RENYI ENTROPY OF THE IDEAL GAS IN FINITE MOMENTUM INTERVALS

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Coincidence probabilities of multiparticle events, as measured in finite momentum intervals for Bose and Fermi ideal gas, are calculated and compared with the exact expressions given in statistical physics.

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1. Introduction

Recently, we have proposed to measure the coincidence probabilities in the multiparticle systems produced in high-energy processes in order to obtain an information about the entropy created in the collision [1]. The coincidence probability of order l is defined as [1–3]

$$C_l \equiv \sum_i [p_i]^l \,, \tag{1}$$

where p_i is the probability for the system to be in the state i and the sum runs over all states of the system. For integer l, this quantity can be measured simply by counting the number (N_l) of l-plets of the *identical events* observed in a given process. Then

$$C_l = \frac{N_l}{N(N-1)\dots(N-l+1)} \,, \tag{2}$$

where N is the total number of events in the sample (the denominator in (2) represents the total number of l-plets of observed events).

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C_l's are simply related to Renyi entropies [3]

$$H_l = -\frac{\log C_l}{l-1} \tag{3}$$

from which, in turn, one may obtain the standard Shannon entropy by extrapolation to $l = 1$:

$$S = \lim_{l \to 1} H_l \, . \tag{4}$$

It is thus clear that the suggested measurements touch the very fundamental properties of the system.

As we have already remarked several times [1], the difficulty in performing a measurement of the coincidence probabilities is the continuous distribution of the particle momenta observed in the high-energy experiments. Due to this property of the spectra, the coincidence measurement, if taken literally, is not possible. Therefore some discretization procedure, *i.e.*, a division of the available phase-space into finite size momentum intervals, is necessary.

It should be realized that such a discretization procedure affects — in an important way — the expected results of the coincidence measurements. Clearly, the larger are the chosen intervals, the less fluctuations can be observed and thus larger coincidence probabilities are expected.

The purpose of the present paper is to discuss how actually the discretization procedure affects the coincidence measurements and how these measurements should be interpreted. To this end we consider in detail the case of ideal gas in equilibrium. The general formulae for Renyi entropies are derived and the limits of very small and very large intervals are quantitatively analyzed.

2. Ideal gas: formulation of the problem

Consider a particle momentum interval Δ, centered at p_x^0, p_y^0, p_z^0 and of size given by the inequalities

$$|p_x - p_x^0| \leq \Delta_x \, , \quad |p_y - p_y^0| \leq \Delta_y \, , \quad |p_z - p_z^0| \leq \Delta_z \, . \tag{5}$$

The fluctuations attached to this bin are given by the multiplicity distribution of particles which happen to fall into Δ. Denoting the probability to find n particles in Δ by $P(n)$ we have

$$C_l(\Delta) = \sum_n [P(n)]^l \, . \tag{6}$$

If M bins are considered, the corresponding coincidence probability is calculated from the formula

$$C_l(\Delta_1, \dots \Delta_M) = \sum_{n_1,\dots,n_M} [P(n_1, \dots n_M)]^l \,, \tag{7}$$

i.e. we have to know the joint particle distribution in all M bins.

In case of the ideal gas, there are no correlations between bins, thus

$$P(n_1, \dots n_M) = \prod_{m=1}^{M} P_m(n_m) \,, \tag{8}$$

so that we have

$$C_l(\Delta_1, \dots \Delta_M) = \sum_{n_1,\dots,n_M} \prod_{m=1}^{M} [P_m(n_m)]^l = \prod_{m=1}^{M} [C_l(\Delta_m)] \tag{9}$$

from which it follows that the Renyi entropies (given by (3)) obey the additivity constraint and thus it is enough to consider one single bin.

We see from these formulae that the coincidence probabilities are determined from the multiplicity distribution in the selected bin Δ. This multiplicity distribution depends on the particle energy levels which are contained in Δ.

Let us denote by e_i the energy levels accessible to one particle. In the ideal gas at equilibrium, the probability to find s_i particles on the level i is given by

$$p_i(s_i) = (1 - u_i) u_i^{s_i} \tag{10}$$

for bosons and

$$p_i(s_i) = \frac{u_i^{s_i}}{1 + u_i} \tag{11}$$

for fermions, where

$$u_i = \mathrm{e}^{-\beta(e_i - \mu)} \,. \tag{12}$$

For bosons $s_i = 0, 1, 2, \dots$, for fermions $s_i = 0, 1$, μ is the chemical potential.

Consider now the interval Δ given by (5). One has to consider two cases.

(i) Large bins. Suppose that the interval is large enough to contain a number I of energy levels e_i. Then the probability of finding precisely n particle in this bin is

$$P(n) = \sum_{s_1+\ldots s_I=n} \prod_{i=1}^{I} p_i(s_i)\,, \tag{13}$$

where the product runs over all "elementary" cells which are inside the bin Δ.

(ii) Small bins. If the interval is so small that its size is smaller than the distance Δ_0 between the energy levels, the probability to find n bosons in it is

$$\begin{aligned} P(n) &= \sum_{k=0}^{\infty} p_i(k)\frac{k!}{n!(k-n)!}v^n(1-v)^{k-n} \\ &= (1-u_i)\sum_{k=n}^{\infty}\frac{k!}{n!(k-n)!}v^n(1-v)^{k-n}u_i^k \\ &= (1-u_i)\,(vu_i)^n\sum_{j=0}^{\infty}\frac{(n+j)!}{n!j!}[(1-v)u_i]^j \\ &= (1-u_i)\,(vu_i)^n\,[1-(1-v)u_i]^{-(n+1)} \\ &= \frac{1-u_i}{1-(1-v)u_i}\left(\frac{vu_i}{1-(1-v)u_i}\right)^n, \end{aligned} \tag{14}$$

where

$$v = \frac{\Delta}{\Delta_0}\,, \quad \langle n\rangle = v\frac{u_i}{1-u_i}\,. \tag{15}$$

For fermions we obtain

$$P(0) = 1 - v\frac{1}{1+u_i}\,, \quad P(1) = v\frac{u_i}{1+u_i}\,, \quad \langle n\rangle = P(1)\,. \tag{16}$$

Note that for non-interacting particles in a box, the distance between the energy levels, Δ_0, is (in each dimension) given by

$$\Delta_0 = \frac{2\pi}{L}\,. \tag{17}$$

Therefore, the size of the interval Δ below which the Eqs. (14) and (16) are applicable depends on the size of the system in the configuration space.

3. Coincidence probabilities in small intervals

In this case the calculation is fairly simple. Using (14) we obtain for bosons

$$
\begin{aligned}
C_l &= \left(\frac{1-u_i}{1-(1-v)u_i}\right)^l \sum_{n=0}^{\infty}\left(\frac{vu_i}{1-(1-v)u_i}\right)^l n \\
&= \frac{(1-u_i)^l}{[1-(1-v)u_i]^l-(vu)^l} \\
&= \frac{1}{\left(1+\frac{vu_i}{1-u_i}\right)^l-\left(\frac{vu_i}{1-u_i}\right)^l} \\
&= \frac{1}{(1+\langle n(\Delta)\rangle)^l-\langle n(\Delta)\rangle^l}\,,
\end{aligned}
\tag{18}
$$

where $\langle n(\Delta)\rangle$ is the average multiplicity in the bin Δ.

Using (16) we have for fermions

$$
C_l = \left(1-\frac{vu_i}{1+u_i}\right)^l + \left(\frac{vu_i}{1+u_i}\right)^l = (1-\langle n(\Delta)\rangle)^l + \langle n(\Delta)\rangle^l\,. \tag{19}
$$

It is interesting to consider the limit when the bin is split into many ($M\to\infty$) pieces of equal size. Using (9) we thus obtain

$$
\Delta_M \equiv \frac{\Delta}{M}\,, \quad \langle n(\Delta_M)\rangle = \frac{\langle n(\Delta)\rangle}{M} \tag{20}
$$

and thus the Renyi entropy calculated for all M bins tends to a finite value

$$
H_l(\Delta)_{M\to\infty} \;\to\; \frac{l}{l-1}\langle n(\Delta)\rangle\,. \tag{21}
$$

This result is a manifestation of the "empty bin effect". It depends only on the assumption that there are no correlations between particles. We see that in this limit one does not obtain any useful information about the system, as everything is determined by the average multiplicity. It is also seen that the extrapolation to $l=1$ does not make sense in this limit.

4. Multiplicity distribution and coincidence probabilities in large intervals

To evaluate the multiplicity distribution of (13), which is necessary for estimate of C_l, as is seen from (6), we employ the technique of the generating function. Denoting

$$
F(z) \equiv \sum_n P(n) z^n \tag{22}
$$

we have

$$F(z) = \prod_i \phi_i(z)\,, \tag{23}$$

where $\phi_i(z)$ is the generating function of the distribution (13)

$$\phi_i(z) = \sum_{s_i} p_i(s_i) z^{s_i}\,. \tag{24}$$

Using (10) and (11) we obtain

$$\phi_i(z) = \phi(z, u_i) = \frac{1-u_i}{1-zu_i} \tag{25}$$

for bosons and

$$\phi_i(z) = \phi(z, u_i) = \frac{1+zu_i}{1+u_i} \tag{26}$$

for fermions.

In order to transform the product in (23) into a sum, we take the logarithm:

$$\log F(z) \equiv f(z) = \sum_i \log[\phi_i(z)] \;\to\; \int\limits_{\Delta} \frac{d^3p d^3x}{(2\pi)^3} \log[\phi(z,u)]\,, \tag{27}$$

where

$$u = u(p) = \exp\left[-\beta(\sqrt{p^2+m^2}-\mu)\right] \tag{28}$$

and where in the second equality of (27) we have explicitely used the assumption that the interval Δ is large enough to contain many energy levels. If this number is not very large, one has to keep the first equality of (27), *i.e.* use explicitely the sum over the energy levels.

Using (24) and (25) we obtain

$$f(z) = \sum_k b_k z^k\,, \tag{29}$$

where

$$b_0 = -\sum_{k=1}^{\infty} b_k\,, \quad b_k = \frac{\lambda^k U_k}{k} \tag{30}$$

for bosons and

$$b_0 = -\sum_{k=1}^{\infty} b_k\,, \qquad b_k = -\frac{(-\lambda)^k U_k}{k} \tag{31}$$

for fermions, with

$$U_k = \sum_i \mathrm{e}^{-k\beta\sqrt{{p_i}^2+m^2}}\,, \quad \lambda = \beta\mu\,. \tag{32}$$

The sum runs over all energy levels contained in the interval Δ. In the limit of large bins we have

$$U_k = \frac{V}{(2\pi)^3}\int\limits_{\Delta} d^3p\mathrm{e}^{-k\beta\sqrt{p^2+m^2}}\,. \tag{33}$$

The average multiplicity in the bin Δ equals $dF(z)/dz|_{z=1}$. This implies

$$\langle n\rangle = \sum_{k=1}^{\infty} kb_k\,. \tag{34}$$

We thus see that everything can be expressed by the expansion coefficients b_k and the average multiplicity $\langle n\rangle$. Unfortunately, the sums (integrals) (32) and (33) cannot be easily evaluated, in general.

To obtain the multiplicity distribution we have to expand

$$F(z) = \mathrm{e}^{f(z)} = \sum_n P(n)z^n\,. \tag{35}$$

This can be done as follows:

Consider the function

$$F_K(z) \equiv \exp\left[\sum_{k=0}^{K} z^k b_k\right]\,. \tag{36}$$

This function approaches $F(z)$ when $K \to \infty$.

It is possible to develop $F_K(z)$. We obtain

$$
\begin{aligned}
F_K(z) &= \sum_{M=0}^{\infty} \frac{1}{M!} \left[\sum_{k=0}^{K} z^k b_k \right]^M \\
&= \sum_{M=0}^{\infty} \frac{1}{M!} \sum_{k_0=0}^{M} \sum_{k_1=0}^{M} \ldots \sum_{k_K=0}^{M} \delta(k_1 + k_2 + \ldots + k_K - M) \\
&\quad \times \frac{M!(b_0)^{k_0}(zb_1)^{k_1} \ldots (z^K b_K)^{k_K}}{k_0! k_1! \ldots k_K!} \\
&= \sum_{k_0=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \ldots \sum_{k_K=0}^{\infty} \\
&\quad \times \frac{(b_0)^{k_0}(b_1)^{k_1}(b_2)^{k_2} \ldots (b_K)^{k_K}}{k_0! k_1! k_2! \ldots k_K!} z^{k_1+2k_2+\ldots+Kk_K} \\
&= \sum_{n=0}^{\infty} P_K(n) z^n \,, \qquad (37)
\end{aligned}
$$

where

$$
\begin{aligned}
P_K(n) &= \mathrm{e}^{b_0} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \ldots \sum_{k_K=0}^{\infty} \delta(k_1 + 2k_2 + \ldots + Kk_K - n) \\
&\quad \times \frac{(b_1)^{k_1}(b_2)^{k_2} \ldots (b_K)^{k_K}}{k_1! k_2! \ldots k_K!} \,. \qquad (38)
\end{aligned}
$$

Actually, the sums in (38) are, for fixed n, limited:

$$
\begin{aligned}
&\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \ldots \sum_{k_K=0}^{\infty} \delta(k_1 + 2k_2 + \ldots + Kk_K - n) \\
&= \sum_{k_K=0}^{[n/K]} \sum_{k_{K-1}=0}^{[(n-Kk_K)/(K-1)]} \ldots \sum_{k_1=0}^{n-2k_2-\ldots-Kk_K} \delta(k_1 + 2k_2 + \ldots + Kk_K - n), \qquad (39)
\end{aligned}
$$

where $[\ldots]$ denotes the integer part of the expression inside the brackets.

One also sees that only $K \leq n$ contributes at a given n. Therefore, in the limit $K \to \infty$ we have

$$
\begin{aligned}
&\sum_{k_K=0}^{[n/K]} \sum_{k_{K-1}=0}^{[(n-Kk_K)/(K-1)]} \ldots \sum_{k_1=0}^{n-2k_2-\ldots-Kk_N} \delta(k_1 + 2k_2 + \ldots + Kk_K - n) \\
&= \sum_{k_n=0}^{1} \sum_{k_{n-1}=0}^{[(n-nk_n)/(n-1)]} \ldots \sum_{k_1=0}^{n-2k_2-\ldots-nk_n} \delta(k_1 + 2k_2 + \ldots + nk_n - n) \qquad (40)
\end{aligned}
$$

and thus finally

$$P(n) \;=\; \mathrm{e}^{b_0} \sum_{k_n=0}^{1} \sum_{k_{n-1}=0}^{[(n-nk_n)/(n-1)]} \cdots \sum_{k_1=0}^{n-2k_2-\ldots-nk_n} \delta(k_1+2k_2+\ldots+nk_n-n) \times \frac{(b_1)^{k_1}(b_2)^{k_2}\ldots(b_n)^{k_n}}{k_1!k_2!\ldots k_n!} \,. \tag{41}$$

These formulae complete the evaluation of the multiplicity distribution. Using (6) we find for the coincidence probabilities

$$C_l = \mathrm{e}^{b_0 l} \prod_{s=1}^{\infty} S_l(s) \,, \tag{42}$$

where

$$S_l(s) = \sum_{k=1}^{\infty} \frac{(b_s)^{lk}}{(k!)^l} \,. \tag{43}$$

Note that for $l = 2$ we have

$$S_2(s) = I_0(2b_s) \,. \tag{44}$$

5. Asymptotic limit of large density

The formulae of the previous section are useful as long as $\langle n \rangle$ is not too large. For large $\langle n \rangle$ the asymptotic formulae may be more appropriate and one can thus try to calculate the expansion coefficients directly using the Cauchy integral formula

$$P(n) = \frac{1}{n!} F^{(n)}(z)|_{z=0} = \frac{1}{2\pi i} \oint_{\Gamma} \frac{F(z)}{z^{n+1}} dz \,. \tag{45}$$

This integral is best calculated along a circle of radius r:

$$P(n) = \mathrm{e}^{a\hat{b}_0} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \exp\left[-in\phi - n\log r + a\sum_{k=1}^{\infty} b_k \mathrm{e}^{ik\phi}\right] \,. \tag{46}$$

To apply the saddle-point method, we denote

$$b_k = a\hat{b}_k \;\rightarrow\; \langle n \rangle = a\sum_k k\hat{b}_k \tag{47}$$

with $a \to \infty$ and $\hat{b}_k$ finite. The saddle-point method gives:

$$\begin{aligned} h(\phi) &= -in\phi - n\log r + a\sum_{k=1}^{\infty}\hat{b}_k r^k \mathrm{e}^{ik\phi}\,, \\ h'(\phi) &= -in + ia\sum_{k=1}^{\infty}k\hat{b}_k r^k \mathrm{e}^{ik\phi}\,, \\ h''(\phi) &= -a\sum_{k=1}^{\infty}k^2\hat{b}_k r^k \mathrm{e}^{ik\phi}\,. \end{aligned} \tag{48}$$

The condition $h'(\phi) = 0$ implies

$$a\sum_{k=1}^{\infty}k\hat{b}_k r^k \mathrm{e}^{ik\phi} = n\,, \tag{49}$$

and thus we obtain $\phi_0 = 0$, and r must be a solution of the equation

$$a\sum_{k=1}^{\infty}k\hat{b}_k r^k = n\,. \tag{50}$$

From (48) we deduce that the second derivative at $\phi = 0$ is

$$h''(0) = -a\sum_{k=1}^{\infty}k^2\hat{b}_k r^k\,, \tag{51}$$

and thus the saddle point method gives

$$P(n) = \mathrm{e}^{a\hat{b}_0}\frac{1}{2\pi}\mathrm{e}^{h(0)}\sqrt{\frac{2\pi}{-h''(0)}} = \frac{1}{\sqrt{2\pi a\sum_{k=1}^{\infty}k^2\hat{b}_k r^k}}\frac{1}{r^n}\mathrm{e}^{a\sum_{k=1}^{\infty}\hat{b}_k r^k}\,. \tag{52}$$

The condition (50) can be rewritten as

$$\sum_{k=1}^{\infty}k\hat{b}_k r^k = x\,, \quad x = \frac{n}{a}\,. \tag{53}$$

From which it follows that r is a function of x. And the expression (52) becomes

$$P(n) = \mathrm{e}^{a\hat{b}_0}\frac{\sqrt{x}}{\sqrt{2\pi n\sum_{k=1}^{\infty}r^k/k^2}}\frac{1}{r^n}\exp\left[\frac{n}{x}\sum_{k=1}^{\infty}r^k/k^4\right]\,. \tag{54}$$

The check of this procedure is to verify if $\sum P(n) = 1$.

We thus write

$$\sum_n P(n) \approx \mathrm{e}^{a\hat{b}_0} \frac{\sqrt{a}}{\sqrt{2\pi}} \int\limits_0^\infty dx \frac{1}{\sqrt{\sum_{k=1}^\infty k^2 \hat{b}_k r^k}} \frac{1}{r^{ax}} \mathrm{e}^{a\sum_{k=1}^\infty \hat{b}_k r^k} \tag{55}$$

with the condition

$$\sum_{k=1}^\infty k\hat{b}_k r^k = \frac{n}{a} \equiv x \;\rightarrow\; \frac{r'}{r} \sum_{k=1}^\infty k^2 \hat{b}_k r^k = 1\,. \tag{56}$$

We use the saddle point method:

$$\sum_n P(n) \approx \mathrm{e}^{a\hat{b}_0} \frac{\sqrt{a}}{\sqrt{2\pi}} \int\limits_0^\infty dx \frac{1}{\sqrt{\sum_{k=1}^\infty k^2 \hat{b}_k r^k}} \mathrm{e}^{g(x)} \tag{57}$$

with

$$\begin{aligned} g(x) &= -ax\log r + a\sum_{k=1}^\infty \hat{b}_k r^k\,, \\ g'(x) &= -a\log r - ax\frac{r'}{r} + a\frac{r'}{r}\sum_{k=1}^\infty k\hat{b}_k r^k = -a\log r\,, \end{aligned} \tag{58}$$

where in the last step we have used (56). The condition $g' = 0$ gives

$$a\log r = 0 \;\rightarrow\; r_0 \equiv r(x_0) = 1 \;\rightarrow\; x_0 = \sum_{k=1}^\infty k\hat{b}_k\,. \tag{59}$$

Furthermore

$$g''(x) = -a\frac{r'}{r} \;\rightarrow\; g''(x_0) = -a\frac{1}{\sum_{k=1}^\infty k^2 \hat{b}_k}\,. \tag{60}$$

Consequently the saddle point value of the integral (55) is

$$\mathrm{e}^{a\hat{b}_0} \frac{\sqrt{2\pi\sum_{k=1}^\infty k^2\hat{b}_k}}{\sqrt{a}} \frac{\sqrt{a}}{\sqrt{2\pi\sum_{k=1}^\infty k^2 \hat{b}_k}} \mathrm{e}^{a\sum_{k=1}^\infty \hat{b}_k} = \mathrm{e}^{a\hat{b}_0}\mathrm{e}^{a\sum_{k=1}^\infty \hat{b}_k} = 1\,. \tag{61}$$

Since this procedure does work, we are encouraged to calculate the Renyi entropies. First, coincidence coefficients:

$$C_l = \sum_n [P(n)]^l = \mathrm{e}^{la\hat{b}_0} \frac{a}{(2\pi a)^{l/2}} \int \frac{dx}{\left(\sum_{k=1}^{\infty} k^2 \hat{b}_k r^k\right)^{l/2}} \frac{1}{r^{alx}} \mathrm{e}^{al\sum_{k=1}^{\infty} \hat{b}_k r^k} . \quad (62)$$

The saddle point method can be used in a similar way as before. The result is

$$\begin{aligned} C_l &= \frac{(2\pi \sum_{k=1}^{\infty} k^2 \hat{b}_k)^{1/2}}{(al)^{1/2}} \frac{a}{(2\pi a \sum_{k=1}^{\infty} k^2 \hat{b}_k)^{l/2}} \\ &= \frac{1}{l^{1/2}} \frac{1}{\left(2\pi a \sum_{k=1}^{\infty} k^2 \hat{b}_k\right)^{(l-1)/2}} , \end{aligned} \quad (63)$$

and thus

$$\begin{aligned} H_l &= \frac{1}{2}\frac{\log l}{l-1} + \frac{1}{2}\log\left(2\pi a \sum_{k=1}^{\infty} k^2 \hat{b}_k\right) \\ &= \frac{1}{2}\frac{\log l}{l-1} + \frac{1}{2}\log\left(2\pi \langle n\rangle \frac{\sum_{k=1}^{\infty} k^2 \hat{b}_k}{\sum_{k=1}^{\infty} k\hat{b}_k}\right) . \end{aligned} \quad (64)$$

This is actually a general formula for any ideal gas. For the photon gas we obtain

$$H_l = \frac{1}{2}\frac{\log l}{l-1} + \frac{1}{2}\log\left(2\pi\langle n\rangle \frac{\zeta(2)}{\zeta(3)}\right) . \quad (65)$$

The extrapolation to $l = 1$ gives

$$H_1 \equiv S = \frac{1}{2}\left[1 + \log\left(2\pi\langle n\rangle\frac{\zeta(2)}{\zeta(3)}\right)\right] , \quad (66)$$

which is to be compared with standard expression $S = \langle n\rangle\zeta(4)/\zeta(3)$ (see the last position of Ref. [1]). Remember that this last expression is obtained when $\Delta \to \infty$ and fluctuations occur on the scale of "elementary" cells.

The most important conclusion from this analysis is that the Renyi entropies obtained from these measurements depend only logarithmically on the number of particles.

6. Comparison with exact formulae from statistical physics

It is interesting to confront our results with those obtained in standard statistical physics approach. We only consider the case of Bose gas.

The l-th Renyi entropy of the Bose gas is

$$H_l = \frac{1}{1-l} \sum_i \log \frac{(1-u_i)^l}{1-u_i^l} \tag{67}$$

with u_i given by (12). Furthermore, the average number of particles can be expressed as

$$\langle n \rangle = \sum_i \frac{u_i}{1-u_i} \,. \tag{68}$$

In the limit of very small interval Δ the sum in (67) and (68) reduce to one single term and thus the elimination of u from $C_l = (1-u)^l/(1-u^l)$ and $\langle n \rangle = u/(1-u)$ leads immediately to Eq. (18).

In the case of a large interval Δ, the result is, however, markedly different. The reason is that the statistical physics calculation takes into account all fluctuations which may occur in the system, whereas the coincidence measurements ignore fluctuations on the scale smaller than Δ. This can be directly demonstrated using Eqs. (67) and (68). Changing sums into integrals we obtain

$$\langle n(\Delta) \rangle = \frac{V}{(2\pi)^3} \int d^3p \frac{u(p)}{1-u(p)} = \sum_{k=1}^{\infty} k b_k \tag{69}$$

and

$$\begin{aligned} H_l(\Delta) &= \frac{1}{1-l} \frac{V}{(2\pi)^3} \int\limits_{\Delta} d^3p \log \frac{(1-u(p))^l}{1-u(p)^l} \\ &= \frac{1}{1-l} \frac{V}{(2\pi)^3} \int_{\Delta} d^3p \left[-l \sum_{k=1}^{\infty} \frac{u(p)^k}{k} + \sum_{k=1}^{\infty} \frac{u(p)^{kl}}{k} \right] \\ &= \frac{l}{l-1} \sum_{k=1}^{\infty} \left[b_k - b_{kl} \right] \,. \end{aligned} \tag{70}$$

One sees that this formula is very different from those obtained in Section 4.

The case of the photon gas can be discussed explicitely. Then we have $\mu = 0$ and thus

$$\begin{aligned} b_k &= \frac{V}{(2\pi)^3 k}\int\limits_{\Delta} d^3p\mathrm{e}^{-k\beta p} = \frac{V}{2\pi^2 k}\int\limits_0^{\Delta} p^2 dp\mathrm{e}^{-k\beta p} \\ &= \frac{V}{\pi^2\beta^3 k^4}\left[1-\mathrm{e}^{-kd}\left(1+kd+\frac{(kd)^2}{2}\right)\right], \end{aligned} \tag{71}$$

where $d \equiv \Delta\beta$. Using this we obtain from (70)

$$\langle n\rangle = \frac{V}{\pi^2\beta^3}\left[\zeta(3) - g_3(d) - dg_2(d) + \frac{d^2}{2}\log\left(1-\mathrm{e}^{-d}\right)\right], \tag{72}$$

and

$$H_l = \frac{l}{l-1}\frac{V}{\pi^2\beta^3}\left[G(d) - \frac{1}{l^4}G(ld)\right] \tag{73}$$

with

$$G(x) = \zeta(4) - g_4(x) - xg_3(x) - \frac{x^2}{2}g_2(x)\,, \quad g_p(x) = \sum_{k=1}^{\infty}\frac{\mathrm{e}^{-kx}}{k^p}\,. \tag{74}$$

The limit of high average number of particles, $\langle n\rangle$, can be achieved either by taking a large volume V or by taking the high temperature ($\beta \to 0$). In the limit $V \to \infty$ and fixed temperature one obtains the standard thermodynamical expressions.

To obtain a relation between H_l and $\langle n\rangle$ in the high temperature limit ($\Delta\beta \to 0$) from (72) and (73) is however very tricky. It is much simpler to deal directly with the integrals (69) and (70). Indeed, the leading term for $\langle n(\Delta)\rangle$ (remember that also $\Delta p \ll 1$) is

$$\langle n(\Delta)\rangle = \frac{V}{(2\pi)^3}\int\limits_{\Delta} d^3p\frac{\mathrm{e}^{-\beta p}}{1-\mathrm{e}^{-\beta p}} \approx \frac{V}{2\pi^2}\int\limits_0^{\Delta} dpp^2\frac{1}{\beta p} = \frac{V}{\beta^3}\frac{1}{(2\pi)^2}(\beta\Delta)^2\,, \tag{75}$$

and for $H_l(\Delta)$

$$\begin{aligned} H_l(\Delta) &= \frac{1}{1-l}\frac{V}{(2\pi)^3}\int\limits_{\Delta} d^3p\log\frac{(1-\mathrm{e}^{-\beta p})^l}{1-\mathrm{e}^{-l\beta p}} \approx \frac{V}{2\pi^2}\frac{1}{1-l}\int\limits_0^{\Delta} dpp^2\log\frac{(\beta p)^l}{l\beta p} \\ &= \frac{V}{2\pi^2}\frac{1}{\beta^3}(\beta\Delta)^3\left[\frac{\log l}{l-1} - \log\beta\Delta + \frac{1}{3}\right]. \end{aligned} \tag{76}$$

But, in this limit, we also have $-\log \beta\Delta \gg 1/3$. Thus from (75) and (76) we get

$$\langle n(\Delta)\rangle \approx \frac{\sigma}{(2\pi)^2}\frac{1}{\beta\Delta} , \tag{77}$$

and

$$H_l(\Delta) \approx \frac{\sigma}{2\pi^2}\left[\frac{\log l}{l-1} + \log\left((2\pi)^2\frac{\langle n(\Delta)\rangle}{\sigma}\right)\right], \tag{78}$$

where $\sigma = V\Delta^3$. This limiting formula is different from the one obtained for high density of photons *with coincidences calculated for the large interval* Δ (65). So, the fluctuations on a scale much smaller than Δ — as is the case in the standard statistical physics — do introduce important corrections.

7. Conclusions

We have given several examples of how a discretization procedure affects the coincidence measurements performed for ideal gases.

We find that — as expected — the larger are the intervals of discretization the less fluctuations can be observed, which result in larger coincidence probabilities.

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REFERENCES

[1] A. Bialas, W. Czyz, J. Wosiek, *Acta Phys. Pol.* **B30**, 107 (1999); A. Bialas, W. Czyz, *Phys. Rev.* **D61**, 074021 (2000); A. Bialas, W. Czyz, *Acta Phys. Pol.* **B31**, 687 (2000); A. Bialas, W. Czyz, *Acta Phys. Pol.* **B32**, 2793 (2001); A. Bialas, W. Czyz, A. Ostruszka, *Acta Phys. Pol.* **B34**, 69 (2003); A. Bialas, W. Czyz, *Acta Phys. Pol.* **B31**, 2803 (2000).

[2] S.K. Ma, *Statistical Mechanics*, World Scientific, Singapore 1985; S.K. Ma, *J. Stat. Phys.* **26**, 221 (1981).

[3] A. Renyi, it On Measures of Entropy and Information, in Proc. 4th Berkeley Symp. on Math. Stat. Prob. 1960, Vol.1, University of California Press, Berkeley, Los Angeles 1961, p. 547.