FOURTH-ORDER SQUEEZING IN SUPERPOSED COHERENT STATES

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We study the fourth-order squeezing in the most general case of superposition of two coherent states by considering $\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle$ where $X_\theta = X_1\cos\theta + X_2\sin\theta$, $X_1 + iX_2 = a$ is annihilation operator, θ is real, $|\psi\rangle = Z_1\,|\alpha\rangle + Z_2\,|\beta\rangle$, $|\alpha\rangle$ and $|\beta\rangle$ are coherent states and Z_1, Z_2, α, β are complex numbers. We find the absolute minimum value 0.050693 for an infinite combinations with $\alpha - \beta = 1.30848\exp[\pm i(\pi/2) + i\theta]$, $Z_1/Z_2 = \exp(\alpha^*\beta - \alpha\beta^*)$ with arbitrary values of $\alpha+\beta$ and θ. For this minimum value of $\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle$, the expectation value of photon number can vary from the minimum value 0.36084 (for $\alpha+\beta = 0$) to infinity. We note that the variation of $\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle$ near the absolute minimum is less flat when the expectation value of photon number is larger. Thus the fourth-order squeezing can be observed at large intensities also, but settings of the parameters become more demanding.

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A state is said to be squeezed if variance of a quadrature amplitude is less than that for vacuum state [1] and this is a purely quantum phenomenon which can not be explained on the basis of classical probability concept. Earlier, study [2] of squeezing was largely in academic interest because of this point but now its utility in reducing noise has been well realized [1, 3]. With the development of techniques for making higher order correlation measurements in quantum optics, the higher order moments of the radiation have also become interesting. From the point of view of noise reduction, higher-order squeezing [4] (*i.e.*, reduction in expectation value of the N-th power ($N > 2$) of fluctuations in one of the quadrature amplitude operators, below the level associated with the coherent state of the radiation or the vacuum state) may be particularly interesting because of a larger possible fractional reduction in the expectation value of a higher power of fluctuation than that for the second power studied in second order squeezing. Higher order squeezing was earlier studied [5] by Buzek, Vidiella-Barranco and Knight in some special cases of superposition of two coherent states, like the even, odd and the Yurke-stoler [6] coherent states. In this paper we study the fourth order squeezing of the most general hermitian quadrature amplitude operator, $X_\theta \equiv X_1 \cos\theta + X_2 \sin\theta$, where θ is a real angle, $X_{1,2}$ are the Hermitian quadrature amplitude operators defined by $X_1 + iX_2 = a$, and a is the annihilation operator, in the most general superposition of two coherent states in a single mode of radiation of the form $|\psi\rangle = Z_1 |\alpha\rangle + Z_2 |\beta\rangle$. Here, $Z_{1,2}$ are complex numbers and $|\alpha\rangle$, $|\beta\rangle$ are the coherent states defined by $a|\alpha\rangle = \alpha|\alpha\rangle$, $a|\beta\rangle = \beta|\beta\rangle$. The complex numbers Z_1, Z_2, α, β and the real θ are regarded completely arbitrary subject to the only constraint given by

$$\langle\psi \mid \psi\rangle = |Z_1|^2 + |Z_2|^2 + 2\mathrm{Re}\left[Z_1^* Z_2 \exp\left\{-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^*\beta\right\}\right] = 1\,. \tag{1}$$

We show that expectation value of fourth power of $\Delta X_\theta \equiv X_\theta - \langle\psi| X_\theta |\psi\rangle$ in the state $|\psi\rangle$ given [4] by

$$\langle\psi| (\Delta X_\theta)^4 |\psi\rangle = \langle\psi| : (\Delta X_\theta)^4 : |\psi\rangle + \frac{3}{2}\langle\psi| : (\Delta X_\theta)^2 : |\psi\rangle + \frac{3}{16}\,, \tag{2}$$

where an operator put between two colons denotes its normal value, has an absolute minimum value 0.050693 (which is less than the value 0.1875 for the coherent state) for an infinite combinations with $Z_1/Z_2 = \exp(\beta\alpha^* - \beta^*\alpha)$, $\alpha - \beta = 1.30848\exp[\pm i(\pi/2) + i\theta]$, and with arbitrary values of $\alpha + \beta$ and θ. For this minimum value of the variance of X_θ, the expectation value of photon number in state $|\psi\rangle$ can vary from the minimum value 0.36084 (for $\alpha + \beta = 0$) to infinity. We find that large higher-order squeezing can be observed at large intensities also, but settings of the parameters become more important in this case. Single mode radiation coherent state $|\alpha\rangle$ defined by $a|\alpha\rangle = \alpha|\alpha\rangle$ is given by [7]

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \left(\alpha^n/\sqrt{n!}\right) |n\rangle = D(\alpha)\ |0\rangle\ , \tag{3}$$

where $|n\rangle$ is the occupation number state and $D(\alpha)$ is the displacement operator given by $D(\alpha) = \exp(\alpha a^+ - \alpha^* a)$. It is easily seen that

$$D^+(\alpha)\, X_\theta\, D(\alpha) = X_\theta + \alpha_r \cos\theta + i\alpha_i \sin\theta\,, \tag{4}$$

where $\alpha_r + \mathrm{i}\alpha_i = \alpha$, and hence expectation values of powers of ΔX_θin any state $|\psi\rangle$ will be same as that in $D(\alpha)\,|\psi\rangle$. This observation and the relation [7]

$$D(\alpha)\,D(\beta) = \exp(\alpha\beta^* - \beta\alpha^*)\,D(\alpha+\beta) \tag{5}$$

suggests that we can write the state $|\psi\rangle = Z_1\,|\alpha\rangle + Z_2\,|\beta\rangle$ as

$$|\psi\rangle = D\left(\frac{1}{2}[\alpha+\beta]\right)\,|\psi_1\rangle\,, \quad |\psi_1\rangle = Z_1'\,|\xi\rangle + Z_2'\,|-\xi\rangle\,, \quad \xi = \frac{1}{2}(\alpha-\beta)\,, \tag{6}$$

where $Z_{1,2}' = Z_{1,2}\exp[\pm\frac{1}{2}(\alpha\beta^* - \beta\alpha^*)]$. The state $|\psi_1\rangle$ can be written in terms of superposition of the even and odd coherent states defined by

$$|\xi,\pm\rangle = K_\pm(|\xi\rangle \pm |-\xi\rangle); \quad K_\pm = (2[1\pm \mathrm{e}^{-2|\xi|^2}]\,)^{-\frac{1}{2}} \tag{7}$$

in the form

$$|\psi_1\rangle = \cos\frac{\chi}{2}\;|\xi,+\rangle + \sin\frac{\chi}{2}\,\mathrm{e}^{i\phi}\;|\xi,-\rangle\;, \tag{8}$$

where $0 \le \chi \le \pi,\ -\pi < \phi \le \pi$. Here we have taken coefficient of $|\xi,+\rangle$ as real without any loss of generality. For the states $|\xi,\pm\rangle$, we have

$$a\,|\xi,\pm\rangle = K\xi_\pm\,|\xi,\mp\rangle \quad \text{and} \quad a^2\,|\xi,\pm\rangle = \xi^2\,|\xi,\pm\rangle\;, \tag{9}$$

where $\xi_\pm = \xi(1 \mp e^{-2|\xi|2})$ and $K = (1-\exp(-4\,|\xi|^2))^{-\frac{1}{2}}$.
We calculate $\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle = \langle\psi_1|\,(\Delta X_\theta)^4\,|\psi_1\rangle$ using the result given in Eq. (2). We note that

$$\begin{aligned}\langle\psi_1|:X_\theta^4:|\psi_1\rangle = \frac{1}{8}\{&\mathrm{Re}[\langle\psi_1|\,a^4\,|\psi_1\rangle\,e^{-4i\theta}] + 4\mathrm{Re}[\langle\psi_1|\,a^+a^3\,|\psi_1\rangle\,e^{-2i\theta}]\\ &+3(\langle\psi_1|\,a^{+2}a^2\,|\psi_1\rangle)\,\})\,,\end{aligned} \tag{10}$$

$$\langle\psi_1|:X_\theta^3:|\psi_1\rangle = \frac{1}{4}\{\mathrm{Re}[\langle\psi_1|\,a^3\,|\psi_1\rangle\,e^{-3i\theta}] + 3\mathrm{Re}[\langle\psi_1|\,a^+a^2\,|\psi_1\rangle\,e^{-i\theta}]\}, \tag{11}$$

$$\langle\psi_1|:X_\theta^2:|\psi_1\rangle = \frac{1}{2}\{\mathrm{Re}[\langle\psi_1|\,a^2\,|\psi_1\rangle\,e^{-2i\theta}] + \langle\psi_1|\,a^+a\,|\psi_1\rangle\}\,. \tag{12}$$

If we write $\xi = A\exp(i\theta_\xi)$ and $\delta = \theta_\xi - \theta$, straight forward calculations lead to the expectation of the fourth power of fluctuation in X_θ,

$$\begin{aligned}\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle \;&=\; \langle\psi_1|\,(\Delta X_\theta)^4\,|\psi_1\rangle\\ &=\; \frac{3}{16} + \langle\psi_1|{:}X_\theta^4{:}|\psi_1\rangle + 6(\langle\psi_1|{:}X_\theta^2{:}|\psi_1\rangle)(\langle\psi_1|{:}X_\theta{:}|\psi_1\rangle)^2\\ &\quad -3(\langle\psi_1|{:}X_\theta{:}|\psi_1\rangle)^4 - 4\,\langle\psi_1|{:}X_\theta^3{:}|\psi_1\rangle\,\langle\psi_1|{:}X_\theta{:}|\psi_1\rangle\\ &\quad +\frac{3}{2}[\langle\psi_1|{:}X_\theta^2{:}|\psi_1\rangle - (\langle\psi_1|{:}X_\theta{:}|\psi_1\rangle)^2]\end{aligned} \tag{13}$$

and the final result is

$$\begin{aligned}\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle = \frac{3}{16} &+ \tfrac{1}{8}A^4(\cos 4\delta + 4f\cos 2\delta + 3) - 3m^4 \\ &+ \frac{3}{2}\left\{\tfrac{1}{2}A^2(f + \cos 2\delta) - m^2\right\} - 4KmA^3\sin\chi(\cos\phi\cos^3\delta \\ &+ \sin\phi\sin^3\delta\, e^{-2A^2}) + 3A^2m^2(f + \cos 2\delta)\,, \qquad (14)\end{aligned}$$

$f \equiv K^2(1 + \mathrm{e}^{-4A^2} - 2\cos\chi e^{-2A^2})$; $m \equiv KA\sin\chi(\cos\phi\cos\delta - \sin\phi\sin\delta e^{-2A^2})$.

The corresponding results for second order squeezing is simpler [8] and permits finding minima with ϕ, χ and δ analytically. We calculate the minimum value of$\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle$ given above using a computer programming. We get the minimum value 0.050693 of $\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle$ at $A = 0.65424$, $\chi = 0$, $\delta = \pi/2$ and it is independent of ϕ. The variations of $\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle$ with A, χ and δ near this minima are shown in figures 1(a), 1(b), and 1(c) respectively.

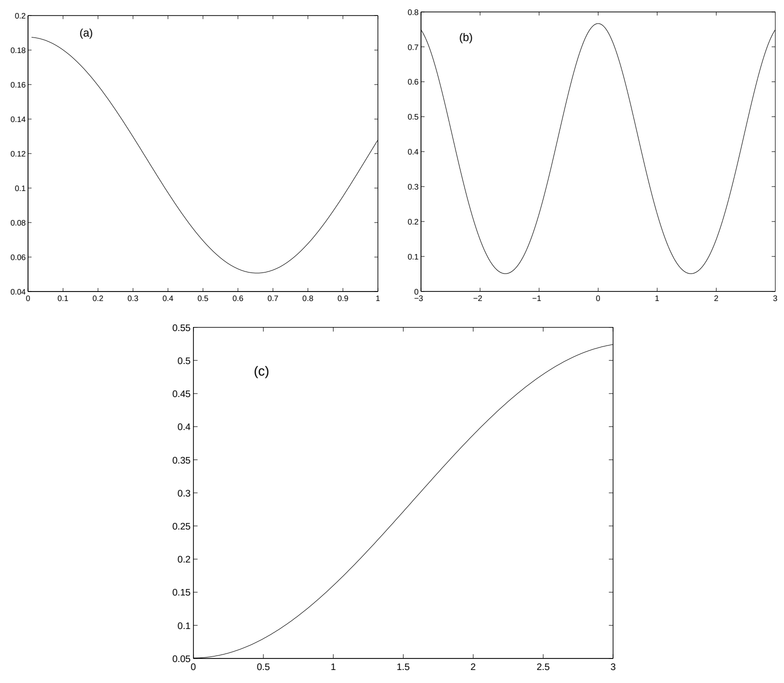

Fig. 1. (a)The variation of $\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle$ (on Y-axis) with A (on X-axis)at $\chi = 0$, $\delta = \pi/2$. (b)The variation of $\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle$ (on Y-axis) with δ (on X-axis)at $A = 0.65424$, $\chi = 0$. (c)The variation of $\langle\psi|\,(\Delta X_\theta)^4\,|\psi\rangle$ (on Y-axis) with χ (on X-axis) at $A = 0.65424, \delta = \pi/2$.

In terms of Z_1, Z_2, α, β and θ, we conclude, therefore, that the maximum forth-order squeezing of X_θ in the state $|\psi\rangle$ occurs for an infinite combinations with $Z_1/Z_2 = \exp(\alpha^*\beta - \alpha\beta^*)$ and $\alpha - \beta = 1.30848 \exp[\pm i(\pi/2) + i\theta]$, and with arbitrary values of $\alpha + \beta$ and θ. For this state the average photon number is 0.36084. Since, the action of displacement operator $D(\alpha)$ on $|\psi\rangle$ does not affect the expectation value of ΔX_θ but can change photon number, it is obvious that this large squeezing can occur at arbitrarily large intensities. To illustrate this point we consider the state in the form

$$\left|\psi'\right\rangle = K'(|x\rangle + \left|xe^{i\gamma}\right\rangle) \quad ; \quad K' = \{2[1 + e^{-x^2(1-\cos\gamma)}\cos(x^2\sin\gamma\,)]\}^{-\frac{1}{2}}\,. \tag{15}$$

This state has the expectation value of fourth power of ΔX_θ,

$$\begin{aligned}
&\left\langle\psi'\right| (\Delta X_\theta)^4 \left|\psi'\right\rangle = \frac{3}{16} + \frac{1}{8}K'^2x^4[6 + 4\cos 2\theta + \cos 4\theta + \cos(4\gamma - 4\theta)\\
&+C_1\{\cos(4\gamma - 4\theta + C_2) + \cos(4\theta + C_2) + 4\cos(\gamma + 2\theta + C_2)\\
&+4\cos(3\gamma - 2\theta + C_2) + 4\cos(2\gamma - 2\theta) + 6\cos(2\gamma + C_2)\}]\\
&+3K'^6x^4M^2[2 + \cos 2\theta + \cos(2\gamma - 2\theta) + C_1\{\cos(2\gamma - 2\theta + C_2)\\
&+\cos(2\theta + C_2) + 2\cos(\gamma + C_2)\}] - 3K'^8x^4M^4 - K'^4x^4M[4\cos^3\theta\\
&+4\cos^3(\gamma - \theta) + C_1\{\cos(3\gamma - 3\theta + C_2) + \cos(3\theta + C_2)\\
&+3\cos(2\gamma - \theta + C_2) + 3\cos(\gamma + \theta + C_2)\}] + \tfrac{3}{2}[\tfrac{1}{2}K'^2x^2[2 + \cos 2\theta\\
&+\cos(2\gamma - 2\theta) + C_1\{\cos(2\gamma - 2\theta + C_2) + \cos(2\theta + C_2)\\
&+2\cos(\gamma + C_2)\}] - K'^4x^2M^2]\,,
\end{aligned} \tag{16}$$

where

$$C_1 \equiv e^{-x^2(1-\cos\gamma)};\ C_2 \equiv x^2\sin\gamma$$

and

$$M \equiv \cos\theta + \cos(\gamma - \theta) + C_1\{\cos(\gamma - \theta + C_2) + \cos(\theta + C_2)\}\,.$$

We see that this state may have the minimum fourth order moment of X_θ as 0.050693, if

$$x(1 - e^{i\gamma}) = 1.30848e^{i(\pm\frac{\pi}{2}+\theta)}\,. \tag{17}$$

This is satisfied for an infinite combinations of x, γ and θ. For example $\gamma = 0.0029$, $\theta = \gamma/2$ and $x = 451.28$, the minimum value of the expectation of the fourth-order fluctuations is 0.050693. The Variation of $\langle\psi'|\,(\Delta X_\theta)^4\,|\psi'\rangle$ with x for $\gamma = 0.0029$ and $\theta = \gamma/2$ and is shown in Fig. 2(a). As compared to earlier results, the variation is fast. For the same minimum value of $\langle\psi'|\,(\Delta X_\theta)^4\,|\psi'\rangle$ variation with A shown in Fig. 1(a) is much slower than the variation with x shown in Fig. 2(a). Both A and x are related to the

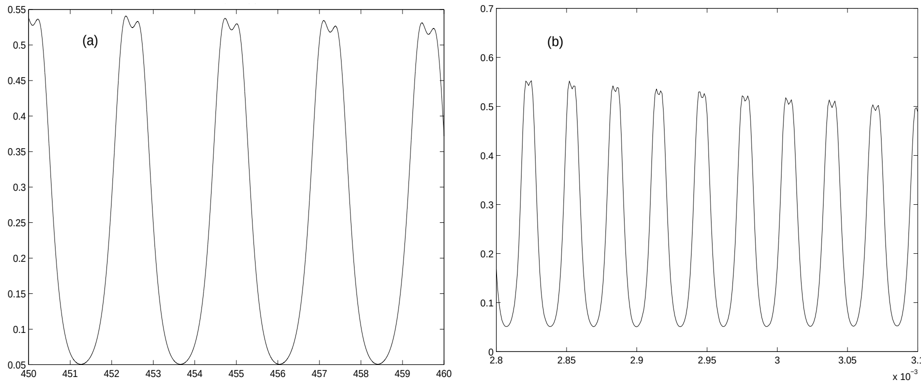

Fig. 2. (a) The variation of $\langle\psi'|\,(\Delta X_\theta)^4\,|\psi'\rangle$ (on Y-axis) with x (on X-axis) at $\gamma = 0.0029$ and $\theta = 0.00145$. (b) The variation of $\langle\psi'|\,(\Delta X_\theta)^4\,|\psi'\rangle$ (on Y-axis) with γ (on X-axis) at $x = 451.28$, $\theta = 0.00145$.

average photon number. Figure 2(b) shows that the phase angle γ plays an important role for the maximum fourth-order squeezing. Thus large higher-order squeezing can be produced with high intensity also, but, for its observation settings of parameters become more demanding.

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REFERENCES

[1] See *e.g.*, review article D.F. Walls, *Nature* **306**, 141 (1983); M.O. Scully, M.S. Zubairy, *Quantum Optics*, Cambridge University Press, 1997, pp.60–63.

[2] B.R. Mollow, R.J. Glauber, *Phys. Rev.* **160**, 1076 (1967); N. Chandra, H. Prakash, Indian J. Pure Appl. Phys. **2,** 677, 688, 767 (1971); N. Chandra, H. Prakash, *Lett. Nuovo. Cim.* **4**, 1196 (1970).

[3] See *e.g.*, Hans-A. Bachor, *A Guide to Experiments in Quantum Optics*, WILEY-VCH, Weingheim Press, 1998, pp.228–230.

[4] C.K. Hong, L. Mandel, *Phys. Rev. Lett.* **54**, 323 (1985).

[5] V. Buzek, A. Vidiella-Barranco, P.L. Knight, *Phys. Rev.* **A45**, 6570 (1992).

[6] B. Yurke, D. Stoler, *Phys. Rev. Lett.* **57**, 13 (1986).

[7] R. J. Glauber, *Phys. Rev.* **131**, 2766 (1963).

[8] H. Prakash, P. Kumar, *Physica A* **319**, 305 (2003)