MATHISSON'S SPINNING ELECTRON: NONCOMMUTATIVE MECHANICS AND EXOTIC GALILEAN SYMMETRY, 66 YEARS AGO

P.A. Horváthy

Laboratoire de Mathématiques et de Physique Théorique
Université de Tours
Parc de Grandmont, 37 200 Tours, France

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The acceleration-dependent system with noncommuting coordinates, proposed by Lukierski, Stichel and Zakrzewski [*Ann. Phys.* **260**, 224 (1997)] is derived as the non-relativistic limit of Mathisson's classical electron [*Acta Phys. Pol.* **6**, 218 (1937)], further discussed by Weyssenhoff and Raabe [*Acta Phys. Pol.* **9**, 7 (1947)]. The two-parameter centrally extended Galilean symmetry of the model is recovered using elementary methods. The relation to Schrödinger's Zitternde Elektron is indicated.

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1. Introduction

Non-commutative (quantum) mechanics, where the position coordinates satisfy

$$\big\{X_1, X_2\big\} = \theta\,, \tag{1.1}$$

has been at the center of recent interest [1]. In the plane and in the non-relativistic context, such theories are closely related to the "exotic" Galilean symmetry associated with the two-fold central extension of the planar Galilei group [2]. A model which provides a physical realization of this symmetry has been presented by Lukierski, Stichel and Zakrzewski [3] who considered the acceleration-dependent Lagrangian

$$L = \frac{m\dot{\vec{x}}^2}{2} + \frac{\kappa}{2}\,\dot{\vec{x}} \times \ddot{\vec{x}}\,. \tag{1.2}$$

My aim here is to point out that the model of Lukierski *et al.* can actually be derived from that published by Mathisson in '37 [4], and further discussed by Weyssenhoff and Raabe [5]. Not surprisingly, their theory shows interesting analogies also with Schrödinger's *Zitternde Elektron* [6].

This Note is dedicated to the memory of these outstanding physicists who, with an extreme courage, tried to continue their scientific activity under those terrible years of World War II.

2. The Mathisson electron

Two years before the outbreak of World War II, Mathisson [4] proposed to describe a classical electron with the relativistic equations

$$\begin{aligned} m\dot{u}^{\alpha} &+ \frac{1}{c^2}S^{\alpha\sigma}\ddot{u}_{\sigma} = f^{\alpha}\,, \\ \dot{S}^{\alpha\beta} &- \frac{1}{c^2}S^{\alpha\sigma}\dot{u}_{\sigma}u^{\beta} + \frac{1}{c^2}S^{\beta\sigma}\dot{u}_{\sigma}u^{\alpha} = 0\,, \end{aligned} \tag{2.1}$$

where m is the mass, u^{α} the four-velocity, f^{α} the force; the dot means differentiation w.r.t. proper time. The antisymmetric tensor $S^{\alpha\beta}$ represents the spin of the electron and is assumed to satisfy the orthogonality condition

$$S^{\alpha\beta}u_{\beta} = 0\,. \tag{2.2}$$

In the rest frame, the spatial components of $S^{\alpha\beta}$ form therefore a three-vector $\vec{S}$.

In the non-relativistic limit, $\vec{S}$ becomes a constant of the motion. In the absence of external force, the motion is [apart of free motion along the direction of $\vec{S}$], in the plane perpendicular to $\vec{S}$ and satisfies the third-order equation

$$m\ddot{x}_i = -\kappa\varepsilon_{ij}\dddot{x}_j\,, \tag{2.3}$$

where the new constant κ has been defined by the Jackiw–Nair Ansatz [7]

$$s = \kappa c^2, \tag{2.4}$$

$s = |\vec{S}|$ being the length of the spin vector. Eq. (2.3) is precisely the equation of motion put forward by of Lukierski *et al.* [3].

From now on we drop the coordinate parallel to $\vec{S}$ and focus our attention on the motion in the plane.

3. Conserved quantities

The equations of motion (2.3) are associated with the Lagrangian (1.2). Then Lukierski *et al.* derive the conserved quantities associated to the space–time symmetries applying the higher-order version of Noether's theorem. Let us now reproduce their results using elementary methods.

• An obvious first integral of (2.3) is the momentum,

$$P_i = m\dot{x}_i + \kappa\varepsilon_{ij}\ddot{x}_j\,. \tag{3.1}$$

Eq. (2.3) is in fact $\dot{P}_i = 0$.

• Multiplying (2.3) by the velocity, $\dot{\vec{x}}$, yields a total time derivative, where we recognize the conserved *energy*,

$$H = \frac{m\dot{\vec{x}}^2}{2} + \kappa\,\dot{\vec{x}} \times \ddot{\vec{x}}. \tag{3.2}$$

• Similarly, taking the vector product of (2.3) with $\vec{x}$ yields the conserved *angular momentum*,

$$J = m\vec{x} \times \dot{\vec{x}} + \frac{\kappa}{2}\dot{\vec{x}}^2 - \kappa\vec{x} \cdot \ddot{\vec{x}}\,. \tag{3.3}$$

• A Galilean boost $\vec{x} \to \vec{x} + \vec{b}t$ shifts the momentum as $\vec{P} \to \vec{P} + m\vec{b}$. A rest frame where the momentum vanishes can be found, providing us with the conserved *boost* vector

$$K_i = mx_i - t\big(m\dot{x}_i + \kappa\varepsilon_{ij}\ddot{x}_j\big) + \kappa\varepsilon_{ij}\dot{x}_j\,. \tag{3.4}$$

Somewhat surprisingly, one more conserved quantity can be found.

• The vector product of (2.3) with the acceleration, $\ddot{\vec{x}}$, yields the square of the acceleration,

$$I = \frac{\kappa^3}{2m^2}\big(\ddot{\vec{x}}\big)^2\,, \tag{3.5}$$

where a constant factor has been included for later convenience.

• Curiously, multiplying (2.3) by $\dddot{\vec{x}}$ yields once again the same quantity, namely $(m/2)\big(\ddot{\vec{x}}\big)^2 = (m/\kappa)^3 I$.

The construction of this new quantity reminds one that of angular momentum and of energy. Its precise origin will be clarified below.

Let us observe that, owing to the conservation of I, $\ddot{\vec{x}} = 0$ can be consistently required. Then the conserved quantities found above reduce to those of an "elementary exotic particle" studied in [8].

4. Zitterbewegung and center-of-mass decomposition

The equation of motion (2.3) is integrated at once. Putting indeed

$$Q_i = -\left(\frac{\kappa}{m}\right)^2 \varepsilon_{ij}\ddot{x}_j\,, \tag{4.1}$$

Eq. (2.3) becomes

$$\dot{Q}_i = \frac{m}{\kappa}\varepsilon_{ij}Q_j\,, \tag{4.2}$$

showing that the acceleration rotates uniformly with angular velocity m/κ. Putting $Q = Q_1 + iQ_2$, $Q(t) = Q_0 \mathrm{e}^{-i(m/\kappa)t}$. This is plainly consistent with the conservation of the magnitude of the acceleration, Eq. (3.5). Then

$$X_i = x_i + \varepsilon_{ij}Q_j \tag{4.3}$$

moves freely,

$$\ddot{X}_i = 0\,. \tag{4.4}$$

In conclusion, the motion has been separated into the free motion of the center of mass coordinate $\vec{X}$, combined with the "Zitterbewegung" [uniform rotation] of the internal coordinate $\vec{Q}$.

A key feature of Mathisson's electron is that the internal variable $\vec{Q}$ measures in fact the extent of how much the momentum, $\vec{P}$, differs from [m-times] the velocity, $\dot{\vec{x}}$,

$$\vec{Q} = \frac{\kappa}{m^2}(m\dot{\vec{x}} - \vec{P})\,. \tag{4.5}$$

Re-writing the conserved quantities in terms of the new coordinates confirms the above interpretation. In fact,

$$\begin{aligned}
\vec{P} &= m\dot{\vec{X}}\,,\\
H = H_{\mathrm{CM}} + H_{\mathrm{int}} &= \frac{m\dot{\vec{X}}^2}{2} - \frac{m^3}{2\kappa^2}\vec{Q}^2\,,\\
J = J_{\mathrm{CM}} + J_{\mathrm{int}} &= m\vec{X}\times\dot{\vec{X}} + \frac{\kappa}{2}\dot{\vec{X}}^2 + \frac{m^2}{2\kappa}\vec{Q}^2\,,\\
K_i &= m(X_i - \dot{X}_i t) + \kappa\varepsilon_{ij}X_j\,,\\
I &= \frac{m^2}{2\kappa}\vec{Q}^2\,.
\end{aligned} \tag{4.6}$$

Mathisson's electron is hence a composite system. Note that in (4.6) the center of mass behaves precisely as an elementary exotic particle [8]; the internal coordinate only contributes to the energy and the angular momentum. In fact, $H_{\mathrm{int}} = -\dfrac{m}{\kappa}I$ and $J_{\mathrm{int}} = I$. The new conserved quantity found in (3.5) is hence the internal angular momentum and also the internal energy [which are linked in a 2-dimensional phase space].

Let us now observe that the equations of motion (4.2)–(4.4) are consistent with the Poisson structure associated with the symplectic form

$$\Omega = \Omega_{\mathrm{CM}} + \Omega_{\mathrm{int}} = dP_i\wedge dX_i + \frac{\kappa}{2m^2}\varepsilon_{ij}dP_i\wedge dP_j + \frac{m^2}{\kappa}\varepsilon_{ij}dQ_i\wedge dQ_j\,. \tag{4.7}$$

The 6 dimensional phase space is hence the direct sum of the four-dimensional "exotic" phase space of the center of mass with coordinates $\vec{X}$ and $\vec{P}$, with the two-dimensional internal phase space of the $\vec{Q}$, endowed with a canonical symplectic structure.

The Poisson structure can be used to calculate the algebraic structure of the symmetries. Consistently with Lukierski *et al.* [3], we find that $\vec{P}, H, J, \vec{K}$, supplemented with the central charges m and κ, realize the "exotic" [two-fold centrally extended] planar Galilei group. The structure relations of this latter only differ from those of the usual Galilei group in that the Poisson bracket of the boost components yields the "exotic" central charge,

$$\{K_1, K_2\} = \kappa\,. \tag{4.8}$$

Similarly, the center-of-mass coordinates have a nonvanishing Poisson bracket,

$$\{X_1, X_2\} = \frac{\kappa}{m^2}\,, \qquad \{Q_1, Q_2\} = -\frac{\kappa}{m^2}\,. \tag{4.9}$$

Both the center-of-mass and the internal coordinates are hence noncommuting, *cf.* (1.1) with $\theta = (\kappa/m^2)$ [while the original coordinates x_i commute]. This is similar to what happens in the Landau problem where the guiding center coordinates are noncommuting, with $\theta = 1/eB$.

The additional conserved quantity I in (3.5) is actually associated with the *internal symmetries* of the system. The translations and boosts form indeed an invariant subgroup K of the Galilei group. The quotient G/K, which consists of rotations and time translations, is hence a group that can be made to act separately on the center-of-mass and the internal space. We can, *e.g.*, rotate the internal coordinate $\vec{Q}$ alone and leave the center-of-mass coordinate $\vec{X}$ fixed. This is plainly a symmetry, and the associated conserved quantity is the internal angular momentum $J_{\rm int} = I$. (A physical rotation moves both the external and internal coordinates, yielding the total angular momentum in (4.6).) The internal energy arises in a similar way. In conclusion, the non-relativistic limit of the Mathisson electron admits the direct product of the "exotic" Galilei group with the internal rotations and time translations, $\mathrm{SO}(2) \times \boldsymbol{R}$, as symmetry. Here the action of the Galilei group is transitive on the submanifolds $I = \mathrm{const}$ *i.e.*, $\vec{Q}^2 = \mathrm{const}$.

The same statement is valid for any composite nonrelativistic system, *i.e.* one upon which the Galilei group acts by symmetries but not transitively [9].

5. Relation to Schrödinger's Zitternde Elektron

The results of Section 4 remind those Schrödinger derived in his original paper on Zitterbewegung [6]. Schrödinger starts in fact with the Dirac

Hamiltonian

$$H = c\vec{\alpha}\cdot\vec{P} + m^2c^2\beta\,, \tag{5.1}$$

where $\vec{\alpha}$ and β denote the usual Dirac matrices. In the Heisenberg picture, the operators satisfy

$$\frac{d\vec{P}}{dt} = 0\,, \qquad \frac{dH}{dt} = 0\,, \qquad \frac{d\vec{x}}{dt} = c\vec{\alpha}\,. \tag{5.2}$$

The last equation can be rewritten as $-i\frac{d\vec{\eta}}{dt} = 2H\vec{\eta}$, $(\hbar = 1)$, where $\vec{\eta} = \vec{\alpha} - cH^{-1}\vec{P}$. This can be integrated as $\vec{\eta}(t) = \mathrm{e}^{2iHt}\vec{\eta}_0 = \vec{\eta}_0\mathrm{e}^{-2iHt}$, where $\vec{\eta}_0$ is a constant operator. Hence

$$\frac{d\vec{x}}{dt} = c^2H^{-1}\vec{P} + c\vec{\eta}_0\mathrm{e}^{-2iHt}\,,$$

which can again be integrated to yield

$$\vec{x}(t) = \left\{\vec{X}_0 + c^2H^{-1}\vec{P}t\right\} + \tfrac{1}{2}ic\vec{\eta}_0H^{-1}\mathrm{e}^{-2iHt}\,, \tag{5.3}$$

where $\vec{X}_0$ is a constant operator. The structure is clearly the same as in (4.3), with the operator

$$\vec{X}(t) = \vec{X}_0 + c^2H^{-1}\vec{P}t \tag{5.4}$$

representing the freely moving center-of-mass, and the second term describing the internal Zitterbewegung. The precise relation is more subtle, though. Intuitively, dropping the third component and working in the plane, putting $s = 1/2$ and $s/c^2 \simeq \kappa$ [which would require the spin to diverge as $c \to \infty$ rather then remain a constant], setting $c\vec{\alpha} \simeq \dot{\vec{x}}$ and replacing $H \simeq mc^2$, would transform (5.3) formally into (4.3). In fact,

$$\vec{X}(t) \simeq \vec{X}_0 + \frac{\vec{P}}{m}t\,, \quad mc\vec{\eta} \simeq \frac{m^2}{\kappa}\vec{Q}\,, \quad \mathrm{e}^{-i2Ht} = \mathrm{e}^{-i(H/s)t} \simeq \mathrm{e}^{-i(m/\kappa)t}\,. \tag{5.5}$$

Note that

$$mc\vec{\eta} = mc\vec{\alpha} - mc^2H^{-1}\vec{P} \simeq m\dot{\vec{x}} - \vec{P}\,. \tag{5.6}$$

consistently with (4.5).

A distinctive feature of Schrödinger's Zitternde Elektron is that the center-of-mass coordinates satisfy the nontrivial commutation relation

$$[X_i, X_j] = -ic^2E^{-2}\varepsilon_{ijk}S_k\,, \tag{5.7}$$

where $E = c\sqrt{\vec{P}^2 + m^2c^2}$ and $\vec{S} = -(i/4)\vec{\alpha}\times\vec{\alpha}$ is the spin operator. If we assume that the spin is polarized in the third direction, $S_3 = -s$, and we consider the non-relativistic limit $E \simeq mc^2 + \vec{P}^2/2m$ together with the Ansatz (2.4), we find for the planar components

$$[X_1, X_2] \simeq i\frac{s}{c^2m^2} = i\frac{\kappa}{m^2}\,, \tag{5.8}$$

cf. (1.1) with $\theta = \kappa/m^2$. Let us remark that our procedure here is in fact the quantum version of the subtle non-relativistic limit proposed by Jackiw and Nair [7].

6. The relativistic description of Weyssenhoff and Raabe

Mathisson's classical electron was further elaborated by Weyssenhoff and Raabe in a paper published after the War [5]. They posit the equations

$$\begin{aligned} \dot{p}^\alpha &= 0\,, \qquad p^\alpha = mu^\alpha + \frac{1}{c^2}S^{\alpha\beta}\dot{u}_\beta\,, \\ \dot{S}^{\alpha\beta} &= p^\alpha u^\beta - p^\beta u^\alpha\,, \\ S^{\alpha\sigma}u_\sigma &= 0\,, \end{aligned} \tag{6.1}$$

where $m = -\frac{1}{c^2}u_\beta p^\beta$. Eliminating p^α yields the relativistic Mathisson equations (2.1) once again. Eqs. (6.1) imply that m is constant of the motion, $\dot{m} = 0$, identified as the rest-mass of the particle. $S_{\alpha\beta}S^{\alpha\beta} = s^2$ is also a constant of the motion. They also observe that, owing to $\dot{p}^\alpha = 0$, the quantity M defined by $p_\alpha p^\alpha = M^2c^2$ is another constant of the motion. It is worth noting that the position satisfies again a third-order equation analogous to (2.3), namely

$$m\ddot{x}_\alpha = -\frac{1}{c^2}S_{\alpha\sigma}\dddot{x}_\sigma. \tag{6.2}$$

Then Weyssenhoff and Raabe proceed to integrate the free relativistic equations of motion. In a suitable inertial frame (called the proper system) the spatial components, P_i, of the vector p^α can be made to vanish, so that its time component is Mc. In this frame $\vec{S}$ is constant. The mass is $m = M/\sqrt{1-(\vec{v}/c)^2}$ where $v_i = u_i\sqrt{1-(\vec{v}/c)^2}$ denotes the three-velocity. Hence the time component of the four-velocity is also constant so that the four-acceleration is proportional to the three-acceleration, $\vec{a} = d^2\vec{x}/dt^2$. Transforming from proper time to t, $\vec{P} = 0$ reduces finally to

$$M\vec{v} + \frac{1}{c^2}\vec{S}\times\vec{a} = 0\,. \tag{6.3}$$

The particle moves hence along a circle in the plane perpendicular to $\vec{S}$, with uniform angular velocity

$$\frac{mc^2}{s}\Big(1 - \frac{\vec{v}^2}{c^2}\Big)\,. \tag{6.4}$$

In a general Lorentz frame, the motion is a superposition of such a motion with a uniform translation.

Our clue is to observe that in the non-relativistic limit these formulæ reduce, with the Jackiw–Nair Ansatz $s = \kappa c^2$ *cf.* (2.4), to those we derived in Section 4.

It is worth mentioning that the equations of Weyssenhoff and Raabe have again and again re-emerged in the course of the years. Consider, for example, (6.1) in *five* dimensions and for $s = \frac{1}{2}$. Multiplication of p^β with $S_{\alpha\beta}$ allows us to express the five-vector $\dot{u}_\alpha$ as

$$\dot{u}_\alpha = \frac{4}{c^2} S_{\alpha\sigma} p^\sigma \tag{6.5}$$

which, together with the remaining relations in (6.1) and the constraint $u_\alpha u^\alpha = 1$, are precisely the equations proposed by Barut and Zanghi [10] as a "Kaluza–Klein" description of a classical Dirac electron.

7. Conclusion

In this Note we have shown that the non-relativistic limit of Mathisson's classical spinning electron yields the acceleration-dependent model of Lukierski *et al.* [3]. This latter has non-commuting coordinates and realizes the "exotic" Galilean symmetry.

Our results confirm once again the relation between the relativistic spin and the non-relativistic "exotic" structure, advocated by Jackiw and Nair [7]. Their rule (2.4) is, however, a rather strange one since it requires the spin to diverge as $c \to \infty$ so that s/c^2 remains finite. For this reason, the use of a Dirac equation valid for the fixed value $s = \frac{1}{2}$ [as in Section 5 above] is clearly illegitimate, and should be replaced by some anyon equation, valid for any real spin s [11].

Another intriguing feature of this procedure is the following. While the relativistic model is associated with an irreducible representation of the Poincaré group, its dequantized & non-relativistic limit, namely the model of Lukierski *et al.*, only carries a reducible representation of the Galilei group: irreducibility is lost in the procedure.

A final remark concerns the spin constraint (2.2) which appears to lie at the very root of the Zitterbewegung. Trading it for

$$S^{\alpha\beta} p_\beta = 0 \tag{7.1}$$

would in fact eliminate the Zitterbewegung altogether and lead to models of the type discussed in [12].

I am indebted to Professor J. Lukierski for sending me copies of those old *Acta Physica Polonica* papers, and also to Professor A. Staruszkiewicz, who provided me with some biographic data.

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[4] M. Mathisson, *Das Zitternde Elektron und seine Dynamik*, *Acta Phys. Pol.* **6**, 218 (1937). Myron Mathisson (1897-1940), of Jewish origin, taught mathematical physics at Warsaw University as a *Privatdozent*. He also worked in Kraków, benefitting of a kind of "private scholarship" created especially for him by Weyssenhoff. Then he spent one year in Kazan, in the Soviet Union. In 1939 he escaped to Britain, where he died. He was remembered by P.A.M. Dirac in the Obituary reproduced below, published in *Nature* **146**, 613 (1940): "The death of Dr. Myron Mathisson on September 13 at the early age of fourty-three has cut short an interesting line of research. Mathisson had been engaged for many years in studying the general dynamical laws governing the motion of a particle, with possibly a spin or a moment, in a gravitational or electromagnetic field, and had developed a powerful method of his own for passing from field equations to particle equations. The subject is of particular interest at the present time, as it has now become clear that quantum mechanics cannot solve the difficulties that arise in connexion with the interaction of point particles with fields, and a deeper classical analysis of the problem is needed. It is much to be regretted that Mathisson's death has occured before the relations between his method and those of other workers on the subject have been

completely elucidated. Mathisson carried out his work at the Universities of Warsaw and Kazan and at an institute which he started in Cracow, and, since the spring of 1939, at Cambridge."
[Sources: A short history of Theoretical Physics at Hoża 69 . . . , and personal communication of Prof. A. Staruszkiewicz.]

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Mr. Raabe was a highly gifted young physicist with whom I outlined in all its main features the contents of this paper in 1940/41 in Lwów. We tried to pursue our work in 1942 in Cracow, but unfortunately in June 1942 Mr. Raabe fell victim of a man-hunt in the streets of Cracow; he died four months later in the German concentration camp Oświęcim [Auschwitz]."
Jan Weyssenhoff (1889-1972) came from a prominent Baltic-German aristocratic family, which remained Catholic and became Polish in the XVIIth century. He was a gentleman in the old sense of the word, who used his personal fortune and his wealthy friends to help other colleagues. His father was a successful writer. His mother came from a very wealthy Jewish banking family which owned, among other things, the Warsaw–Vienna railway. Weyssenhoff studied in Kraków and in Zürich, where he also met Einstein, who refers to him in his work on Brownian Motions [available in Dover Publications]. He was also interested in the Hall effect and wrote his Ph. D. on the theory of paramagnetism. He returned to his country in 1919. He got involved in the study of relativistic spinning particles and fluids in 1937. Between 1939 and 1941 he worked at the Polytechnical University of Lwów, occupied by the Soviet army and attached to Ukraine. In 1942 he returned to Kraków, and was followed by Raabe, who lived in his flat and whom he helped also to get documents, *e.g.*, a "Kennkarte".
He also organized secret seminars on physics in his home. Unlike his young collaborator, he survived the war and continued his scientific work until his death in Kraków, in 1972.

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