POISSON–LIE STRUCTURES ON THE GALILEI GROUP

Y. Brihaye

Department of Mathematical Physics, University of Mons
Av. Maistriau, 7000 Mons, Belgium

E. Kowalczyk[†] and P. Maślanka[‡]

Department of Theoretical Physics II, University of Łódź
Pomorska 149/153, 90-236 Łódź, Poland

(Received September 25, 2002)

The complete list of Poisson–Lie structures on the Galilei group acting in 4-dimensional space–time is presented.

PACS numbers: 02.20.Qs, 02.40.Hw, 02.40.Vh

1. Introduction

Quantum groups have emerged in physics in connection with an attempt to understand the symmetries underlying exact solvability of certain quantum–mechanical and statistical models; they appeared to be quite powerful in this respect.

Therefore, it is natural to ask whether their range of applicability as a mathematical tool for describing physical symmetries is wider and covers, in particular, the most important case of space–time symmetries. The theory of Hopf algebras offers a variety of structures which can be viewed as deformations of classical space–time symmetry groups. For example, a number of deformed Poincaré groups were considered [1]. They possess many attractive features. However, if one is going to take seriously the very idea of quantum space time symmetries, many conceptual problems arise the solution of which seems to be quite difficult. It is rather obvious that one should concentrate on deformations of relativistic symmetries because it is a high energy/small distance region where the deviations from the predictions of

[†] Supported by Łódź University grant.
[‡] Supported by the Polish State Committee for Scientific Research (KBN) grant no. 5P03B05620.

"classical" theory should occur. However, as far as the conceptual problems are concerned, the nonrelativistic region provides a similar challenge. On the other hand it seems that the study of nonrelativistic deformed symmetries is slightly simpler. One is not here faced with some complications typical of relativistic quantum theory as, for example, the nonexistence of particle number conserving nontrivial interactions.

In the present paper as a first step towards the understanding of nonrelativistic quantum space–time symmetries we classify all nonequivalent Poisson–Lie structures on the Galilei group acting in 4-dimensional space–time. The method we use (*cf.* [2]) is based on solving directly the co-cycle condition; it has been already used for finding Poisson–Lie structures on the Galilei group of 2-dimensional space–time [3,4].

We find families of structures which cannot be related to each other by the automorphisms of the Galilei group; contrary to the case of the Poincaré group [5] many of them are not of the co-boundary type.

The paper is organized as follows. In Sec. 2 we sketch a general strategy while the results are presented in Sec. 3. All technical details are relegated to a number of appendices.

Let us conclude the introduction with some details concerning the Galilei group [6].

The generic element g of the ten-parameter Galilei group G is denoted by

$$g = (t, \vec{a}, \vec{v}, R)\,. \tag{1}$$

We shall denote by

$$\begin{aligned} T &= \{(t, \vec{0}, \vec{0}, I)\}\,, \\ S &= \{(0, \vec{a}, \vec{0}, I)\}\,, \\ V &= \{(0, \vec{0}, \vec{v}, I)\}\,, \\ \mathcal{R} &= \{(0, \vec{0}, \vec{0}, R)\}\,. \end{aligned}$$

The subgroups of time translations, space-translations, pure Galilei transformations (boosts), rotations, respectively.

The group law is expressed by

$$\begin{aligned} g'' &= g \cdot g' = (t, \vec{a}, \vec{v}, R)(t', \vec{a'}, \vec{v'}, R') \\ &= (t' + t, \vec{a} + R\vec{a'} + \vec{v}t', \vec{v} + Rv', RR')\,. \end{aligned} \tag{2}$$

The identity for the group is

$$e = (0, \vec{0}, \vec{0}, I) \tag{3}$$

and the inverse of the generic element is given by

$$g^{-1} = (t, \vec{a}, \vec{v}, R)^{-1} = (-t, -R^{-1}(\vec{a} - t\vec{v}), -R^{-1}\vec{v}, R^{-1})\,. \tag{4}$$

The generators $H, \vec{P}, \vec{K}$ and $\vec{J}$ of the Galilei Lie algebra $\mathcal{G}$ are defined with the help of the exponential parametrization

$$g \;=\; \mathrm{e}^{-itH}\,\mathrm{e}^{i\vec{a}\vec{P}}\,\mathrm{e}^{i\vec{v}\vec{K}}\,\mathrm{e}^{i\vec{\theta}\vec{J}} \tag{5}$$

and they obey the following commutation rules (only the non vanishing ones are written up)

$$\begin{aligned}
[J_i, J_j] &= i\varepsilon_{ijk} J_k\,,\\
[J_i, K_j] &= i\varepsilon_{ijk} K_k\,,\\
[J_i, P_j] &= i\varepsilon_{ijk} P_k\,,\\
[K_i, H] &= iP_i\,,
\end{aligned} \tag{6}$$

here and in the sequel the summation over repeated indices is understood $(i, j, k = 1, 2, 3)$.

The automorphism group of G consists of the inner automorphisms together with two outer ones generated by space and time dilations.

$$\begin{aligned}
(t, \vec{a}, \vec{v}, R) &\;\to\; (t, a\vec{a}, a\vec{v}, R)\,,\\
(t, \vec{a}, \vec{v}, R) &\;\to\; (bt, \vec{a}, b^{-1}\vec{v}, R)\,.
\end{aligned} \tag{7}$$

In what follows we shall need the right invariant vector fields on G. Denoting by $\hat{X}$ the right invariant vector field corresponding to the element X of Lie algebra $\mathcal{G}$ we have

$$\begin{aligned}
\hat{H} &= -i\frac{\partial}{\partial t}\,,\\
\hat{P}_i &= i\frac{\partial}{\partial a_i}\,,\\
\hat{K}_i &= i\left(t\frac{\partial}{\partial a_i} + \frac{\partial}{\partial v_i}\right)\,,\\
\hat{J}_i &= -i\varepsilon_{ijk} a_j \frac{\partial}{\partial a_k} - i\varepsilon_{ijk} v_j \frac{\partial}{\partial v_k} - i\varepsilon_{ijk} R_{jl}\frac{\partial}{\partial R_{kl}}\,.
\end{aligned} \tag{8}$$

2. Poisson–Lie structures on the Galilei group — the general strategy

Let us recall the notion of Poisson–Lie group [7, 8]. It is a Lie group $\tilde{G}$ which has a Poisson structure $\{\ ,\ \}$ such that the multiplication map $m : \tilde{G} \times \tilde{G} \to \tilde{G}$ is a Poisson map (of course $\tilde{G} \times \tilde{G}$ is given the product Poisson structure).

Poisson–Lie structures can be described explicitly as follows. Let $\tilde{\mathcal{G}}$ be the Lie algebra of $\tilde{G}$; denote by $\{X_i\}$ an arbitrary basis in $\tilde{\mathcal{G}}$ and let c_{ij}^k be the corresponding structure constants. One defines a mapping $\eta : \tilde{G} \to \wedge^2\tilde{\mathcal{G}}$,

$$\eta(g) \equiv \eta^{ij}(g) X_i \otimes X_j \,, \qquad \eta^{ij}(g) = -\eta^{ji}(g) \,. \tag{9}$$

Let $\{X_i^{\mathrm{R}}\}$ be the realization of $\tilde{\mathcal{G}}$ in terms of right invariant vector fields on $\tilde{G}$. The Poisson bracket on $\tilde{G}$ given by

$$\{\Phi, \Psi\} \;=\; \eta^{ij}(X_i^{\mathrm{R}}\Phi)(X_j^{\mathrm{R}}\Psi) \tag{10}$$

defines the Poisson–Lie structure on $\tilde{G}$ provided the following conditions are obeyed
(i) Poisson–Lie property (co-cycle condition)

$$\eta(g'g) \;=\; \eta(g') + Ad(g')\eta(g) \,, \tag{11}$$

(ii) Jacobi identity

$$\eta^{il}X_l^{\mathrm{R}}\eta^{jk} + \eta^{kl}X_l^{\mathrm{R}}\eta^{ij} + \eta^{jl}X_l^{\mathrm{R}}\eta^{ki} - c_{lp}^j\eta^{il}\eta^{pk} - c_{lp}^i\eta^{kl}\eta^{pj} - c_{lp}^k\eta^{jl}\eta^{pi} = 0 \,. \tag{12}$$

The inverse is also true: any Poisson–Lie structure on $\tilde{G}$ can be written in the above form. The infinitesimal analogues of Poisson–Lie groups are Lie bialgebras. For any $X \in \tilde{\mathcal{G}}$ define

$$\delta(X) \;\equiv\; \frac{d}{dt}\eta\left(e^{itX}\right)\Big|_{t=0} \,. \tag{13}$$

Then it can be easily shown that $\delta : \tilde{\mathcal{G}} \to \wedge^2\tilde{\mathcal{G}}$ has the following properties which are the infinitesimal counterparts of *(i)* and *(ii)*:
(i') co-cycle condition

$$\delta([X,Y]) \;=\; [X \otimes I + I \otimes X, \delta(Y)] + [\delta(X), I \otimes Y + Y \otimes I] \,, \tag{14}$$

(ii') co-Jacobi identity

$$\sum_{\text{c.p.}} (\delta \otimes id) \circ \delta(X) \;=\; 0 \,, \tag{15}$$

where c.p. means the summation over cyclic permutation of the factors in $\tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}}$.

Every Poisson–Lie structure on $\tilde{G}$ defines a Lie bialgebra structure on $\tilde{\mathcal{G}}$. The inverse is true provided $\tilde{G}$ is connected and simple connected [9].

Two Piosson–Lie structures on $\tilde{G}$ will be called equivalent if there exists an automorphism of $\tilde{G}$ which is a Poisson map.

The main aim of the present paper is to classify, up to equivalence, all Poisson–Lie structures on the Galilei group G acting in 4-dimensional space–time. We adopt the following, rather straightforward, strategy.

First, we write out η in the form

$$\begin{aligned}\eta(g) &= \Psi_i(g)\hat{H}\wedge\hat{J}_i + \Phi_i(g)\hat{H}\wedge\hat{P}_i + \Gamma_i(g)\hat{H}\wedge\hat{K}_i + \Lambda_i(g)\varepsilon_{ijk}\hat{P}_j\wedge\hat{P}_k \\ &\quad + \Upsilon_{ij}(g)\hat{P}_i\wedge\hat{K}_j + \Sigma_{ij}(g)\hat{P}_i\wedge\hat{J}_j + \Xi_i(g)\varepsilon_{ijk}\hat{K}_j\wedge\hat{K}_k \\ &\quad + \Omega_{ij}(g)\hat{K}_i\wedge\hat{J}_j + \Pi_i(g)\varepsilon_{ijk}\hat{J}_j\wedge\hat{J}_k\,, \end{aligned} \tag{16}$$

where $g\equiv(t,\vec{a},\vec{v},R)$ is an arbitrary element of the Galilei group G.

Inserting the expansion (16) into the co-cycle condition (11) one obtains the following set of functional equations for the coefficients Ψ_i, Φ_i *etc.*

$$\begin{aligned}\Psi(gg') &= \Psi_i(g) + R_{il}\Psi_l(g')\,, \\ \Phi_i(gg') &= \Phi_i(g) + R_{il}\Phi_l(g') + \varepsilon_{ink}(v_nt-a_n)R_{kl}\Psi_l(g') - tR_{il}\Gamma_l(g')\,, \\ \Gamma_i(gg') &= \Gamma_i(g) + R_{il}\Gamma_l(g') - \varepsilon_{ink}v_nR_{kl}\Psi_l(g')\,, \\ \Lambda_i(gg') &= \Lambda_i(g) + R_{il}\Lambda_l(g') - \tfrac{1}{2}\varepsilon_{imn}v_mR_{nl}\Phi_l(g') \\ &\quad + \tfrac{1}{2}\,[(\vec{v}^2t - a_mv_m)\delta_{in} - v_n(v_it-a_i)]R_{nl}\Psi_l(g') \\ &\quad + \tfrac{1}{2}\,t\varepsilon_{imn}v_mR_{nl}\Gamma_l(g') - \tfrac{1}{2}\,tR_{il}\varepsilon_{lmn}\Upsilon_{mn}(g') \\ &\quad + \tfrac{1}{2}\,[(tv_i-a_i)\delta_{mp} - R_{im}R_{np}(tv_n-a_n)] \\ &\quad \times(\Sigma_{pm}(g') - \tfrac{1}{2}\,t\Omega_{pm}(g')) + t^2R_{il}\Xi_l(g') \\ &\quad + (t^2v_pv_i - v_pa_it - v_ia_pt + a_pa_i)R_{pm}\Pi_m(g')\,, \\ \Upsilon_{ij}(gg') &= \Upsilon_{ij}(g) + R_{im}R_{jn}\Upsilon_{mn}(g') + v_i\varepsilon_{jnk}v_nR_{kl}\Psi_l(g') \\ &\quad - v_iR_{jl}\Gamma_l(g') - \varepsilon_{jnl}v_nR_{ip}R_{lk}\Sigma_{pk} - 2t\varepsilon_{ijs}R_{sl}\Xi_l(g') \\ &\quad + [\varepsilon_{inl}(a_n-v_nt)R_{jp} + \varepsilon_{jnl}v_ntR_{ip}]R_{lk}\Omega_{pk}(g') \\ &\quad + 2[\varepsilon_{ijn}v_n(a_p-v_pt)R_{pm} - \varepsilon_{njr}a_nv_rR_{im}]\Pi_m(g')\,, \\ \Sigma_{ij}(gg') &= \Sigma_{ij}(g) + R_{ip}R_{jk}\Sigma_{pk}(g') - v_iR_{jl}\Psi_l(g') - tR_{ip}R_{jk}\Omega_{pk}(g') \\ &\quad + 2[(a_j-tv_j)R_{im} - (a_l-tv_l)R_{lm}\delta_{ij}]\Pi_m(g')\,, \\ \Xi_i(gg') &= \Xi_i(g) + R_{in}\Xi_n(g') - \tfrac{1}{2}\,(v_i\delta_{pm} - R_{im}R_{np}v_n)\Omega_{pm}(g') \\ &\quad + v_iv_pR_{pm}\Pi_m(g')\,, \\ \Omega_{ij}(gg') &= \Omega_{ij}(g) + R_{ip}R_{jk}\Omega_{pk}(g') + 2(R_{im}v_j - R_{lm}v_l\delta_{ij})\Pi_m(g')\,, \\ \Pi_i(gg') &= \Pi(g) + R_{im}\Pi_m(g')\,. \end{aligned} \tag{17}$$

In spite of their complicated structure they can be solved in the following way (*cf.* [2–4]). One decomposes the general element $g=(t,\vec{a},\vec{v},R)$ into

the product of four elements belonging to the subgroups of time and space translations, boosts and rotations, (see Sec. 1).

$$(t,\vec{a},\vec{v},R) \;=\; (t,\vec{0},\vec{0},I)\cdot(0,\vec{a},\vec{0},I)\cdot(0,\vec{0},\vec{v},I)\cdot(0,\vec{0},\vec{0},R)\,. \qquad (18)$$

According to the condition *(i)* one can successively calculate $\eta(g)$ using the above decomposition provided the form of η for all four subgroups is known. In order to find the latter we specialize Eqs. (17) to those subgroups. The resulting equations can be easily solved; this is done in Appendix A. However, in obtaining the final form of η we apply Eq. (17) with some definite order of multiplication — for example using

$$(t,\vec{a},\vec{v},R) = (t,\vec{0},\vec{0},I)\cdot((0,\vec{a},0,I)((0,0,\vec{v},I)(0,\vec{0},\vec{0},R)))\,,$$

so there could be further constraints on parameters entering η following from associativity. Therefore, we reinsert η into Eq. (11) with arbitrary g and g' to find all missed relations between parameters. In this way we produce the general solution to Eq. (11) described in Appendix B (see Eq. (B.1)).

There remains to solve *(ii)* which imposes additional relations between coefficients of η. It is very tedious to try to solve Eq. (12) directly, so we adopt a different method. From our general form of η we calculate δ and impose *(ii')* which, in this context, is equivalent to *(ii)*. On the other hand it is well known that *(ii')* can be restated as the condition that the dual map δ^* defines Lie algebra structure on $\mathcal{G}^*$. Therefore, we first calculate δ and the commutators on $\mathcal{G}^*$ resulting from it and then solve the Jacobi identities. This is still a complicated problem but it can be simplified by using boost and translation automorphisms of Galilei group/algebra. Once this is done there remains only to use the residual automorphisms to put our solutions in canonical position. The more detailed discussion is given in Appendix C.

At the end, having the form of η (Eq. (16)) and the form of right invariant vector fields (Eq. (8)), using Eq. (10) and taking into account that for all families of solutions $\Pi_i(g)\equiv 0$ (see Eq. (B.1) and Sec. 3) one can easily calculate the following fundamental Poisson–Lie brackets.

$$\begin{aligned}
\{R_{ab},R_{cd}\} &= 0\,,\\
\{v_a,R_{bc}\} &= \varepsilon_{bjl}R_{lc}\Omega_{aj}\,,\\
\{a_a,R_{bc}\} &= \varepsilon_{bjl}R_{lc}(\Sigma_{aj}+t\Omega_{aj})\,,\\
\{t,R_{bc}\} &= -\varepsilon_{bjl}R_{lc}\Psi_j\,,\\
\{t,v_a\} &= \Gamma_a-\varepsilon_{ajl}v_l\Psi_j\,,\\
\{v_a,v_b\} &= -2\varepsilon_{abj}\Xi_j+\varepsilon_{bjl}v_l\Omega_{aj}-\varepsilon_{ajl}v_l\Omega_{bj}\,,\\
\{a_a,v_b\} &= -\Upsilon_{ab}+\varepsilon_{bjl}v_l\Sigma_{aj}-2t\varepsilon_{abj}\Xi_j+t\varepsilon_{bjl}v_l\Omega_{aj}-\varepsilon_{ajl}a_l\Omega_{bj}\,,
\end{aligned}$$

$$\begin{aligned}\{a_a, a_b\} &= -2\varepsilon_{abj}\Lambda_j - t\Upsilon_{ab} + t\Upsilon_{ba} - 2t^2\varepsilon_{abj}\Xi_j + \varepsilon_{bjl}a_l\Sigma_{aj} \\ &\quad -\varepsilon_{ajl}a_l\Sigma_{bj} + t\varepsilon_{bjl}a_l\Omega_{aj} - t\varepsilon_{ajl}a_l\Omega_{bj}\,, \\ \{t, a_a\} &= -\varepsilon_{ina}a_n\Psi_i + \Phi_a + \Gamma_a\,. \end{aligned} \tag{19}$$

3. Poisson–Lie structures on the Galilei group — the results

By applying the procedure outlined above we solve the relevant Jacobi identities for dual algebra (making use of boost and translation automorphisms) and we arrive at the following families of Poisson–Lie structures (for all cases $\vec{n} = 0$).

I $\vec{\alpha}$-arbitrary, $\beta \neq 0, \vec{\gamma} = 0, \vec{\phi} = 0, \nu = 0, \vec{\xi} = 0, \theta = 0, \rho = 0, \sigma_{ij} = 0,$
$\chi_{ij} = 0, \omega_{ij} = 0$
free parameters: $\vec{\alpha}, \beta \neq 0$

II $\vec{\alpha} \neq 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = F\vec{\alpha}, \vec{\lambda} = L\vec{\alpha}, \nu$–arbitrary, $\vec{\xi} = 0, \theta = 0, \rho = 0, \sigma_{ij} = 0,$
$\omega_{ij} = W(\vec{\alpha^2}\delta_{ij} - \alpha_i\alpha_j), \chi_{ij} = B(\alpha_i\alpha_j - \frac{1}{3}\vec{\alpha^2}\delta_{ij}) + 2Wv\varepsilon_{ijk}\alpha_k$
free parameters: $F, L, \nu, W \neq 0, B$

III $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = F\vec{\mu}, \vec{\lambda} = L\vec{\mu}, \nu = 0, \vec{\xi} = 0, \theta = 0, \rho = 0, \sigma_{ij} = 0,$
$\omega_{ij} = W(\delta_{ij} - \mu_i\mu_j), \chi_{ij} = B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ijk}\mu_k$
free parameters: $F \neq 0, L, B, C, W \neq 0, \vec{\mu}, ||\vec{\mu}|| = 1$

IV $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = 0, \vec{\lambda} = L\vec{\mu}, \nu = 0, \vec{\xi} = X\vec{\mu}, \theta$–arbitrary, $\rho = 0, \sigma_{ij} = 0,$
$\omega_{ij} = W(\delta_{ij} - \mu_i\mu_j), \chi_{ij} = B(\mu_i\mu_j - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ijk}\mu_k$
free parameters: $L, X, \theta, W \neq 0, B, C, \vec{\mu}, ||\vec{\mu}|| = 1$

V $\vec{\alpha} \neq 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = F\vec{\alpha}, \vec{\lambda} = L\vec{\alpha}, \nu$–arbitrary, $\vec{\xi} = 0, \theta = 0, \rho = 0,$
$\sigma_{ij} = 0, \omega_{ij} = 0, \chi_{ij} = B(\alpha_i\alpha_j - \frac{1}{3}\vec{\alpha^2}\delta_{ij})$
free parameters: $\vec{\alpha} \neq 0, F, L, \nu, B$

VI $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = 0, \nu = 0, \vec{\xi} = 0, \theta$–arbitrary, $\rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0,$
$\vec{\lambda}$ and χ_{ij}– arbitrary except that $\varepsilon_{abc}\chi_{ab}\lambda_c = 0$
free parameters: $\vec{\lambda}, \chi_{ab}, \theta$

VII $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \phi_i = F\varepsilon_{imn}\chi_{mn}, \vec{\lambda} = 0, \nu = 0, \vec{\xi} = 0, \theta = 0,$
$\rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0,$ χ_{ij}– arbitrary except that $\varepsilon_{imn}\chi_{mn} \neq 0$
free parameters: $F \neq 0, \chi_{mn}$

VIII $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \phi_i \neq F\varepsilon_{imn}\chi_{mn}$ and $\vec{\phi} \neq 0, \vec{\lambda} = L\vec{\phi}, \nu = 0, \vec{\xi} = 0,$
$\theta = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0,$ χ_{ij}– arbitrary
free parameters: $\vec{\phi} \neq 0, L, \chi_{ij}$

IX $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\vec{\phi}$–arbitrary, $\vec{\lambda}=0,\nu\neq 0,\vec{\xi}=0,\theta=0,\rho=0$, $\sigma_{ij}=0,\omega_{ij}=0,\chi_{ij}$–arbitrary
free parameters: $\vec{\phi},\nu\neq 0,\chi_{ij}$

X $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\vec{\phi}=v,\vec{\lambda}$–arbitrary, $\nu=0,\vec{\xi}\neq 0,\theta$–arbitrary, $\rho=0,\sigma_{ij}=0,\omega_{ij}=0,\chi_{ij}$–arbitrary except that $\varepsilon_{abc}\chi_{ab}\xi_c=0$
free parameters: $\vec{\lambda},\vec{\xi}\neq 0,\theta,\chi_{ij}$

XI $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\vec{\phi}=0,\vec{\lambda}=L\vec{\mu},\nu=0,\vec{\xi}=0,\theta$–arbitrary, $\rho=-\frac{1}{3}S,\sigma_{ij}=S(\mu_i\mu_j-\frac{1}{3}\delta_{ij}),\omega_{ij}=0,\chi_{ij}=B(\mu_i\mu_j-\frac{1}{3}\delta_{ij})$
free parameters: $S\neq 0,L,B,\theta,\vec{\mu},||\vec{\mu}||=1$

XII $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\vec{\phi}=0,\vec{\lambda}=0,\nu=0,\vec{\xi}=0,\theta$–arbitrary, $\rho=-\frac{1}{3}S,\sigma_{ij}=S(\mu_i\mu_j-\frac{1}{3}\delta_{ij}),\omega_{ij}=0,\chi_{ij}=B(\mu_i\mu_j-\frac{1}{3}\delta_{ij})+C\varepsilon_{ijk}\mu_k$
free parameters: $S\neq 0,B,C\neq 0,\theta,\vec{\mu},||\vec{\mu}||=1$

XIII $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\vec{\phi}=0,\vec{\lambda}=0,\nu\neq 0,\vec{\xi}=0,\theta=0$, $\rho=-\frac{1}{3}S,\sigma_{ij}=S(\mu_i\mu_j-\frac{1}{3}\delta_{ij}),\omega_{ij}=0,\chi_{ij}=B(\mu_i\mu_j-\frac{1}{3}\delta_{ij})+C\varepsilon_{ijk}\mu_k$
free parameters: $S\neq 0,\nu=0,B,C,\vec{\mu},||\vec{\mu}||=1$

XIV $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\vec{\phi}=0,\vec{\lambda}=L\vec{\mu},\nu=0,\vec{\xi}=X\vec{\mu},\theta$–arbitrary, $\rho=-\frac{1}{3}S,\sigma_{ij}=S(\mu_i\mu_j-\frac{1}{3}\delta_{ij}),\omega_{ij}=0,\chi_{ij}=B(\mu_i\mu_j-\frac{1}{3}\delta_{ij})$
free parameters: $S\neq 0,X\neq 0,L,\theta,B,\vec{\mu},||\vec{\mu}||=1$

XV $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\vec{\phi}=F\vec{\mu},\vec{\lambda}=0,\nu\neq 0,\vec{\xi}=0,\theta=0$, $\rho=-\frac{1}{3}S,\sigma_{ij}=S(\mu_i\mu_j-\frac{1}{3}\delta_{ij}),\omega_{ij}=0,\chi_{ij}=B(\mu_i\mu_j-\frac{1}{3}\delta_{ij})+C\varepsilon_{ijk}\mu_k$
free parameters: $S\neq 0,F\neq 0,\nu\neq 0,B,C,\vec{\mu},||\vec{\mu}||=1$

XVI $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\vec{\phi}=F\vec{\mu},\vec{\lambda}=0,\nu=0,\vec{\xi}=0,\theta=0$, $\rho=-\frac{1}{3}S,\sigma_{ij}=S(\mu_i\mu_j-\frac{1}{3}\delta_{ij}),\omega_{ij}=0,\chi_{ij}=B(\mu_i\mu_j-\frac{1}{3}\delta_{ij})+C\varepsilon_{ijk}\mu_k$
free parameters: $S\neq 0,F\neq 0,B,C,\vec{\mu},||\vec{\mu}||=1$

XVII $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\vec{\phi}=F\vec{\mu},\vec{\lambda}=L\vec{\mu},\nu=0,\vec{\xi}=0,\theta=0$, $\rho=-\frac{1}{3}S,\sigma_{ij}=S(\mu_i\mu_j-\frac{1}{3}\delta_{ij}),\omega_{ij}=0,\chi_{ij}=B(\mu_i\mu_j-\frac{1}{3}\delta_{ij})$
free parameters: $S\neq 0,F\neq 0,B,L,\vec{\mu},||\vec{\mu}||=1$

XVIII $\vec{\alpha}=0,\beta=0,\vec{\gamma}\neq 0,\vec{\phi}=0,\vec{\lambda}=L\vec{\gamma},\nu=0,\vec{\xi}=X\vec{\gamma},\theta=0$, $\rho=-\frac{1}{3}\vec{\gamma}^2\eta,\sigma_{ij}=-\varepsilon_{ijk}\gamma_k+\eta(\gamma_i\gamma_j-\frac{1}{3}\delta_{ij}\vec{\gamma}^2),\omega_{ij}=0,\chi_{ij}=\delta_{0\eta}\gamma_i w_j$, $\vec{w}$-arbitrary except that $\vec{w}\vec{\gamma}=0$
free parameters: $X,L,\eta,\vec{w}$.

Let us note that all Poisson–Lie structures with $\beta=0,\ \nu=0$ and $\theta=0$ are co-boundary and the corresponding r–matrix reads

$$
\begin{aligned}
r \;=\; & i\phi_k H \wedge P_k + i\gamma_k H \wedge K_k + i\alpha_k H \wedge J_k \\
& + i\varepsilon_{ijk}\lambda_k P_i \wedge P_j + i(\sigma_{ij} - \rho\delta_{ij}) P_i \wedge J_j \\
& + i\chi_{ij} P_i \wedge K_j - i(2\omega_{ij} - \omega_{nn}\delta_{ij}) J_i \wedge K_j \\
& + i\varepsilon_{ijk}\xi_k K_i \wedge K_j \,. \qquad (20)
\end{aligned}
$$

Now there remains only to classify the orbits under the action of residual automorphisms corresponding to the rotations and scaling and put our structure in the canonical form. This is a straightforward although very tedious task. The result can be summarized as follows. There are 69 families of inequivalent Poisson–Lie structures which have been grouped for convenience into eight groups. Each group is described by the appropriate tables which are given below. They provide the main result of our paper.

Let us note that for all groups $\vec{n} = 0$. In the last column (labeled by #) we indicate the number of essential parameters.

(I) $\quad \rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0, \chi_{ij} = 0, \vec{\gamma} = 0\,,$

TABLE I

N	$\vec{\alpha}$	β	$\vec{\phi}$	$\vec{\lambda}$	ν	$\vec{\xi}$	θ	#
1	$(0,0,\alpha)$	1	0	0	0	0	0	1
	$\alpha \geq 0$							
2	(0,0,1)	0	(0,0,1)	(0,0,L)	ν	0	0	2
3	(0,0,1)	0	0	(0,0,$\pm$1)	ν	0	0	1
4	(0,0,1)	0	0	0	1	0	0	0
5	(0,0,1)	0	0	0	0	0	0	0
6	0	0	(0,0,1)	(0,0,$\pm$1)	0	0	0	0
7	0	0	(0,0,1)	0	0	0	0	0
8	0	0	(0,0,1)	0	1	0	0	0
9	0	0	0	$\lambda_1 = 0$	0	(0,0,1)	θ	2
				$\lambda_2^2 + \lambda_3^2 = 1$				
10	0	0	0	(0,0,1)	0	0	$\pm$1	0
11	0	0	0	(0,0,1)	0	0	0	0
12	0	0	0	0	0	(0,0,1)	θ	1
13	0	0	0	0	0	0	$\pm$1	0
14	0	0	0	0	0	0	0	0

(II) $\beta = 0, \vec{\gamma} = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = W(\delta_{ij} - \delta_{i3}\delta_{j3})$,
$\chi_{ij} = B(\delta_{i3}\delta_{j3} - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ij3}$,

TABLE II

N	$\vec{\alpha}$	$\vec{\phi}$	$\vec{\lambda}$	ν	$\vec{\xi}$	θ	B	C	W	#
15	(0,0,1)	(0,0,F)	(0,0,L)	ν	0	0	B	$2v$	1	4
16	(0,0,1)	(0,0,F)	(0,0,L)	ν	0	0	1	0	0	3
17	0	(0,0,$\pm$1)	(0,0,L)	0	0	0	B	C	1	3
18	0	$\vec{\phi}$	0	1	0	0	0	1	0	3
19	0	0	(0,0,L)	0	(0,0,X)	θ	$2B^2+6C^2=3$		1	4
20	0	0	(0,0,$\pm$1)	0	(0,0,X)	θ	0	0	1	2
21	0	0	0	0	(0,0,X)	θ	0	0	1	2

(III) $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \rho = -\frac{1}{3}, \sigma_{ij} = (\delta_{i3}\delta_{j3} - \frac{1}{3}\delta_{ij}), \omega_{ij} = 0$,
$\chi_{ij} = B(\delta_{i3}\delta_{j3} - \frac{1}{3}\delta_{ij}) + C\varepsilon_{ij3}$,

TABLE III

N	$\vec{\phi}$	$\vec{\lambda}$	ν	$\vec{\xi}$	θ	B	C	#
22	(0,0,1)	(0,0,L)	0	0	0	B	0	2
23	(0,0,1)	0	$\nu \neq 0$	0	0	B	0	2
24	(0,0,1)	0	ν	0	0	B	$C \neq 0$	3
25	0	(0,0,L)	0	(0,0,$\pm$1)	θ	B	0	3
26	0	(0,0,L)	0	0	$\pm$1	B	0	2
27	0	(0,0,L)	0	0	0	1	0	1
28	0	(0,0,L)	0	0	0	0	0	1
29	0	0	1	0	0	B	C	2
30	0	0	0	0	θ	$2B^2+6C^2=3$ $C\neq 0$		2

(IV) $\vec{\alpha}=0,\beta=0,\vec{\phi}=0,\nu=0,,\theta=0,\rho=-\frac{1}{3},\omega_{ij}=0\,,$
$\chi_{ij}=\delta_{0\eta}\delta_{i3}w_j,w_3=0,\sigma_{ij}=-\varepsilon_{ij3}+\eta(\delta_{i3}\delta_{j3}-\frac{1}{3}\delta_{ij})\,,$

TABLE IV

N	$\vec{\gamma}$	$\vec{\lambda}$	$\vec{\xi}$	#
31	(0,0,1)	(0,0,L)	(0,0,$\pm$1)	1
32	(0,0,1)	(0,0,L)	0	1

(V) $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\nu=0,\rho=0,\sigma_{ij}=0,\omega_{ij}=0,\mathrm{tr}\chi=0,$
$\sum\limits_{ij}^{3}\chi_{ij}^2=1,\chi_{ij}=\mathrm{diag}(\chi_{11},\chi_{22},\chi_{33})\,,$

(Va) $\chi_{11}\neq\chi_{22}\neq\chi_{33}\,,$

TABLE Va

N	$\vec{\phi}$	$\vec{\lambda}$	$\vec{\xi}$	θ	#
33	0	$\|\|\vec{\lambda}\|\|=1$	0	θ	4
34	0	$\vec{\lambda}$	$\|\|\vec{\xi}\|\|=1$	θ	7
35	$\|\|\vec{\phi}\|\|=1$	$\vec{\lambda}=L\vec{\phi}$	0	0	4

(Vb) $\chi_{11}=\chi_{22}\neq\chi_{33}\,,$

TABLE Vb

N	$\vec{\phi}$	$\vec{\lambda}$	$\vec{\xi}$	θ	#
36	0	$\lambda_1=0$	0	θ	2
		$\lambda_2^2+\lambda_3^2=1$			
37	0	$\vec{\lambda}$	$\xi_1=0$	θ	5
			$\xi_2^2+\xi_3^2=1$		
38	$\phi_1=0$	$\vec{\lambda}=L\vec{\phi}$	0	0	2
	$\phi_2^2+\phi_3^2=1$				

(VI) $\vec{\alpha}=0,\beta=0,\vec{\gamma}=0,\nu=0,\rho=0,\sigma_{ij}=0,\omega_{ij}=0,tr\chi=0\,,$
$\sum\limits_{ij}^{3}\chi_{ij}^2=1,\chi_{32}=-\chi_{23}\neq0,\chi_{12}=\chi_{21},\chi_{13}=\chi_{32}\,,$

(VIa) $\chi_{22} \neq \chi_{33}$,

TABLE VIa

N	$\vec{\phi}$	$\vec{\lambda}$	$\vec{\xi}$	θ	#
39	0	$\lambda_1 = 0$	0	θ	6
		$\lambda_2^2 + \lambda_3^2 = 1$			
40	0	$\vec{\lambda}$	$\xi_1 = 0$	θ	9
			$\xi_2^2 + \xi_3^2 = 1$		
41	$\|\|\vec{\phi}\|\| = 1$	$\vec{\lambda} = L\vec{\phi}$	0	0	7
	$\phi_2^2 + \phi_3^2 = 1$				

(VIb) $\chi_{22} = \chi_{33}$,

TABLE VIb

N	$\vec{\phi}$	$\vec{\lambda}$	$\vec{\xi}$	θ	#
42	0	$\lambda_1 = \lambda_2 = 0$	0	θ	4
		$\lambda_2 = \pm 1$			
43	0	$\vec{\lambda}$	$\xi_1 = \xi_2 = 0$	θ	7
			$\xi_3 = \pm 1$		
44	(F,0,1)	$\vec{\lambda} = L\vec{\phi}$	0	0	5

(VII) $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\lambda} = 0, \vec{\xi} = 0, \rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0$,
$\mathrm{tr}\chi = 0, \sum_{ij}^{3} \chi_{ij}^2 = 1$,

(VIIa) $\chi_{11} \neq \chi_{22} \neq \chi_{33}, \chi_{13} = -\chi_{31}, \chi_{23} = -\chi_{32}, \chi_{12} = -\chi_{21}$,

TABLE VIIa

N	$\vec{\phi}$	ν	θ	#
45	0	0	$\pm 1, 0$	4
46	$\vec{\phi}$	1	0	7

(VIIb) $\chi_{11} = \chi_{22} \neq \chi_{33}, \chi_{13} = -\chi_{31} \neq 0, \chi_{23} = \chi_{32} = 0, \chi_{12} = -\chi_{21}$,

TABLE VIIb

ϕ	ν	θ	#	
47	0	0	$\pm 1, 0$	2
48	$\vec{\phi}$	1	0	5

(VIIc) $\chi_{11} = \chi_{22}, \chi_{13} = \chi_{31} = \chi_{23} = \chi_{32} = 0, \chi_{12} = -\chi_{21}$,

TABLE VIIc

N	$\vec{\phi}$	ν	θ	#
49	0	0	$\pm 1, 0$	1
50	$(0, \phi_2, \phi_3)$	1	0	3

(VIII) $\vec{\alpha} = 0, \beta = 0, \vec{\gamma} = 0, \vec{\phi} = (0, 0, 1), \vec{\lambda} = 0, \nu = 0, \vec{\xi} = 0, \theta = 0$,

$$\rho = 0, \sigma_{ij} = 0, \omega_{ij} = 0, tr\chi = 0, \sum_{ij}^{3} \chi_{ij}^2 = 1, \chi_{12} = -\chi_{21} = 1/F \, ,$$

(VIIIa) $\chi_{11} \neq \chi_{22}, \chi_{13} = \chi_{31}, \chi_{23} = \chi_{32}$,

(VIIIb) $\chi_{11} = \chi_{22}, \chi_{13} = \chi_{31} = 0, \chi_{23} = \chi_{32}$.

Now, inserting appropriate values of parameters (listed above) to Eq. (B.1) and using Eq. (19) one can easily calculate the fundamental Poisson brackets for all nonequivalent structures.

4. Summary

We have obtained all Lie–Poisson structures on the Galilei group acting in 4-dimensional space–time and classified them up to the equivalence implied by group automorphisms. The resulting set of structures appears to be quite rich; in particular, it includes many non–co-boundary structures, to be contrasted with the Poincare group case [5]. In spite of that, part of them can be surely obtained from those on Poincare group by a contraction procedure.

The next step to be done is to quantize the Lie–Poisson structures. In general, the consistent quantization is not an easy task. However, the preliminary study already done by us shows, that most of the cases described here are quantization friendly.

The problem of finding the behavior of the co-boundary Poisson structures of the Poincaré group under contraction to the Galilei group is under investigation.

Appendix A

The co-cycle condition for subgroups

Let us specify the Eqs. (17) for the subgroups of rotations, boosts, space and time translations. They read, respectively,

$$\begin{aligned}
\Psi_i(RR') &= \Psi_i(R) + R_{il}\Psi_l(R')\,,\\
\Phi_i(RR') &= \Phi_i(R) + R_{il}\Phi_l(R')\,,\\
\Gamma_i(RR') &= \Gamma_l(R) + R_{il}\Gamma_l(R')\,,\\
\Lambda_i(RR') &= \Lambda_i(R) + R_{il}\Lambda_l(R')\,,\\
\Upsilon_{ij}(RR') &= \Upsilon_{ij}(R) + R_{im}R_{jl}\Upsilon_{ml}(R')\,,\\
\Sigma_{ij}(RR') &= \Sigma_{ij}(R) + R_{im}R_{jl}\Sigma_{ml}(R')\,,\\
\Xi_i(RR') &= \Xi_i(R) + R_{im}\Xi_m(R')\,,\\
\Omega_{ij}(RR') &= \Omega_{ij}(R) + R_{im}R_{jl}\Omega_{ml}(R')\,,\\
\Pi_i(RR') &= \Pi_i(R) + R_{il}\Pi_l(R')\,.
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\Psi_i(\vec{v}+\vec{v'}) &= \Psi_i(\vec{v}) + \Psi_i(\vec{v'})\,,\\
\Phi_i(\vec{v}+\vec{v'}) &= \Phi_i(\vec{v}) + \Phi_i(\vec{v'})\,,\\
\Gamma_i(\vec{v}+\vec{v'}) &= \Gamma_i(\vec{v}) + \Gamma_i(\vec{v'}) - \varepsilon_{ink}v_n\Psi_k(\vec{v'})\,,\\
\Lambda_i(\vec{v}+\vec{v'}) &= \Lambda_i(\vec{v}) + \Lambda_i(\vec{v'}) - \tfrac{1}{2}\,\varepsilon_{imn}v_m\Phi_n(\vec{v'})\,,\\
\Upsilon_{ij}(\vec{v}+\vec{v'}) &= \Upsilon_{ij}(\vec{v}) + \varepsilon_{jnk}v_n\Psi_k(\vec{v'})v_i - \Gamma_j(\vec{v'})v_i - \varepsilon_{jnk}v_n\Sigma_{ik}(\vec{v'})\,,\\
\Sigma_{ij}(\vec{v}+\vec{v'}) &= \Sigma_{ij}(\vec{v}) + \Sigma_{ij}(\vec{v'}) - v_i\Psi_j(\vec{v'})\,,\\
\Xi_i(\vec{v}+\vec{v'}) &= \Xi_i(\vec{v}) + \Xi_i(\vec{v'}) - \tfrac{1}{2}\,(v_i\Omega_{mm}(\vec{v'}) - v_p\Omega_{pi}(\vec{v'})) + \Pi_m(\vec{v'})v_mv_i\,,\\
\Omega_{ij}(\vec{v}+\vec{v'}) &= \Omega_{ij}(\vec{v}) + \Omega_{ij}(\vec{v'}) + 2\Pi_i(\vec{v'})v_j - 2\Pi_m(\vec{v'})v_m\delta_{ij}\,,\\
\Pi_i(\vec{v}+\vec{v'}) &= \Pi_i(\vec{v}) + \Pi_i(\vec{v'})\,.
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\Psi_i(\vec{a}+\vec{a'}) &= \Psi_i(\vec{a}) + \Psi_i(\vec{a'})\,,\\
\Phi_i(\vec{a}+\vec{a'}) &= \Phi_i(\vec{a}) + \Phi_i(\vec{a'}) - \varepsilon_{ink}a_n\Psi_k(\vec{a'})\,,\\
\Gamma_i(\vec{a}+\vec{a'}) &= \Gamma_i(\vec{a}) + \Gamma_i(\vec{a'})\,,\\
\Lambda_i(\vec{a}+\vec{a'}) &= \Lambda_i(\vec{a}) + \Lambda_i(\vec{a'}) + \Pi_m(\vec{a'})a_ma_i - \tfrac{1}{2}\,(a_i\delta_{lm} - a_l\delta_{im})\Sigma_{lm}(\vec{a'})\,,\\
\Upsilon_{ij}(\vec{a}+\vec{a'}) &= \Upsilon_{ij}(\vec{a}) + \Upsilon_{ij}(\vec{a'}) + \varepsilon_{ink}a_n\Omega_{jk}(\vec{a'})\,,\\
\Sigma_{ij}(\vec{a}+\vec{a'}) &= \Sigma_{ij}(\vec{a}) + \Sigma_{ij}(\vec{a'}) + 2(\Pi_i(\vec{a'})a_j - \Pi_l(\vec{a'})a_l\delta_{ij})\,,
\end{aligned}$$

$$
\begin{aligned}
\Xi_i(\vec{a}+\vec{a'}) &= \Xi_i(a)+\Xi_i(a)\,,\\
\Omega_{ij}(\vec{a}+\vec{a'}) &= \Omega_{ij}(\vec{a})+\Omega_{ij}(\vec{a'})\,,\\
\Pi_i(\vec{a}+\vec{a'}) &= \Pi_i(\vec{a})+\Pi_i(\vec{a'})\,. \qquad \text{(A.3)}\\
\Psi_i(t+t') &= \Psi_i(t)+\Psi_i(t')\,,\\
\Phi_i(t+t') &= \Phi_i(t)+\Phi_i(t')-t\Gamma_i(t')\,,\\
\Gamma_i(t+t') &= \Gamma_i(t)+\Gamma_i(t')\,,\\
\Lambda_i(t+t') &= \Lambda_i(t)+\Lambda_i(t')-\tfrac{1}{2}\,t\varepsilon_{imn}\Upsilon_{mn}(t')+t^2\,\Xi_i(t')\,,\\
\Upsilon_{ij}(t+t') &= \Upsilon_{ij}(t)+\Upsilon_{ij}(t')-2t\varepsilon_{ijl}\Xi_l(t')\,,\\
\Sigma_{ij}(t+t') &= \Sigma_{ij}(t)+\Sigma_{ij}(t')-t\Omega_{ij}(t')\,,\\
\Xi_i(t+t') &= \Xi_i(t)+\Xi_i(t')\,,\\
\Omega_{ij}(t+t') &= \Omega_{ij}(t)+\Omega_{ij}(t')\,,\\
\Pi_i(t+t') &= \Pi_i(t)+\Pi_i(t')\,. \qquad \text{(A.4)}
\end{aligned}
$$

Note that all Eqs. (A.1) have the same structure

$$T_{i_1\ldots i_k}(RR') \;=\; T_{i_1\ldots i_k}(R)+R_{i_1j_1}\ldots R_{i_kj_k}T_{j_1\ldots j_k}(R')\,. \qquad \text{(A.5)}$$

They can be solved by integrating over R' with respec to Haar measure on SO(3)

$$T_{i_1\ldots i_k}(R) \;=\; (R_{i_1j_1}\ldots R_{i_kj_k}-\delta_{i_1\ldots j_1}\ldots\delta_{i_kj_k})c_{j_1\ldots j_k}\,. \qquad \text{(A.6)}$$

This result agrees with the general theorem stating that all semi-simple Lie groups are coboundaries. On the other hand it follows immediately from Eqs. (A.2)–(A.4) that all functions entering there are polynomials in the relevant parameters. This allows us to write out explicitly the general solutions.

$$
\begin{aligned}
\Psi_i(\vec{v}) &= 0\,,\\
\Phi_i(\vec{v}) &= 0\,,\\
\Gamma_i(\vec{v}) &= a_{ij}v_j\,,\\
\Lambda_i(\vec{v}) &= b_{ij}v_j\\
\Upsilon_{ij}(\vec{v}) &= c_{ijk}v_k-\tfrac{1}{2}\,(\delta_{ik}a_{jl}+\delta_{ji}a_{lk}-a_{lj}\delta_{ki}+\tfrac{1}{2}\,\varepsilon_{jkl}\varepsilon_{inm}a_{nm})v_kv_l\,,\\
\Sigma_{ij}(\vec{v}) &= (\tfrac{1}{2}\,\delta_{jk}\varepsilon_{inm}a_{nm}-\varepsilon_{ijn}a_{kn})v_k\,,\\
\Xi_i(\vec{v}) &= d_{ik}v_k+\left(\tfrac{1}{4}e_{jik}-\tfrac{1}{8}(e_j\delta_{ik}+e_k\delta_{ij})\right)v_jv_k\,,\\
\Omega_{ij}(\vec{v}) &= (e_{ijk}+\tfrac{1}{2}\,(e_k\delta_{ji}-e_i\delta_{jk}))v_k\,,\\
\Pi_i(v) &= 0\,, \qquad \text{(A.7)}
\end{aligned}
$$

where $e_{ijk}=e_{kji},\ e_{iik}=0$;

$$
\begin{aligned}
\Psi_i(\vec{a}) &= 0\,,\\
\Phi_i(\vec{a}) &= f_{ij}a_j\,,\\
\Gamma_i(\vec{a}) &= g_{ij}a_j\,,\\
\Lambda_i(\vec{a}) &= h_{ij}a_j + \tfrac{1}{4}(h_{jik} - \tfrac{1}{2}\,(h_j\delta_{ik} + h_k\delta_{ij}))a_j a_k\,,\\
\Upsilon_{ij}(\vec{a}) &= k_{ijk}a_k\\
\Sigma_{ij}(\vec{a}) &= (h_{ijk} + \tfrac{1}{2}\,(h_k\delta_{ij} - h_i\delta_{jk}))a_k\,,\\
\Xi_i(\vec{a}) &= l_{ij}a_j\,,\\
\Omega_{ij}(\vec{a}) &= 0\,,\\
\Pi_i(\vec{a}) &= 0\,,
\end{aligned}
\tag{A.8}
$$

where $h_{ijk} = h_{kji},\ h_{iik} = 0$;

$$
\begin{aligned}
\Psi(t) &= p_i t\,,\\
\Phi_i(t) &= r_i t - \tfrac{1}{2}\,s_i t^2\,,\\
\Gamma_i(t) &= s_i t\,,\\
\Lambda_i(t) &= u_i t - \tfrac{1}{4}\varepsilon_{imn}x_{mn}t^2 + \tfrac{1}{3}w_i t^3\,,\\
\Upsilon_{ij}(t) &= x_{ij}t - \varepsilon_{ijk}w_k t^2\\
\Sigma_{ij}(t) &= y_{ij}t - \tfrac{1}{2}\,z_{ij}t^2\,,\\
\Xi_i(t) &= w_i t\,,\\
\Omega_{ij}(t) &= z_{ij}t\,,\\
\Pi_i(t) &= m_i t\,.
\end{aligned}
\tag{A.9}
$$

Appendix B

The general solution to the cocycle condition

According to the procedure outlined in Sec. 2 we use the cocycle condition *(i)* together with the decomposition (18) and the expressions written out in Appendix A to produce the Ansatz for $\eta(g)$. Inserting it back into (17) we find the general solution for η of the form.

$$
\begin{aligned}
\Psi_i(g) &= (R_{ij} - \delta_{ij})\alpha_j\,,\\
\Phi_i(g) &= (R_{ij} - \delta_{ij})\phi_j + \beta(a_i - v_i t) - \gamma_j R_{ij}t + \varepsilon_{ijk}\alpha_l R_{jl}(a_k - v_k t)\,,\\
\Gamma_i(g) &= (R_{ij} - \delta_{ij})\gamma_j + \beta v_i + \varepsilon_{ijk}\alpha_l R_{jl}v_k\,,\\
\Lambda_i(g) &= (R_{ij} - \delta_{ij})\lambda_j + (\rho - \tfrac{1}{2}\,\sigma_{nn})(a_i - v_i t) + \tfrac{1}{2}\,\beta\varepsilon_{ijk}a_j v_k\\
&\quad + \tfrac{1}{2}\,\varepsilon_{ijk}\phi_l R_{jl}v_k - \tfrac{1}{2}\,\alpha_j R_{ij}(a_k v_k - \vec{v}^2 t)
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{2}\,\alpha_j R_{kj} v_k (a_i - v_i t) + \xi_j R_{ij} t^2 - \tfrac{1}{2}\,\varepsilon_{ijk}\gamma_l R_{jl} v_k t \\
&-\tfrac{1}{2}\,\varepsilon_{jkl}\chi_{kl} R_{ij} t - \omega_{jl} R_{ij} R_{kl} a_k t + \nu v_i + \omega_{lk} R_{nk} R_{il} v_n t^2 \\
&+\tfrac{1}{2}\,\sigma_{ij} R_{ij} R_{kl} (a_k - v_k t) + n_k R_{mk} (a_i a_m + 2 v_m v_i t) \\
&-n_l (a_k R_{kl} v_i + a_i R_{kl} v_k) t\,, \\
\Upsilon_{ij}(g) \;=\; & (R_{ik} R_{jl} - \delta_{ik}\delta_{jl})\chi_{kl} + \delta_{ij}\theta t - \tfrac{1}{2}\,\delta_{ij}\beta \vec{v}^2 \\
&+\varepsilon_{jkl}\alpha_n R_{ln} v_k v_i - \gamma_k R_{jk} v_i - 2\varepsilon_{ijk}\xi_l R_{kl} t \\
&-\varepsilon_{jkl}\sigma_{ns} R_{in} R_{ls} v_k + \rho\varepsilon_{ijk} v_k + \omega_{nn}\varepsilon_{ijk} a_k \\
&+2\omega_{ns}(\varepsilon_{ikl} R_{js} R_{ln} a_k - \varepsilon_{ijl} R_{ks} R_{ln} v_k t) \\
&+2n_l (R_{jl} v_k - R_{ml} v_m \delta_{ik})\varepsilon_{kin} a_n - 2 n_k v_s v_m R_{mk} \varepsilon_{sij} t\,, \\
\Sigma_{ij}(g) \;=\; & (R_{ik} R_{jl} - \delta_{ik}\delta_{jl})\sigma_{kl} - \beta\varepsilon_{ijk} v_k - \alpha_k R_{jk} v_i \\
&-2\omega_{lk} R_{ik} R_{jl} t + \omega_{nn}\delta_{ij} t \\
&+2n_k (R_{ik} a_j - R_{mk} a_m \delta_{ij} - R_{ik} v_j t + R_{mk} v_m \delta_{ij} t)\,, \\
\Xi_i(g) \;=\; & (R_{ij} - \delta_{ij})\xi_j + \omega_{jl} R_{ij} R_{kl} v_k + n_k R_{mk} v_m v_i\,, \\
\Omega_{ij}(g) \;=\; & 2(R_{ik} R_{jl} - \delta_{ik}\delta_{jl})\omega_{lk} + 2 n_k (R_{ik} v_j - R_{mk} v_m \delta_{ij})\,, \\
\Pi_i(g) \;=\; & (R_{ij} - \delta_{ij}) n_j\,.
\end{aligned}
\tag{B.1}
$$

Appendix C

Jacobi identities

Using the general form of η described in Appendix B we find from Eq. (13):

$$
\begin{aligned}
\delta(H) \;=\; & \gamma_i H \wedge P_i + \tfrac{1}{2}\,(\chi_{kj} - \chi_{jk}) P_j \wedge P_k + (2\varepsilon_{ijk}\xi_k - \theta\delta_{ij}) P_i \wedge K_j \\
&+(2\omega_{ji} - \omega_{nn}\delta_{ji}) P_i \wedge J_j\,, \\
\delta(P_s) \;=\; & (\beta\delta_{is} + \varepsilon_{iks}\alpha_k) H \wedge P_i + \varepsilon_{ijk}(\rho\delta_{is} - \tfrac{1}{2}\,\sigma_{nn}\delta_{is} + \tfrac{1}{2}\,\sigma_{si}) P_j \wedge P_k \\
&+(2\varepsilon_{lis}\omega_{lj} - \varepsilon_{jis}\omega_{nn}) P_i \wedge K_j \\
&+2(n_i\delta_{sj} - n_s\delta_{ij}) P_i \wedge J_j\,, \\
\delta(K_s) \;=\; & (\beta\delta_{is} + \varepsilon_{iks}\alpha_k) H \wedge P_i + (\nu\varepsilon_{sjk} - \tfrac{1}{2}\,(\phi_k\delta_{js} - \phi_j\delta_{ks})) P_j \wedge P_k \\
&+(\rho\varepsilon_{ijs} - \varepsilon_{kjs}\sigma_{ik} - \gamma_j\delta_{is}) P_i \wedge K_j \\
&-(\beta\varepsilon_{ijs} + \alpha_j\delta_{is}) P_i \wedge J_j + \varepsilon_{ijk}\omega_{is} K_j \wedge K_k \\
&+2(n_j\delta_{sk} - n_s\delta_{jk}) K_j \wedge J_k\,, \\
\delta(J_s) \;=\; & \varepsilon_{sij}\alpha_j H \wedge J_i + \varepsilon_{sij}\phi_j H \wedge P_i + \varepsilon_{ijs}\gamma_j H \wedge K_i \\
&+(\lambda_j\delta_{ks} - \lambda_k\delta_{js}) P_j \wedge P_k + (\varepsilon_{sik}\lambda_{kj} + \varepsilon_{sjk}\chi_{ik}) P_i \wedge K_j \\
&+(\varepsilon_{sik}\sigma_{kj} + \varepsilon_{sjk}\sigma_{ik}) P_i \wedge J_j + (\xi_i\delta_{ks} - \xi_k\delta_{js}) K_j \wedge K_k \\
&+2(\varepsilon_{sik}\omega_{jk} + \varepsilon_{sjk}\omega_{ki}) K_i \wedge J_j + 2 n_k J_k \wedge J_s\,.
\end{aligned}
\tag{C.1}
$$

Let $\tilde{X}_i$ denote the basis in $\mathcal{G}^*$ defined by $< \tilde{X}_i, X_j >= \delta_{ij}$. Then δ imposes the following commutator structure in $\mathcal{G}^*$.

$$
\begin{aligned}
[\tilde{H},\tilde{J}_k] &= \varepsilon_{ikl}\alpha_l\tilde{J}_i\,,\\
[\tilde{H},\tilde{P}_k] &= \gamma_k\tilde{H}+(\beta\delta_{ik}+\varepsilon_{ikl}\alpha_l)\tilde{P}_i+\varepsilon_{ikl}\phi_l\tilde{J}_i\,,\\
[\tilde{H},\tilde{K}_k] &= (\beta\delta_{ik}+\varepsilon_{ikl}\alpha_l)\tilde{K}_i+\varepsilon_{ikl}\gamma_l\tilde{J}_i\,,\\
[\tilde{K}_k,\tilde{J}_l] &= 2(n_k\delta_{li}-n_i\delta_{kl})\tilde{K}_i+2(\varepsilon_{ikn}\omega_{ln}+\varepsilon_{iln}\omega_{nk})\tilde{J}_i\,,\\
[\tilde{P}_l,\tilde{P}_m] &= (\chi_{lm}-\chi_{ml})\tilde{H}+2\varepsilon_{klm}[\rho\delta_{ki}+\tfrac{1}{2}\,(\sigma_{ik}-\sigma_{nn}\delta_{ik})]\tilde{P}_i\\
&\quad+2[\nu\varepsilon_{ilm}+\tfrac{1}{2}\,(\phi_l\delta_{im}-\phi_m\delta_{li})]\tilde{K}_i+2(\lambda_l\delta_{im}-\lambda_m\delta_{ik})\tilde{J}_i\,,\\
[\tilde{P}_k,\tilde{K}_l] &= (2\xi_n\varepsilon_{nkl}-\theta\delta_{kl})\tilde{H}+(2\varepsilon_{nki}\omega_{nl}-\omega_{nn}\varepsilon_{lki})\tilde{P}_i\\
&\quad+(\rho\varepsilon_{kli}-\varepsilon_{lin}\sigma_{kn}-\delta_{ki}\gamma_l)\tilde{K}_i+(\varepsilon_{ikn}\chi_{nl}+\varepsilon_{iln}\chi_{kn})\tilde{J}_i\,,\\
[\tilde{P}_k,\tilde{J}_l] &= (2\omega_{ik}-\omega_{nn}\delta_{lk})\tilde{H}+2(n_k\delta_{li}-n_i\delta_{kl})\tilde{P}_i\\
&\quad-(\beta\varepsilon_{kli}+\alpha_l\delta_{ki})\tilde{K}_i+(\varepsilon_{ikn}\sigma_{nl}+\varepsilon_{iln}\sigma_{kn})\tilde{J}_i\,,\\
[\tilde{K}_m,\tilde{K}_n] &= 2\varepsilon_{kmn}\omega_{ki}\tilde{K}_i+2(\xi_m\delta_{ni}-\xi_n\delta_{mi})\tilde{J}_i\,,\\
[\tilde{J}_k,\tilde{J}_l] &= 2(n_k\delta_{li}-n_l\delta_{ki})\tilde{J}_i\,.
\end{aligned}
\tag{C.2}
$$

Now we have to solve the Jacobi identities for the structure described in (C.2). This is still a complicated task so we apply a mixed procedure consisting in solving part of Jacobi identities directly and applying the group of automorphisms to simplify the remaining ones.

Therefore, we first give the action of the automorphism group on the parameters of δ. Let us start with inner automorphisms. They read

$$
\begin{aligned}
\vec{\alpha_k}=\vec{\alpha_i} \qquad & \gamma_k'=\gamma_k-\beta v_k+\varepsilon_{kil}v_i\alpha_l\,,\\
\beta'=\beta \qquad & \lambda_k'=\lambda_k-vv_k-\tfrac{1}{2}\,\varepsilon_{kil}\phi_i v_\alpha\,,\\
\vec{\phi}'=\vec{\phi} \qquad & \xi_k'=\xi_k-\omega_{kn}v_n-v_k(v_nn_n)\,,\\
\theta'=\theta \qquad & \rho'=\rho-\tfrac{1}{3}\alpha_n v_n\,,\\
\nu'=\nu \qquad & \omega_{ij}'=\omega_{ij}+v_in_j+(v_nn_n)\delta_{ij}\,,\\
\vec{n}'=\vec{n} \qquad & \sigma_{ia}'=\sigma_{ia}+\beta\varepsilon_{ial}v_l+\alpha_a v_i-\tfrac{1}{3}\alpha_n v_n\delta_{ai}\,,\\
& \chi_{ab}'=\chi_{ab}+v_a\gamma_b+\varepsilon_{bnm}v_n\alpha_m v_a-\tfrac{1}{2}\,\varepsilon_{abk}\sigma_{nk}v_n\\
& -\rho\varepsilon_{abn}v_n-\tfrac{1}{2}\,\varepsilon_{alm}\sigma_{bl}v_m-\tfrac{1}{2}\,\varepsilon_{blm}\sigma_{al}v_m\\
& -\tfrac{1}{3}(v_n\gamma_n+\varepsilon_{lnm}\sigma_{nl}v_m)\delta_{ab}\,,
\end{aligned}
\tag{C.3}
$$

for boosts,

$$
\begin{aligned}
&\vec{n}'=\vec{n},\ \ \vec{\alpha}'=\alpha,\ \ \beta'=\beta\,,\\
&\nu'=\nu,\ \ \vec{\xi}'=\vec{\xi},\ \ \theta'=\theta\,,
\end{aligned}
$$

$$
\begin{aligned}
\phi_i' &= \phi_i - \beta a_i - \varepsilon_{ink}\alpha_n a_k\,,\\
\gamma_k' &= \gamma_k - 2\varepsilon_{knm}a_m n_n\,,\\
\lambda_k' &= \lambda_k - \tfrac{1}{2}\,\sigma_{nk}a_n - \rho a_k - \vec{a^2}n_k - (n_i a_i)n_k\,,\\
\rho' &= \rho - \tfrac{4}{3}a_k n_k \sigma_{ab}' = \sigma_{ab} + \tfrac{2}{3}(a_k n_k)\delta_{ab} - 2n_a a_b\,,\\
\chi_{ab}' &= \chi_{ab} - \varepsilon_{abn}\omega_{nk}a_k + \varepsilon_{amk}\omega_{mb}a_k\\
&\quad + \varepsilon_{bmk}\omega_{ma}a_k + \tfrac{2}{3}\varepsilon_{mnk}\omega_{mn}a_k\delta_{ab}\,,\\
\omega_{ab}' &= \omega_{ab}\,,
\end{aligned}
\tag{C.4}
$$

for space translations and

$$
\begin{aligned}
&\vec{n'} = \vec{n},\ \vec{\alpha} = \alpha,\ \beta' = \beta, \vec{\gamma'} = \vec{\gamma}\,,\\
&\nu' = \nu,\ \vec{\xi'} = \vec{\xi},\ \theta' = \theta, \omega_{ij}' = \omega_{ij}\,,
\end{aligned}
$$

$$
\begin{aligned}
\vec{\phi'} &= \vec{\phi} + t\vec{\gamma}\,, & \lambda_i' &= \lambda_i - \tfrac{1}{2}\,t\varepsilon_{imn}\chi_{nm} + t^2\xi_i\,,\\
\rho' &= \rho + \tfrac{1}{3}t\omega_{nn}\,, & \sigma_{ab}' &= \sigma_{ab} + 2t\omega_{ba} - \tfrac{2}{3}t\omega_{nn}\delta_{ab}\,,\\
\chi_{ab}' &= \chi_{ab} + 2t\varepsilon_{abk}\xi_k\,,
\end{aligned}
$$

for time translations, respectively.

Under the rotations the parameters transform as tensors of appropriate rank.

Besides, there are two outer automorphisms, which correspond to rescaling of space and time unit, ($\vec{a} \to a\vec{a},\ t \to bt$). They read

$$
\begin{aligned}
n' = \vec{n}, &\quad \alpha' = \tfrac{1}{b}\vec{\alpha}, &&\quad \beta' = \frac{1}{b}\beta\,,\\
\vec{\gamma} = \frac{1}{a}\gamma, &\quad \vec{\phi} = \tfrac{1}{ab}\vec{\phi}, &&\quad \vec{\lambda} = \frac{1}{a^2}\vec{\lambda}\,,\\
\nu' = \frac{1}{ab}\nu, &\quad \vec{\xi} = \tfrac{b^2}{a^2}\vec{\xi}, &&\quad \theta' = \frac{b^2}{a^2}\theta\,,\\
\rho' = \frac{1}{a}\rho, &\quad \sigma_{ab}' = \tfrac{1}{a}\sigma_{ab}, &&\quad \chi_{ab}' = \frac{b}{a^2}\chi_{ab}\,,\\
&\quad \omega_{ab}' = \tfrac{b}{a}\omega_{ab}\,.
\end{aligned}
\tag{C.5}
$$

We shall not enter into all details. Let us rather give a sketch of the procedure.

By solving the Jacobi identities for the subalgebra generated by $\tilde{H}, \tilde{J}_l, \tilde{K}_n$ we find the following six families of constraints on parameters $\vec{n}, \beta, \vec{\alpha}, \omega_{ij}, \vec{\gamma}$, and $\vec{\xi}$.

(a) $\vec{n}=0, \vec{\alpha}$–arbitrary, $\beta \neq 0, \vec{\gamma}$–arbitrary, $\vec{\xi}=0, \omega_{ij}=0$,

(b) $\vec{n}=0, \vec{\alpha} \neq 0, \beta=0, \vec{\gamma}$–arbitrary, $\vec{\xi}=\xi\vec{\alpha}+W(\vec{\alpha}\times\vec{\gamma}), \omega_{ij}$ $=W(\vec{\alpha^2}\delta_{ij}-\alpha_i\alpha_j), W\neq 0$,

(c) $\vec{n}=0, \vec{\alpha}=0, \beta=0, \vec{\xi}$–arbitrary, $\omega_{ij}=W(\delta_{ij}-\mu_i\mu_j)+V\varepsilon_{ijk}\mu_k$, $||\vec{\mu}||=1$,

$$\vec{\gamma}=\begin{cases} \gamma\vec{\mu} & \text{if } |W|+|V|\neq 0, \\ \text{arbitrary} & \text{if } W=V=0, \end{cases}$$

(d) $\vec{n}=0, \vec{\alpha}\neq 0, \beta=0, \omega_{ij}=W\delta_{ij}, \vec{\gamma}=\frac{1}{W}(\vec{\alpha}\times\vec{\xi}), W\neq 0, \vec{\xi}$–arbitrary,

(e) $\vec{n}=0, \vec{\alpha}\neq 0, \beta=0, \omega_{ij}=0, \vec{\xi}||\vec{\alpha}, \vec{\gamma}\perp\vec{\alpha}$,

(f) $\vec{n}=0, \vec{\alpha}=0, \beta=0, \omega_{ij}=W\delta_{ij}, \vec{\gamma}=0, \vec{\xi}$–arbitrary.

Now we have to solve the remaining Jacobi identities. First of all let us note that there are ambiguities in determining the matrices σ_{ij} and χ_{ij}. Namely, both can be redefined by adding arbitrary multiples of the unit matrix. In order to remove this ambiguity we put $\mathrm{tr}\sigma=\mathrm{tr}\chi=0$. Now we use the automorphisms generated by boosts, space and time translations to simplify the Jacobi identities. For example in the cases (a),(d) and (e) we may use the boost to put $\vec{\gamma}=0$ from the very beginning. On the other hand in the case (b) by solving the Jacobi identity for $\tilde{H}, \tilde{P}_i, \tilde{P}_k$ we find $\vec{\gamma}\vec{\alpha}=0$ and again the boost can be used to put $\vec{\gamma}=0$. Proceeding in this way we obtain the eighteen families of solutions described in Sec. 3.

The authors thank Prof. P. Kosiński and Prof. S. Giller for many helpful discussions.

REFERENCES

[1] P. Podleś, S.L. Woronowicz, *Commun. Math. Phys.* **178**, 61 (1996).
[2] J. Sobczyk, *J. Phys.* **A29**, 2887 (1996).
[3] E. Kowalczyk, *Acta. Phys. Pol. B* **28**, 1893 (1997).
[4] A. Opanowicz, *J. Phys.* **A31**, 8387 (1998).
[5] S. Zakrzewski, *Commun. Math. Phys.* **185**, 285 (1997).

[6] J. Levy-Leblond, *J. Math. Phys.* **4**, 776 (1961).

[7] V. Chari, A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press (1994).

[8] L.A. Takhatajan, *Lectures on Quantum Groups*, in *Introduction to Quantum Groups and Integrable Massive Models of Quantum Field Theory* Eds. M.L. Ge and B.H. Zhao, p. 69, World Scientific, Singapore 1990.

[9] V.G. Drinfeld, *Quantum Groups* in Proc. ICM, Berkeley 1986, vol. 1. p. 789.