

RELATION BETWEEN WIGNER FUNCTIONS AND THE SOURCE FUNCTIONS USED IN THE DESCRIPTION OF BOSE–EINSTEIN CORRELATIONS IN MULTIPLE PARTICLE PRODUCTION

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Relations between Wigner functions and the source functions used in models of Bose–Einstein correlations in multiple particle production are derived and discussed. These relations are model dependent. In particular it is important whether the particles are emitted simultaneously and if not, whether the production amplitudes corresponding to different moments of time can interfere with each other.

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1. Introduction

When discussing Bose–Einstein correlations among identical pions¹ produced in multiple particle production processes, one often uses the source function $S(X, K)$ (*cf. e.g.* [1] and references quoted there) related to the single particle density matrix in the momentum representation by the for-

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¹ Here and in the following we call for definiteness the identical bosons pions the results, however, are valid for any identical bosons.

mula²

$$\rho(\mathbf{p}, \mathbf{p}') = \int d^4 X e^{iqX} S(X, K), \quad (1)$$

where $q = p - p'$ and $K = (p + p')/2$ are four-vectors constructed from the on-shell single particle four momenta p and p' . The four components of X are integration variables and, therefore, there is much freedom in their interpretation. It is usual, however, to identify $S(X, K)$ with the space-time distribution of the sources producing the pions with momentum K and then X is interpreted as the space-time position of the source. A source functions combines conveniently the information and/or prejudice about the space-time (X) distribution of the sources and about the momentum distribution of the produced particles. It yields the density matrix, which can be used to get the (two- as well as more particle) correlation function. Comparing the predicted correlation functions with experiment one can improve the model used to find $S(X, K)$, fix its parameters *etc.* Note that the density matrix on the left hand side of formula (1) does not depend on time. Since after freeze-out the particles propagate freely (we do not discuss the final state interactions here), this means that for times after the freeze-out period the elements of the density matrix should be interpreted as the matrix elements of the time dependent (Schrödinger picture) density operator between the time dependent states $|\mathbf{p}, t\rangle = e^{i\hat{E}(\mathbf{p})t}|\mathbf{p}\rangle$, or equivalently as the matrix element of the time independent (interaction picture) density operator between the time independent states $|\mathbf{p}\rangle$.

Formula (1) looks similar to the formula relating the Wigner function $W(\mathbf{X}, \mathbf{K})$ (*cf. e.g.* [2] and references contained there) to the density matrix:

$$\rho(\mathbf{p}, \mathbf{p}', t) = \int d^3 X e^{-iq \cdot \mathbf{X}} W(\mathbf{X}, \mathbf{K}, t). \quad (2)$$

In fact the building block of the source function was originally a Wigner function (*cf.* [3] formula (7)). Because of the similarity between formulae (1) and (2), the source function is often referred to as a Wigner function, a kind of Wigner function, a pseudo-Wigner function *etc.* (*cf. e.g.* Ref. [4], from which all these names have been taken). A discussion of the actual relation between the source function and the Wigner functions, however, seems to be missing in the literature. Let us begin with some general remarks.

Formula (2) contains the time dependent density matrix, *i.e.* the time dependent density operator in the representation of the time independent

² Often an equivalent formula with the density matrix replaced by the average $\langle \hat{a}_p^\dagger \hat{a}_{p'} \rangle = \rho(\mathbf{p}', \mathbf{p})$ is used. Also some authors include on the left hand side a factor $\sqrt{EE'}$. Since this factor does not affect our argument and is easy to introduce at any stage, we skip it.

states $|\mathbf{p}\rangle$. Thus

$$\rho(\mathbf{p}, \mathbf{p}', t) = e^{-iq_0 t} \rho(\mathbf{p}, \mathbf{p}'). \quad (3)$$

Relation (1) follows from (2), if we put

$$S(X, K) = W(\mathbf{X}, \mathbf{K}, t_0) \delta(X_0 - t_0). \quad (4)$$

This choice can be always made and has a simple physical interpretation. Choose a time t_0 after freeze-out, when all the pions are already present, but none have been registered yet. The state at t_0 yields the initial condition from which any distribution at registration can be calculated. Assuming that all the pions appeared simultaneously at t_0 may be poor physics, but this does not change the fact that the predictions at registration time are correct. There is an infinite choice of other source functions, which also satisfy relation (1). Since this relation is the only link between a source function and physical reality, there can be no objection of principle against the choice (4). From the practical point of view, however, another source function, which can be calculated from some model, may be more convenient and, of course, it is just as good, if it reproduces the density matrix equally well. Note that the identity of functions in (4) does not mean that their argument have the same physical interpretation. In fact, on the left hand side X and K denote the position of the source in space-time and the four-momentum of the pion, while on the right hand side $\mathbf{X} = \frac{1}{2}(\mathbf{x} + \mathbf{x}')$ and $\mathbf{K} = \frac{1}{2}(\mathbf{p} + \mathbf{p}')$ are defined by reference to the arguments of the density matrices of the pion in the coordinate representation and in the momentum representation respectively, while t_0 is the time. It is often reasonable to assume that $\mathbf{X}$ and $\mathbf{K}$ on both sides are good approximations to the position at $t = t_0$ and momentum for $t \geq t_0$ of the pion, but this is certainly not always the case.

Let us conclude this section with some more remarks. If one assumes that all the particles were created simultaneously at some common freeze-out time $t = t_0$, then a "realistic" source function should be proportional to $\delta(X_0 - t_0)$ and the proportionality coefficient is the Wigner function as in (4). The source function cannot be equal to a Wigner function, because the dimensions are different. The delta function in formula (4) brings into the dimension the necessary factor 1/time. What could be equal to the source function is what we will refer to as the differential Wigner function defined by the relation

$$\tilde{W}(\mathbf{X}, \mathbf{K}, t_0) = \frac{W_{dt_0}(\mathbf{X}, \mathbf{K}, t_0)}{dt_0}. \quad (5)$$

Here W_{dt_0} is the Wigner function at time $t > t_0$ for the particles from sources created in the time interval dt_0 around time t_0 . Note that we are

using the Wigner functions normalized to the numbers of particles and not to one. When the amplitudes of the pions produced at different times add incoherently, formula (2) can be rewritten in terms of the differential Wigner function $\tilde{W}$ as:

$$\rho(p, p') = \int d^4X e^{iqX} \tilde{W}(X, \mathbf{K}), \quad (6)$$

where the fourth component of the vector $X = (\mathbf{X}, X_0)$ is t_0 . Comparing this formula with formula (1) we find a particular solution for the source function

$$S(X, K) = \tilde{W}(X, \mathbf{K}). \quad (7)$$

We stress that this is only one out of the infinity of source functions, which when substituted into formula (1) yield the correct density matrix and that for the sources incoherence in the creation time has been assumed.

2. GGLP models

In the seminal paper of the Goldhabers, Lee and Pais [5] the assumption was made that pion production amplitudes at different space points are incoherent. In this simple case there is no need to distinguish between the sources and the pions at $t = t_0$. A natural generalization (*cf. e.g.* [6]) was to assume more generally that pions production amplitudes at different space-time points are incoherent. Further we refer to such models as GGLP models.

Let us begin with the simpler case, when all the pions appear simultaneously at $t = t_0$. For $t > t_0$ they are assumed to propagate freely. The time independent density matrix for $t > t_0$ is:

$$\rho(\mathbf{p}, \mathbf{p}') = \int d^3x \langle \mathbf{p}, t | \mathbf{x}, t; t_0 \rangle \rho(\mathbf{x}, t_0) \langle \mathbf{x}, t; t_0 | \mathbf{p}' t \rangle. \quad (8)$$

Here $|\mathbf{x}, t; t_0\rangle = \exp[-i(t - t_0)\hat{H}_0]|\mathbf{x}\rangle$ is the time dependent state, evolving according to the free particle Hamiltonian $\hat{H}_0$ from the initial state $|\mathbf{x}\rangle$, which corresponds to a pion localized at $\mathbf{x}$ at time t_0 . One could also say that in space-time the position of the source is $(\mathbf{x}, t_0)$. Let us note for further use the formula

$$\langle \mathbf{p}, t | \mathbf{x}, t; t_0 \rangle = e^{iE(\mathbf{p})t_0} \langle \mathbf{p} | \mathbf{x} \rangle, \quad (9)$$

where $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_\pi^2}$. Formula (8) is perfectly acceptable from the point of view of quantum mechanics, in spite of the fact that according to the picture described in Ref. [5] a pion with momentum $\mathbf{p}$ is produced at point $\mathbf{x}$, which would contradict the uncertainty principle. Since the density matrix is diagonal in $\mathbf{x}$ (in the $\mathbf{x}$, which labels the incoherent states, not

in position at time t) and $t = t' = t_0$, we can identify the components of the vector $\mathbf{x}$ with X_1, X_2, X_3 and the parameter t_0 with X_0 . The matrix element (8) can be rewritten as

$$\rho(\mathbf{p}, \mathbf{p}') = \int \frac{d^3 X}{(2\pi)^3} \exp(iqX) \rho(X). \quad (10)$$

Using (3) and comparing with (2) this yields

$$W(X, \mathbf{K}) = \frac{\rho(X)}{(2\pi)^3}. \quad (11)$$

The source function is given by formula (4), because by assumption all the particle production happens at $t = t_0$. The components of X are the three components of the position vector of the point where the particle was produced and its time of production (freeze-out time). Since after freeze-out the momentum vector of each particle is conserved separately, the momenta entering $\mathbf{K}$ can be taken at any time not earlier than t_0 . The actual time of observation does not occur in the formula at all.

Note that for this class of models the Wigner function does not depend on momenta. With the present interpretation this would lead to disaster when comparing with experiment. The Goldhabers, Lee and Pais [5] proceeded differently. They were interested in n -pion exclusive states for $n = 4, 5, 6$ with no more than two pions of a given charge. Using matrix (8) they constructed the probability distributions $P(p_1, \dots, p_n)$. These were interpreted as momentum distributions only after the sets of momenta contradicting energy and/or momentum conservation were projected out. In practice they interpreted the P -s as integrands of the corresponding phase space integrals. This could not be generalized to inclusive processes, led to rather complicated integrations and gave results, which at that time were considered acceptable, but which soon were shown to be in violent disagreement with experiment [7]. Most of these difficulties were overcome by Kopylov, Podgoretsky and coworkers as described in the next section.

Let us consider now the more general case, when the pions are produced at different moments of time, but there is no interference between the amplitudes corresponding to production at different moments. Then at times larger than the largest freeze-out time the density matrix is given by an integral over the freeze-out time t_0 of the (differential version of the) density matrix (10) corresponding to particles produced at given time t_0 :

$$\rho(\mathbf{p}, \mathbf{p}') = \int \frac{d^4 X}{(2\pi)^3} \exp(iqX) \tilde{\rho}(X). \quad (12)$$

Note that in order to keep the dimensions right $\rho(X)$, which is a density in space, has been replaced by $\tilde{\rho}(X)$ which is a density in space time. In

fact, $\rho(X)$ are diagonal elements of the time independent single pion density matrix in the $|\mathbf{x}, t; t_0\rangle$ representation, while $\tilde{\rho}(X)$ are the diagonal elements of the corresponding differential density matrix related to $\rho(X)$ by a formula analogous to (5). Comparing with (1) it is seen that one solution for $S(X, K)$ is

$$S(X, K) = \frac{\tilde{\rho}(X)}{(2\pi)^3}. \quad (13)$$

Since there is incoherence in time, the relation between the source function and Wigner's function is $S(X, K) = \tilde{W}(X, \mathbf{K})$ (7).

This seems to be the perfect situation: the source function reproduces exactly the distribution of (incoherent) sources in space-time and coincides with the differential Wigner function. However, (13) is only one of the infinity of source functions, which yield the same density matrix. A quite different one can be obtained *e.g.* by calculating the Wigner function corresponding to the density matrix $\rho(\mathbf{p}_1, \mathbf{p}_2, t)$ and substituting it into formula (4). We can conclude only that for this class of (unrealistic) models there is an acceptable source function, which can be interpreted as a differential Wigner function. Therefore, it can be argued that this particular source function is easier to find than others.

3. Including coherence in space

An important extension of the GGLP approach is to assume that at any given creation time there may be coherence between the amplitudes of the pions produced at different points in space, though pions produced at different times t_0 do not interfere. Models of this type, also in the more general version with coherence in time, have been introduced by Kopylov and Podgoretsky [8]. In terms of sources this may be quite complicated. For instance, in the model described by Kopylov and Podgoretsky [8] all the sources are produced simultaneously and live for some time. The averaging over the energy of each source (not of the pion!), however, kills the interference between the amplitudes for the production of the pion by the same source at different moments of time. Therefore one can replace each original source by a set of sources distributed in time and producing the pions instantly. These are not quite satisfactory models, because it is difficult to explain why the creation amplitudes from neighboring time moments should not interfere, while those from neighboring space points do. Moreover, relativistically the creation processes which are at different times in one Lorentz frame may be simultaneous in another Lorentz frame. As seen from formula (4), however, even in the simplest case of simultaneous production, such models can reproduce correctly any single particle density matrix and con-

sequently any single particle momentum distribution without doing phase space integrations. Thus they can be applied to describe inclusive processes.

Including interference in space and summing without interference over time corresponds to the following generalization of formula (8)

$$\rho(\mathbf{p}, \mathbf{p}') = \int dt_0 \int d^3x \int d^3x' \langle \mathbf{p}, t | \mathbf{x}, t; t_0 \rangle \tilde{\rho}(\mathbf{x}, \mathbf{x}', t_0) \langle \mathbf{x}', t; t_0 | \mathbf{p}', t \rangle. \quad (14)$$

There are many equivalent ways of rewriting this relation. For instance, one may use a representation $|\alpha\rangle$, where the (differential) density matrix $\tilde{\rho}(\mathbf{x}, \mathbf{x}', t_0)$ is diagonal. Then (14) gets replaced by

$$\rho(\mathbf{p}, \mathbf{p}') = \int dt_0 \int d\alpha e^{iq_0 t_0} \langle \mathbf{p} | \alpha \rangle \tilde{\rho}(\alpha; t_0) \langle \alpha | \mathbf{p}' \rangle. \quad (15)$$

It is instructive to compare this model, to the GGLP models. At t_0 the localized states $|\mathbf{x}\rangle = |\mathbf{x}, t_0; t_0\rangle$ have been replaced by the states $|\alpha\rangle = |\alpha, t_0; t_0\rangle$. The plane waves $\langle \mathbf{p} | \mathbf{x} \rangle$ have been replaced by the functions $\langle \mathbf{p} | \alpha \rangle$, which are known under a variety of names: as waved packets, as sources, or as currents. The source function for such models was introduced by Pratt [3]. His statement that³ “ $S(X, \mathbf{K})$ can be identified as the probability of emitting a pion of momentum $\mathbf{K}$ from space–time point X ” is, however, only an approximation.

Kopylov and Podgoretsky [8] assumed that (α, t_0) is a point in space–time, so that formula (15) can be rewritten as

$$\rho(\mathbf{p}, \mathbf{p}') = \int d^4x_0 \langle \mathbf{p} | \psi_{x_0} \rangle \tilde{\rho}(x_0) \langle \psi_{x_0} | \mathbf{p}' \rangle e^{iq_0 t_0}. \quad (16)$$

This approach yields an alternative method of describing the distribution of pions in phase space. In the integrand, $\langle \mathbf{p} | \psi_{x_0} \rangle$ is the probability amplitude for finding a pion with momentum $\mathbf{p}$ produced by a source labelled x_0 . When the states $|\psi_{x_0}\rangle$ correspond to particles well localized in space, $\mathbf{p}$ and $\mathbf{x}_0$ give a reasonably good description of the position of the pion in phase space. One could object that this is only a rough description, but the same is true for the Wigner function: $\mathbf{K}$ and $\mathbf{X}$ give only approximately the momentum and position of the pion. An exact determination of the pion position in phase space is possible only in classical physics.

In Ref. [8] the states $|\alpha\rangle = |\psi_{x_0}\rangle$ were supposed to be related by space time translations so that

$$\psi_{x_0}(x) \equiv \langle \mathbf{x} | \psi_{x_0} \rangle = \psi(x - x_0), \quad (17)$$

$$\langle \mathbf{p} | \psi_{x_0} \rangle = e^{ipx_0} \phi(\mathbf{p}), \quad \phi(\mathbf{p}) = \langle \mathbf{p} | \psi_0 \rangle. \quad (18)$$

³ This has been changed to our notation. Pratt has written $g, \vec{p}, x$ where we have written $S, \mathbf{K}, X$.

The second formula is, of course, equivalent to the first. Another choice has been made in the “covariant current ensemble formalism” [10–13]. There each source is labelled by a position in space–time x_0 and a four-momentum p_0 . Usually $\mathbf{x}_0$ and $\mathbf{p}_0$ denote the centers of the wave packet in ordinary space and in momentum space, respectively. The time component of x_0 is t_0 — the freeze-out time of the wave packet. The time component of p_0 is calculated from the condition $p_0^2 = m^2$, where m can [13], but does not have to [4] be the pion mass. The nice feature of this parametrization of the wave packets is that one can substitute a classical trajectory $\mathbf{x}_0(t), \mathbf{p}_0(t)$ for the source and remain in agreement with the Heisenberg uncertainty principle for the pions. The choice for the scalar products is

$$\langle \mathbf{p} | \psi_{x_0, p_0} \rangle = e^{ipx_0} j \left(\frac{pp_0}{m} \right). \quad (19)$$

Thus, the sources are related by space–time translations and when each of the currents j is considered in its rest frame where $p_0 = (m, 0)$, they are identical. Formulae (17) and (19) have been applied also to relativistically covariant models *cf. e.g.* [8], [12].

Another way of rewriting relation (14) is

$$\rho(\mathbf{p}, \mathbf{p}') = \int d^4 X \exp(iqX) \int \frac{d^3 y}{(2\pi)^3} \exp(-i\mathbf{K} \cdot \mathbf{y}) \tilde{\rho}(\mathbf{x}, \mathbf{x}', t_0), \quad (20)$$

where $\mathbf{X} = (\mathbf{x} + \mathbf{x}')/2$, $\mathbf{y} = \mathbf{x} - \mathbf{x}'$ and $X_0 = t_0$. According to formula (1) a possible choice of the source function is

$$S(X, K) = \int \frac{d^3 y}{(2\pi)^3} \exp(-i\mathbf{K} \cdot \mathbf{y}) \tilde{\rho}(\mathbf{x}, \mathbf{x}', t_0). \quad (21)$$

The differential Wigner function is the Wigner transform of the differential density matrix as it should. The relation of this particular source function to the Wigner function is again given by formula (7). We conclude that among the infinitely many source functions which give the same density matrix in the momentum representation there is one, which can be related to the Wigner function as described. Note that this source function depends on $\mathbf{K}$, but does not depend on K_0 .

4. Including coherence in space and time

Finally let us consider the case, when neither coherence in space, nor coherence in time is assumed. Then the density matrix is

$$\rho(\mathbf{p}, \mathbf{p}') = \int d^4 x \int d^4 x' \langle \mathbf{p}, t | \mathbf{x}, t; t_0 \rangle \tilde{\rho}(x, x') \langle \mathbf{x}', t; t'_0 | \mathbf{p}', t \rangle, \quad (22)$$

where $x = (\mathbf{x}, t_0)$ and $x' = (\mathbf{x}', t'_0)$. For an application of a formula of this type see *e.g.* Ref. [9], where $\sum_F M_F(x)M_F(x')$ stand for our $\tilde{\rho}(x, x')$.

Formula (22) can be rewritten as

$$\rho(\mathbf{p}, \mathbf{p}') = \int d^4X \int d^4y \exp[iqX + iKy] \tilde{\rho}(x, x') / (2\pi)^3, \quad (23)$$

where $X = \frac{1}{2}(x + x')$ and $y = x - x'$. Comparison with formula (1) gives as one of the solutions for the source function

$$S(X, K) = \int \frac{d^4y}{(2\pi)^3} \exp[iKy] \tilde{\rho}(x, x'). \quad (24)$$

The right hand side is rather remote from what one usually calls a Wigner function. The differential Wigner function was at least proportional to a Wigner function, though only for the particles from sources produced in the time interval dt_0 around the time $t = t_0$. Function (24) is proportional to the contribution to the Wigner function from the interference of the production amplitude in the time interval dt_0 around $t = t_0$ with the production amplitude in the time interval dt'_0 around $t = t'_0$, integrated over $t'_0 - t_0$ at fixed $t'_0 + t_0$.

Formula (22) rewritten in terms of wave packets reads

$$\rho(\mathbf{p}, \mathbf{p}') = \int dt_0 \int dt'_0 \int d\alpha \langle \mathbf{p}, t | \alpha, t; t_0 \rangle \tilde{\rho}(\alpha, t_0, t'_0) \langle \alpha, t; t'_0 | \mathbf{p}', t \rangle. \quad (25)$$

Note that the state $|\alpha, t; t_0\rangle$ may, but does not have to, be connected to the state $|\alpha, t; t'_0\rangle$ by a smooth Hamiltonian evolution. The source at time t_0 can be, as well, something quite different from the source at time t'_0 . On the other hand, the evolution of $|\alpha, t; t_0\rangle$ in the time t for t later than the freeze-out of the particle is the ordinary free particle evolution. The various models are defined by the choice of α , of the states $|\alpha, t; t_0\rangle$ and of the weight function $\tilde{\rho}(\alpha, t_0, t'_0)$. Examples can be found in Refs. [4, 8, 12].

5. Conclusions

A source function cannot be equal to a Wigner function, because they have different dimensions. Moreover, for a given state of the system its Wigner function is well defined, while its source function is not. The problem is, therefore, to chose some special source function and try to relate it somehow to a Wigner function.

When all the particles are produced simultaneously and when this is assumed to mean that the source function is proportional to a delta function in time, the proportionality coefficient is unambiguously defined as the Wigner

function of the pions at the time of freeze-out (4). When it is assumed that the production amplitudes at different times add incoherently, one can use the source function proportional to the differential Wigner function as given in (13). When the production amplitudes at different times interfere, a source function can be related to a piece of the Wigner function as given by formula (24) and explained below this formula. In this case, however, the use of wave packets (or sources, or currents) may be a more natural way to analyze the phase space distribution of pions.

If, as is often done, a model is defined by postulating the source function (*cf.* [1] for examples), the question about the relation of this source function to a Wigner function cannot be answered without making an assumption about the coherence or incoherence of the amplitudes for pion production at different times. When the source function is proportional to a delta function in time, one can relate it to a Wigner function by relation (4). When it does not depend on K_0 , one can use formula (7). In the later case there is no guarantee that the production process was such as the differential Wigner functions suggest.

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