}}$. We can always represent the solutions to the Schrödinger equations in the two spaces as:

$$H_{\rm intr}\,|\,\Phi(\Gamma;\gamma_1;a)\rangle = E_{\rm intr}(\Gamma;\gamma_1)\,|\,\Phi(\Gamma;\gamma_1;a)\rangle \tag{10}$$

and

$$H_{\rm rot}|\ \Psi(\Gamma;\gamma_2;a)\rangle = E_{\rm rot}\,(\Gamma;\gamma_2)\,|\,\Psi(\Gamma;\gamma_2;a)\rangle. \tag{11}$$

In such a case, as demonstrated in quantum mechanics courses, each eigen-energy is common to all $a = 1, 2, \ \ldots\ s$ eigen-vectors. In other words, eigen-energies are independent of quantum number a and the corresponding solutions are s-fold degenerate. States (8) constructed out of solutions (10) and (11) are also eigen-states of H and we have:

$$H\,|\Gamma\gamma_1\gamma_2\rangle = [E_{\rm intr}(\Gamma;\gamma_1) + E_{\rm coll}(\Gamma;\gamma_2)]\,|\Gamma\gamma_1\gamma_2\rangle\,. \tag{12}$$

Glancing at relation (12) one may think that the corresponding solutions are multi-fold (here: $s \times s$) degenerate as discussed in the standard quantum mechanics courses in the similar-looking context (*cf. e.g.* Ref. [7]). However, although constructed out of degenerate eigen-energy sets they cannot be treated as degenerate because in order to become acceptable as the physical

ones they must belong to the scalar representations as required by condition (1), or more formally by (2). One can thus conclude that condition (1) proposed in [2] always leads to a non-degenerate eigen-value problem.

To illustrate this problem let us consider two examples: an Abelian symmetry group D_2, well known from the studies of the tri-axial quantum rotor and a D_3 non-Abelian symmetry group. In the following we limit ourselves to the case of even–even nuclei.

3.1. Abelian groups: example of the D_2-symmetry

Let us begin with an example of the D_2-symmetry. The corresponding group is composed of 4 elements: three rotations through the angles of π about three mutually perpendicular axes that can be chosen as $\mathcal{O}_x$, $\mathcal{O}_y$ and $\mathcal{O}_z$ and the identity transformation. The rotation operators are denoted C_{2x}, C_{2y} and C_{2z}, where by definition $C_{n\alpha}$ implies a rotation about an axis α through the angle $2\pi/n$. Group D_2 being Abelian, all its irreducible representations are one-dimensional and, moreover, any group-element forms an equivalence class of its own. Since the number of equivalence-classes equals that of the irreducible representations it follows that D_2 has 4 one-dimensional irreducible representations.

Let us first consider the solutions to the collective rotation Hamiltonian. On the one hand it is well known that these solutions can be directly represented by the Wigner functions $\mathcal{D}^I_{MK}$. On the other hand we would like these solutions to transform as irreducible representations of the D_2-group. Since the latter are one-dimensional, it follows that under the action of any of the group elements they must transform onto themselves, possibly up to a phase. It can be verified using the transformation properties of $\mathcal{D}^I_{MK}$ under 180°-rotations that the sought irreducible representation basis vectors are either $[D^I_{M,K} + D^I_{M,-K}]$ or $[D^I_{M,K} - D^I_{M,-K}]$ (as usual we consider $K \geq 0$). Thus it will always be possible to use the corresponding combinations so that the solutions in question can be considered as basis vectors of the irreducible representations Γ of the symmetry group and we can write

$$H_{\rm rot}\,|\,\Psi(\Gamma;\gamma;a)\rangle = E^{\Gamma\gamma}_{\rm rot}\,|\,\Psi(\Gamma;\gamma;a)\rangle\ . \tag{13}$$

In the present case, index a enumerating basis vectors of a given irreducible representation may take only one value and can be omitted.

It is of advantage to be able to label the irreducible representation basis vectors with quantum numbers invariant under the action of all the group elements. One may often obtain such a convenient labeling by considering first a subgroup of a group in question and constructing the basis-states and the corresponding labeling for the subgroup. Here we chose a subgroup

composed of C_{2x} and of the identity. The latter has two irreducible one-dimensional representations only. They can be labeled with the help of the eigen-values μ_x of the C_{2x} operator. Since $(C_{2x})^2 = 1$ we have $\mu_x = \pm 1$.

To obtain a second label let us observe that the operator of the third component of the angular momentum on the body-fixed $\mathcal{O}_z$ axis, $\hat{I}_z$, when acting on the Wigner functions gives K as an eigen-value. Since for integer K we also have $(-1)^K = (-1)^{(-K)}$ it follows that $[D^I_{M,K} + D^I_{M,-K}]$ and $[D^I_{M,K} - D^I_{M,-K}]$ are eigen-vectors of $(-1)^{\hat{I}_z}$ and we obviously have $C_{2z}\,(-1)^{\hat{I}_z}\,C^\dagger_{2z} = (-1)^{\hat{I}_z}$. Since in addition $C_{2x}\,(-1)^{\hat{I}_z}\,C^\dagger_{2x} = (-1)^{\hat{I}_z}$ and $C_{2y}\,(-1)^{\hat{I}_z}\,C^\dagger_{2y} = (-1)^{\hat{I}_z}$ we conclude that the expression $(-1)^K$ is invariant under all the group operations and can serve as another label of the irreducible representation basis vectors. Furthermore, since C_{2y} can always be expressed in terms of products of the other two operations in D_2 there are no other independent quantum numbers possible and consequently the irreducible representations can be labeled by $\Gamma \equiv \{\mu_x, (-1)^K\}$.

In the case of the Wigner wave-function basis all *equivalent* representations are labeled with $\gamma \equiv \{I, M\}$. One can then show that in order to construct all the possible basis vectors taking into account $I = 0, 1, 2, \ldots$ in the form $\psi = D^I_{MK} + \beta D^I_{M,-K}$ we have to set $\beta = \mu_x * (-1)^I$ and it follows that one can introduce a D_2 specific notation $|\,\Psi(\Gamma;\gamma;a=1)\rangle \rightarrow \Psi(\{\mu_x, K\};\{I, M\})$ in the form

$$\Psi(\{\mu_x, K\};\{I, M\}) = \mathcal{N}(I, K)\left[\left(D^I_{M,K}\right)^\star + \mu_x(-1)^I\left(D^I_{M,-K}\right)^\star\right], \quad (14)$$

where standard normalization factor $\mathcal{N}(I, K) = \sqrt{(2I+1)/[2(1+\delta_{K0})]}$. Elementary calculations involving only the transformation properties of the D^I_{MK} functions give:

$$C_{2x}\ \Psi(\{\mu_x, K\};\{I, M\}) = \mu_x\ \Psi(\{\mu_x, K\};\{I, M\}), \quad (15)$$

$$C_{2z}\ \Psi(\{\mu_x, K\};\{I, M\}) = (-1)^K\ \Psi(\{\mu_x, K\};\{I, M\}). \quad (16)$$

Recall that our ultimate goal is to express the general relation (2) by using the irreducible representation bases constructed according to Eq. (8). However, in realistic calculations it may in general neither be possible nor convenient to re-introduce the quantum numbers μ_x and K in the case of the intrinsic Hamiltonian and other, 'practical', means will be needed to find the adequacy between $\Psi(\{\mu_x, K\};\{I, M\})$ and the intrinsic Hamiltonian eigenfunctions Φ. For that purpose it will be instructive to compare the actions of the operators C_{2x} and $\tilde{C}_{2x}$ (similarly, C_{2y} and $\tilde{C}_{2y}$ and C_{2z} and $\tilde{C}_{2z}$) on the wave functions in their respective spaces. The case of the collective-variables has been treated above. Since one of the three rotation operators of the D_2

group can be expressed as a product of the other two it will be sufficient to consider two of those only. Since furthermore D_2 is an Abelian symmetry group the corresponding operators commute among themselves and with the Hamiltonian, and we may always find a representation such that

$$H_{\rm intr}\,|\,\Phi(\Gamma';\gamma';a)\rangle = E_{\rm intr}^{\Gamma'\gamma'}\,|\,\Phi(\Gamma';\gamma';a)\rangle \tag{17}$$

(index a may take only one value and as before will be omitted) and simultaneously

$$C_{2x}\ \Phi(\Gamma';\gamma') \ = \ \nu_x(\Gamma';\gamma')\ \Phi(\Gamma';\gamma')\,, \tag{18}$$

$$C_{2z}\ \Phi(\Gamma';\gamma') \ = \ \nu_z(\Gamma';\gamma')\ \Phi(\Gamma';\gamma')\,, \tag{19}$$

where the eigen-values $\nu_x(\Gamma';\gamma') = \pm 1$ and $\nu_z(\Gamma';\gamma') = \pm 1$, since as we know $(C_{2x})^2 = 1$ and $(C_{2z})^2 = 1$.

An adequacy (at this stage unknown) between parameter Γ' and the irreducible representation index $\Gamma = \{\mu_x, (-1)^K\}$ can be established by requiring that $\Gamma' = \Gamma$ if

$$\nu_x(\Gamma';\gamma') \ = \ \mu_x\,, \tag{20}$$

$$\nu_z(\Gamma';\gamma') \ = \ (-1)^K\,, \tag{21}$$

[*cf.* Eqs (16) and (16)]. The above condition provides missing information necessary to construct the physical wave functions with the redundant variable approach since from now on the correspondence between $\Phi(\Gamma;\gamma_1;a)$ and $\Phi(\Gamma;\gamma_2;a)$ in Eq. (8) and the wave functions in Eqs (14) and (17), respectively, is one-to-one.

The illustrated here D_2 case may serve as an example for other Abelian symmetry groups.

3.2. Non-Abelian groups: example of the D_3-symmetry

Let us consider an even–even nucleus and assume that its Hamiltonian is invariant with respect to symmetry group D_3. It is a non-Abelian group containing one three-fold and three (perpendicular to the latter) two-fold symmetry axes among its associated symmetry elements. It has two non-equivalent one-dimensional irreducible representations and one two-dimensional irreducible representation, *cf. e.g.* Ref. [6]. The one-dimensional irreducible representations can be treated as in the example above so that below we focus on the two-dimensional irreducible representations.

A detailed presentation of a construction of the bases of the D_3-group irreducible representations that are of interest in the present context goes beyond the scope of this paper, and here we only sketch the final results.

It will be convenient to introduce two auxiliary expressions

$$A^{(+)}_{\mu,K} = \begin{cases} 1, \text{ if } \mu + \mathrm{K} = 0, \pm 3, \pm 6, \ \ldots \\ 0 \quad \text{otherwise} \end{cases} \tag{22}$$

and

$$A^{(-)}_{\mu,K} = \begin{cases} 1, \text{ if } \mu - \mathrm{K} = 0, \pm 3, \pm 6, \ \ldots \\ 0 \quad \text{otherwise} \end{cases} \tag{23}$$

where we set $\mu = 0, \pm 1$, and define, for the one-dimensional irreducible representations, the functions

$$\Psi(\Gamma;\gamma) = \mathcal{N}^I_{\mu,K} \left[A^{(+)}_{\mu,K}(D^I_{M,+K})^\star + \mu_x(-1)^I A^{(-)}_{\mu,K}(D^I_{M,-K})^\star\right], \tag{24}$$

where

$$\mathcal{N}^I_{\mu,K} \equiv \sqrt{\frac{2I+1}{2\,(1+\delta_{K0})}} \tag{25}$$

and for the two-dimensional ones

$$\Psi(\Gamma;\gamma;\,a) = \mathcal{N}^I_{\mu,K} \left[A^{(+)}_{\mu,K}(D^I_{M,+K})^\star + a\,(-1)^I A^{(-)}_{\mu,K}(D^I_{M,-K})^\star\right], \tag{26}$$

where index a enumerates the basis states and may take, by definition, the values $a = -1$ and $a = +1$. (The meaning of index Γ will be specified below.)

One can show that the irreducible representations of the D_3-group that are constructed with the help of the Wigner functions can be parametrized using two integer indices μ and μ_x with $\mu = 0, \pm 1$ and $\mu_x = \pm 1$. These same indices can also be used to *label* the irreducible representations of D_3. More precisely: it is possible to demonstrate that the two independent one-dimensional irreducible representations correspond to $\mu = 0$ and are distinguished with $\mu_x = +1$ or $\mu_x = -1$ [the corresponding labels are $\Gamma = \{|\mu|, \mu_x\} \to \Gamma = \{1, +1\}$ or $\Gamma = \{1, -1\}$, *cf.* Eqs (22)–(24)] while for the two-dimensional irreducible representation we have $\mu = \pm 1$ [the corresponding single label $\Gamma = \{|\mu|, -\} \to \Gamma = \{1, -\}$, *cf.* Eq. (26)].

Furthermore, one can show that the structure of the basis states in question depends on K in the following way. In the case of the one-dimensional irreducible representations we must have $K = 0, \pm 3, \pm 6, \ \ldots$ (*i.e.* K remains a multiple of 3). If $K = 0$ we have additionally the condition that I is even when $\mu_x = +1$ and I is odd when $\mu_x = -1$. For $K \neq 0$ all $(I \geq |K|)$-values are allowed for both $\mu_x = +1$ and $\mu_x = -1$.

In the case of two-dimensional irreducible representations we have $K = \pm 1, \pm 2, \ \pm 4, \pm 5, \ \pm 7, \pm 8, \ \ldots$, *i.e.* all integers not divisible by 3 are allowed [the above rules have been incorporated in the definitions in Eqs (22) and (23)].

In the following we focus on the two-dimensional irreducible representations. It remains to construct the relations analogous to Eqs (17)–(21) that as before would allow establishing an adequacy between the basis vectors in the intrinsic and collective spaces so that Eq. (8) can be applied. For that purpose it will be convenient to use the *basis* states in the intrinsic space that can be labeled by the total angular momentum I, its projection K on the intrinsic 'third' axis and by some additional quantum numbers, say, γ: $|\gamma;\ IK\rangle$. Then each of the two vectors that span two-dimensional irreducible representation bases can be written as a linear combination of those states involving $K = \ldots, -5, -2, 1, 4, \ldots$ for one of them and $K = \ldots, -4, -1, 2, 5, \ldots$, for the other, *cf.* the rules established above. Since in the case of the D_3 symmetry-group $\hat{C}_{3z}$ and the intrinsic Hamiltonian must commute we may always find the solutions Φ common to both of them

$$H_{\rm intr}\,|\,\Phi(\Gamma';\gamma';a)\rangle = E^{\Gamma'\gamma'}_{\rm intr}\,|\,\Phi(\Gamma';\gamma';a)\rangle \tag{27}$$

and

$$\hat{C}_{3z}\ \Phi(\Gamma';\gamma',a) = \mathrm{e}^{-ia\,(2\pi)/3}\,\Phi(\Gamma';\gamma',a)\,, \tag{28}$$

where in this case the unique label $\Gamma' = \{1, -\}$ according to the notation introduced above and where as before $a = \pm 1$. Quantum number a defines the eigenvalues that can be interpreted as D_3 'signature', analogous to 'signature' quantum number introduced usually in relation to D_2 symmetry group of a triaxial nucleus.

It is straightforward to show that the collective wave-functions of Eq. (26) are eigenfunctions of the $\hat{C}_{3z}$ operator to the same eigenvalues as those in Eq. (28). Consequently the adequacy between the basis vectors in the intrinsic and collective spaces needed to apply Eq. (8) can again be established by equating the corresponding eigenvalues (*cf.* discussion towards the end of the preceding section).

4. Summary

A reformulation, Eq. (2), of the condition proposed in [2] allows to find the basis in a subspace of the full redundant-variable space $\mathcal{K}\otimes\tilde{\mathcal{K}}$ in which Eq. (2) is automatically fulfilled. For multidimensional representations of the symmetry group this condition leads to a coupling, Eqs (8), among the collective and intrinsic basis states involving sums over the additional quantum numbers required for unique labeling of the basis states of the corresponding *multidimensional irreducible representations*. This space consists of only the non-degenerate eigenstates of the total Hamiltonian.

The formulation discussed here, based on the group-theoretical structures, offers several mathematical advantages (precision, simplifications in

the large scale calculations and algorithms) as well as the physical ones, in particular, allowing to work directly with the conserved quantum numbers and exploiting directly the new symmetries. A discussion of some particularly intriguing ones can be found in these proceedings, Ref. [8].

Condition (2) can be generalized to include also non scalar representations as they may also be given valid physical interpretation. These representations are required to get a full spectrum of nuclear Hamiltonian, but the related approach needs further investigation.

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