Vol. 34 (2003) *ACTA PHYSICA POLONICA B* No 7

SPONTANEOUS FORMATION OF SPACE–TIME STRUCTURES IN PROBABILISTIC CELLULAR AUTOMATA*

D. Makowiec

Institute of Theoretical Physics and Astrophysics, Gdańsk University
80-952 Gdańsk, Wita Stwosza 57, Poland
e-mail: fizdm@univ.gda.pl

(Received October 15, 2002)

The cluster structure of Toom North-East-Center (NEC) voting rule in probabilistic cellular automata stationary states is analyzed. Such structure has its origin in both geometrical connectivity and Toom interactions. The difference between percolation threshold and ferromagnetic phase transition is determined. The value of this difference depends on the way in which NEC rule is applied: synchronous or asynchronous.

PACS numbers: 05.50.+q, 05.70.Jk

1. Motivation

1.1. *PCA versus EMS*

Stochastic cellular automata lead directly to thermodynamics systems in which self-organization of elements to phase transition can be studied. Even one-dimensional cellular automata can serve examples of complex behavior such as *e.g.* the absorbing phase transition [1–3]. Here we concentrate on how probabilistic cellular automata (PCA) model equilibrium statistical mechanics (EMS). Details on this subject one can find in papers of Domany [4], Bennet *et al.*, [5], Lebowitz *et al.*, [6] or Bigelis *et al.*, [7].

Let $\Lambda \subset Z^d$ is a finite cube that contains points of Z^d lattice and has periodic boundary conditions. At each site $i \in \Lambda$ there is a spin variable $\sigma_i = \pm 1$. Hence the configuration space contains $\sigma \in \{-1, 1\}^{\Lambda}$. The dynamics is performed in discrete time steps. All spins are updated synchronously and

* Presented at the XV Marian Smoluchowski Symposium on Statistical Physics, Zakopane, Poland, September 7–12, 2002.

independently at every time unit. The conditional probability that a spin at site i takes the value σ_i' on a given configuration σ is

$$P_i(\sigma_i'|\sigma) = \frac{1}{2}\left[1 + \tanh \varepsilon \sigma_i' h_i(\sigma)\right] , \tag{1}$$

where ε plays the role of the inverse of temperature and h_i is finite range function describing local couplings. Thus the time evolution is a Markov chain on the configuration space with non-zero transition probabilities:

$$P_\Lambda(\eta|\sigma) = \Pi_{i\in\Lambda} P_i(\eta_i|\sigma) .$$

Notice that if ε is large, what means the temperature is close to zero, then $\sigma_i = \text{sign } h_i(\sigma)$ with high probability.

By the general theory of Markov processes, for any ε and Λ there exists a unique stationary measure ν_Λ^ε for PCA. We say that PCA are reversible with respect to a measure ρ iff

$$P_\Lambda(\eta|\sigma)\rho(\sigma) = P_\Lambda(\sigma|\eta)\rho(\eta)$$

for any lattice configurations η, σ. Notice that any measure satisfying the reversibility condition is stationary for PCA. The opposite statement does not need to be true. It is easy to check that if $h(\sigma)$ satisfies some symmetry conditions, the time evolution is reversible with respect to the following measure:

$$\nu_\Lambda^\varepsilon = \frac{1}{Z} \Pi_{i\in\Lambda} \cosh \varepsilon h_i(\sigma)$$

and this measure is a Gibbs measure for the Hamiltonian $H_\Lambda(\sigma) = \ln \nu_\Lambda^\varepsilon$. Let us close with a remark that stationary measures for infinite volume PCA need no longer be unique.

1.2. Why study Toom PCA

Toom's North-East-Center (NEC in short) voting model is one of the simplest cellular automata that are nonergodic and irreversible [5, 6, 8, 9]. The model consists of a square lattice of spins, each of which can be up or down. A spin's future state is decided by a majority vote of spins from a special, because *unsymmetric* neighborhood. This neighborhood consists of the spin itself C_i and its northern N_i and eastern E_i neighbors. The deterministic vote is perturbed by a stochastic noise ε. The stochastic noise is measured in such a way that ε can have a physical interpretation of the inverse of temperature. The dynamics of Toom PCA is driven by the following conditional probability:

$$P(C_i'|(N_i, E_i, C_i)) = \frac{1}{2}\left[1 + \varepsilon C_i' \text{sign}(N_i + E_i + C_i)\right] . \tag{2}$$

The above formula can be seen as the approximation of the low temperature of the equilibrium dynamics of (1), namely

$$P(C_i'|(N_i, E_i, C_i)) = \frac{1}{2}\left[1 + \tanh \varepsilon C_i'(N_i + E_i + C_i)\right] .$$

If ε is close to 1 but different from 1 then all local transitions are allowed. Therefore, on the $\Lambda = L \times L$ torus the model is ergodic — it leads to a unique stationary distribution. However, Toom showed [8] that for ε sufficiently close to 1, the transition rate between the mostly up and mostly down states of entire system tends to zero with increasing L. Thus the infinite system is nonergodic with two stable phases like in the conventional Ising model of ferromagnetic interactions below its critical temperature. If ε is far from 1, namely the temperature is high, the system is ergodic. Therefore, when studying stochastic Toom NEC system we can, similarly to the Ising model, observe the transition between these two limit cases.

The important motivation to study Toom PCA is that Toom PCA are not reversible with respect to their stationary measure. Hence there is not a direct correspondence described in the previous subsection between these PCA and ESM. The irreversibility of Toom PCA arises from the unsymmetric neighborhood. If the Toom system was modified by adding an extra sub step to the dynamics: at each time step after Toom NEC voting, the majority vote of South, West and a Center spins is performed, then such a two step PCA dynamics would be reversible with respect to the stationary measure [4]. However, there are hints that not only reversible PCA lead to ESM [9]. Toom PCA are the simplest stochastic system in which this hypothesis could be tested.

1.3. Toom PCA versus Ising model

The set of stable configurations makes the key difference between the Toom PCA and the Ising model. The stable configurations set consists of configurations that are invariant with respect to the deterministic dynamics. In the Ising system finite size islands of opposite phase are stable. In the Toom system only infinite size objects are stable [10]. However, if the noise is present, the finite islands can persist to live in a Toom stationary state.

The Toom system looses its nonergodicity in the way similar to the Ising system, *i.e.* by passing the continuous phase transition [5]. The numerical study indicates that stationary state of the finite size system at the transition point is not Gibbsian [11]. Hence there is no proper meaning of energy in the Toom NEC system of the finite size.

Though the Toom PCA do not model any ESM system, the following question can be posed: whether the phase transition belongs to the Ising universality class. Details of the research on this subject are presented in [12].

It occurs that if the NEC rule is applied in an asynchronous way then the transition belongs to the Ising universality class. Toom PCA with synchronous updating falls into the weak Ising universality class. Close to the transition the two-point correlation function decays slower than in the Ising system. The critical exponent describing this decay is $\nu = 0.87$ for Toom with synchronous updating while $\nu = 1$ if the system belongs to the Ising universality class. However, taking corrections which respect the slower correlation decay, singularities of other functions as *e.g.* the magnetization and susceptibility are the same as in the Ising model. The similar scaling properties are observed in systems of coupled map lattices [13,14]. It is also known for PCA that model ESM [7] that critical clusters are larger and more stable when a rule is applied synchronously.

The main concern of the presentation is the cluster structure of Toom PCA stationary states. Such a structure has its origin in both the geometrical connectivity and Toom interactions. The aim is to find links between the ferromagnetic transition and the pure geometric problem of percolation. In case of the Ising model these links are established clearly [15–17]. Since our study on universality class of the transition provides distinction between the Toom NEC system with the rule performed synchronously and the system with asynchronous updating, in the following we will consider the cluster structure in these both systems.

1.4. Percolation and critical phenomena

The percolation problem states are produced by throwing down particles or bonds in an independent way. The simplest percolation model is the site percolation in a two-dimensional square lattice. Each site of a lattice is occupied with probability p. Occupied sites that share edges form a cluster. If p is large there is a percolating cluster, *i.e.*, a cluster that spans the lattice from one edge to the other. The percolation displays a threshold phenomena. In the limit of infinite lattice size there is a sharp transition at some density $p_{\rm c}$, called critical density or percolation threshold, with the property that: for $p < p_{\rm c}$ there is never a percolating cluster and for $p > p_{\rm c}$ there is always a percolating cluster. This transition is of a continuous type. The order parameter for this transition is the percolation probability P_∞. The percolation probability means the probability that a site belongs to a percolating cluster. See [18–20] for the description of the critical transition in the site percolation model.

The study of the percolation threshold in the Ising model was started by Fortuin and Kasteleyn [15]. In the mathematically rigorous way, the pure connectivity problem was transferred to the Potts ferromagnetic model — a special case of which is the Ising model. The idea was proposed to link a percolation cluster to correlations in cooperative systems. This leads to a percolation cluster of the Ising ferromagnet that is a set of nearest neighbor parallel spins but additionally connected by the interaction bonds [16, 17]. Each bond is being present with probability $p = 1 - e^{-K}$ where K is the constant which describes interactions with respect to the temperature. The cluster defined in this way has a remarkable property that the connected probability between two sites of clusters agrees precisely with the two-point correlation function. This property allowed to prove that the percolation transition temperature $T_{\rm P}$ is exactly the same as the Curie temperature.

2. Results

2.1. Computer experiment description

We use the standard importance sampling technique to simulate stationary states of the Toom model. We consider a square lattice of linear size $L = 100$ imposed in periodic boundary conditions. The computer experiments are started with all spins *up*. A new configuration is generated from the old one by the following Markov process: for a given ε the evolution rule (2) is employed to each spin in case of synchronous updating, or to a randomly chosen spin when asynchronous updating case is examined. The evolving system is given 10 000 time steps to reach the stationary state. Such time interval is sufficient to find systems in ergodic states [23].

When a system is in a stationary state then at each time step t (where a time step means a Monte Carlo step, namely one simulation step when synchronous updating is performed and L^2 single spin flips in case of asynchronously applied rule) then:

— an expectation value of magnetization $|m|_L$ is computed (since the transition is observed on the lattice with a finite size we calculate the magnitude of the magnetization):

$$|m|_L = \frac{1}{T} \sum_{t=1,\dots,T} \frac{1}{L^2} \left| \sum_{i=1,\dots,L^2} \sigma_i(t) \right| , \tag{3}$$

— a cluster structure of spins being in the state *up* is identified by using the standard algorithm of Hoshen–Kopelman [21]. The following quantities are calculated to characterize a cluster structure of a state, compare [18, 22]:

• n_s probability to have a cluster of size s, so-called $s\_$ cluster [18], normalized by the lattice size:

$$n_s = \frac{1}{T} \sum_{t=1,\dots,T} \frac{1}{L^2} (\text{number of } s\_ \text{ clusters})(t) \, . \tag{4}$$

Thus sn_s means the probability of any lattice site belongs to an $s\_$ cluster.

- $n_\infty(s)$ probability that an $s\_$ cluster normalized is the maximal one on a given configuration:

$$n_\infty(s) = \frac{1}{T}\frac{1}{L^2}(\text{number of configurations : } s\_ \text{ cluster is maximal})\,. \quad (5)$$

Hence the expectation value of the size of the maximal cluster normalized is $s_{\max} = \sum_s s\, n_\infty(s)$.

- P_{perc} probability for a percolating cluster:

$$P_{\text{perc}} = \frac{1}{T}(\text{number of configurations with a percolating cluster})\,. \quad (6)$$

A configuration has a percolating cluster if there is a cluster spanning vertically or horizontally.

In all experiments $T = 10\,000$. To avoid the possibility that a state is attracted by some metastable state, we perform N independent experiments, with N in the range $50, \ldots, 150$.

2.2. Order parameter study

Fig. 1 shows the probability distribution function (pdf) of clusters n_s, Fig. 2 presents the probability $n_\infty(s)$ for a wide range of ε if NEC rule was applied synchronously and asynchronously. The data is presented in log-scale. The rapid change in the shape of both distributions is evident. A dominant phase cluster explodes into the plenty of clusters of all possible sizes.

Fig. 3 compares decay of magnetization $|m|_L$ to the percolation probability P_∞ — the order parameters of the two transitions that take place, when the temperature increases. The percolation probability P_∞ is estimated by the relation $P_\infty = s_{\max}P_{\text{perc}}$. Notice that $|m|_L < P_\infty$. Similarly to the Ising model, the geometric clusters are too large to describe magnetization dependencies in both Toom PCA.

To learn about the critical points in the systems studied we search for the maximum of the variance of the order parameters. Fig. 4 collects results of the study. The data does not point clearly at the shape of the variance. However, by smoothing the data the following results can be read:

	$\varepsilon_{\text{crit}}^{L=\infty}$	$\varepsilon_{\text{crit}}^{L=100}$	$\varepsilon_{\text{perc}}^{L=100}$	$\Delta\varepsilon_{\text{crit}}^{L=100}$
AToom	0.866[12]	0.862	0.864	0.002
SToom	0.822[12]	0.8208	0.8192	0.0018

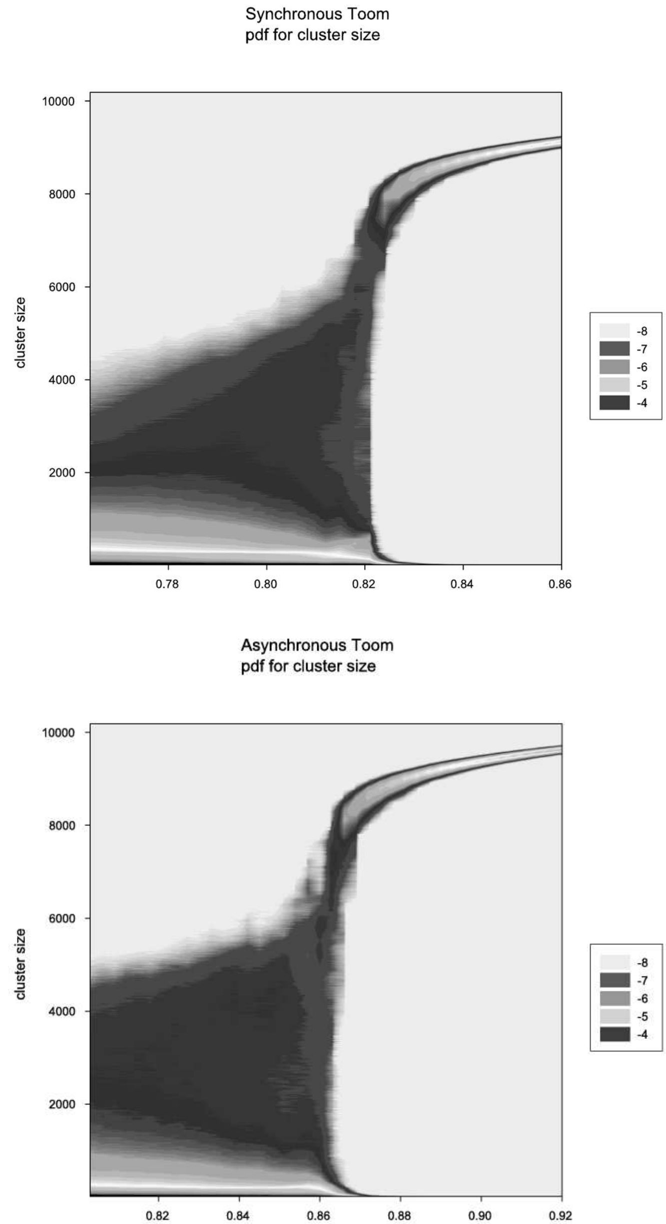
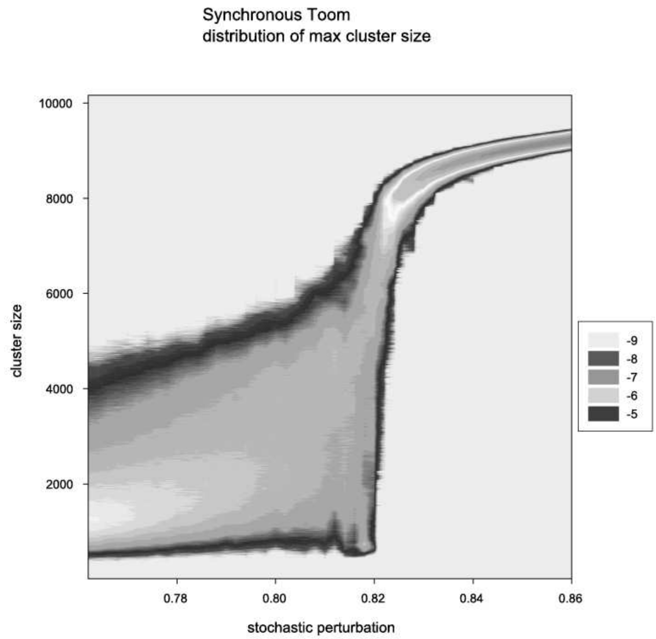


Fig. 1. The log-plot of probability distribution of clusters n_s, see (4), for a different values of stochastic noise ε when (top) synchronous (bottom) asynchronous Toom rule is applied. One can observe how the Gaussian distribution of ferromagnetic phase transfers into the exponential distribution of paramagnetic phase.



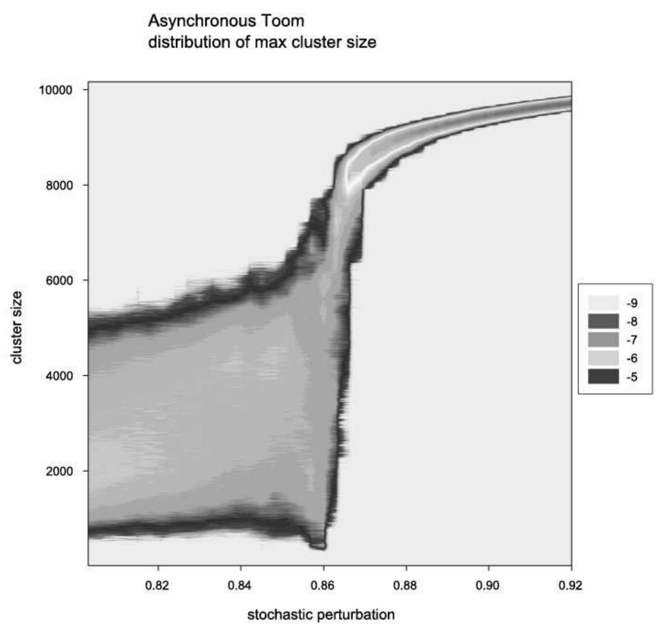


Fig. 2. The log-plot of the probability that a cluster of a given size is the maximal one, see (5) for a different values of stochastic noise ε when (top) synchronous (bottom) asynchronous Toom rule is applied.

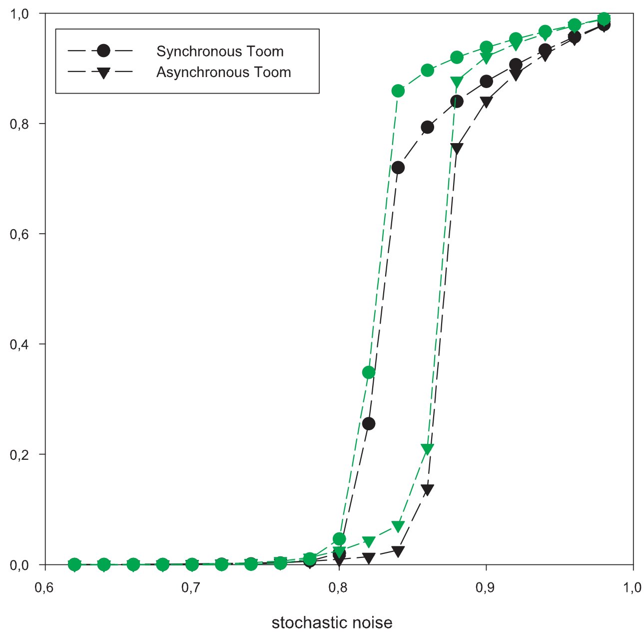


Fig. 3. Magnetization $|m_L|$ black plots, and percolation probability of 1s P_∞- grey plots for Toom PCA with synchronous and asynchronous dynamics.

The two transitions: ferromagnetic and percolation do not coincide in both systems. Moreover a small difference in $\Delta\varepsilon_{\rm crit}^{L=100}$ between asynchronous Toom and synchronous Toom PCA is observed. In Fig. 5 we present n_s and n_∞ in the critical ranges. If the noise is low, $\varepsilon > 0.90$, then n_s and n_∞ are of gaussian type and almost identical since there is a single cluster of 1's on a configuration. If the noise is high, $\varepsilon < 0.70$, then n_s decays exponentially and the distribution of maximal size becomes the right wing of the n_s distribution. In the critical regime both n_s and n_∞ continuously transfer between these two limit distributions. At the magnetic critical point the peak of the dominant phase is still present, while at the percolation threshold we observe unusually large flat wing of the exponential decay. $\Delta\varepsilon_{\rm crit}^{L=100} = 0.002$ in temperature-like-measure of a noise denotes that the deterministic rule of NEC voting in case of percolating threshold needs a perturbation stronger by 0.001 than the ferromagnetic transition. This property hints how strongly the geometrical clusters should be destroyed to obtain both transitions at the same temperature.

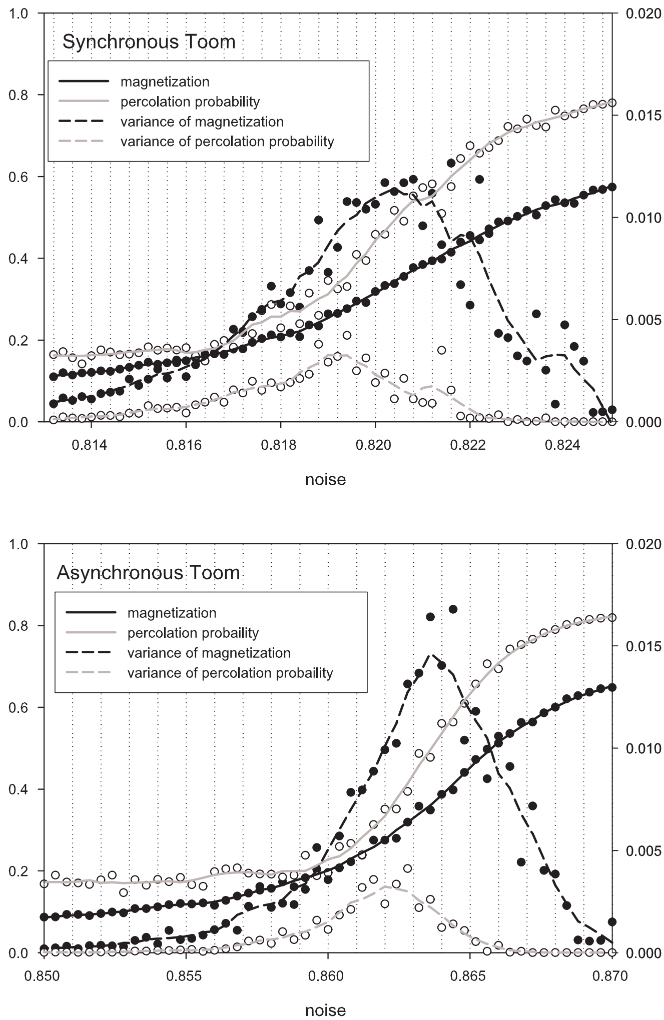


Fig. 4. Criticality of (top) synchronous and (bottom) asynchronous Toom PCA by variance of the order parameters.

3. Conclusion

Presented results are only preliminary. Our further investigations should involve lattices with different sizes to determine $\Delta\varepsilon^{\infty}_{\rm crit}$ between the ferromagnetic critical point and the percolation threshold if the infinite lattice system of both Toom dynamics is considered. Moreover, there is a need to include properties of clusters formed by the opposite phase to have a complete description to the process that takes place at the transition point.

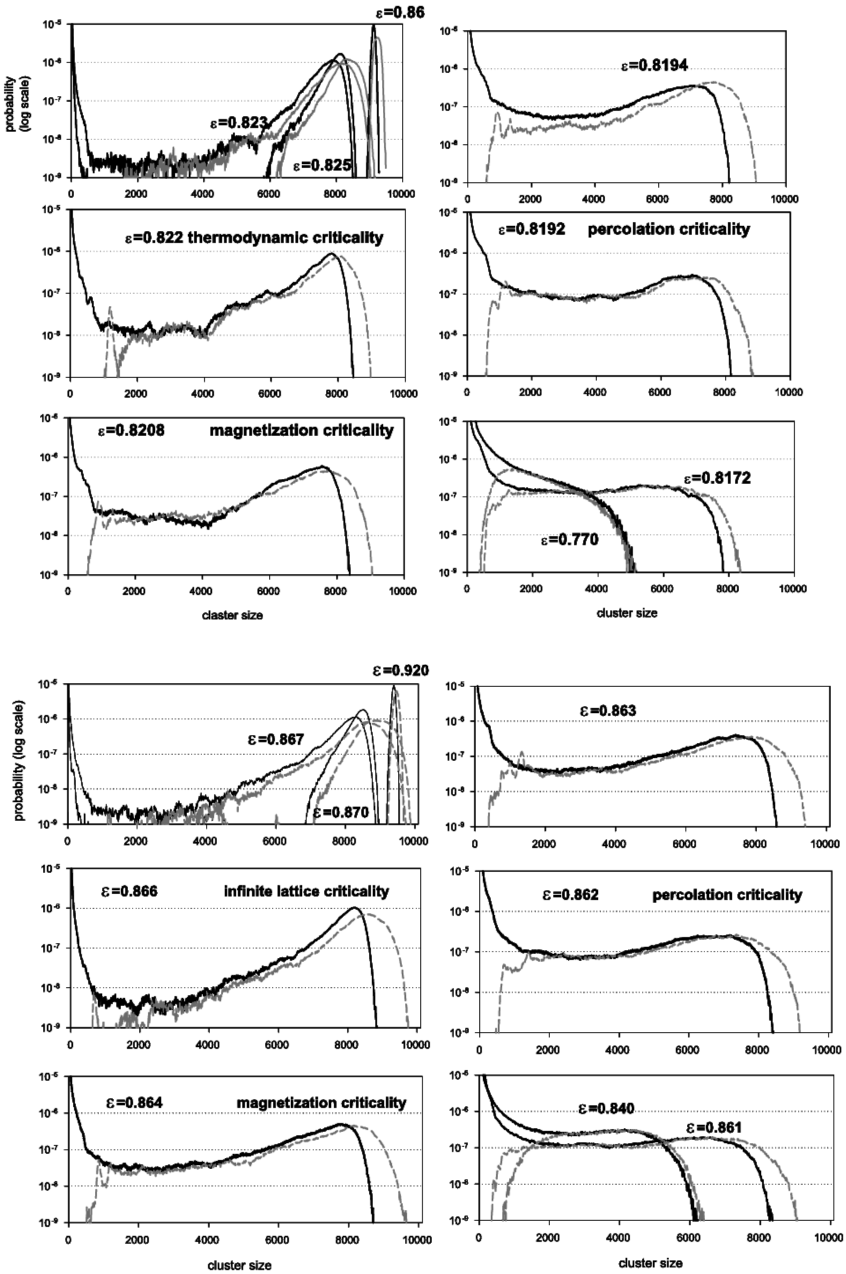


Fig. 5. Properties of probability distributions of cluster size n_s — black plots, and maximal cluster size n_∞ — grey plots, for (top) synchronous and (bottom) asynchronous Toom PCA near critical points.

As it was said in Sec. 1, in the ferromagnetic Ising model the thermodynamic transition can be described in terms of clusters made of parallel spins connected by "fictitious" bonds [17]. In other thermodynamic systems such as *e.g.* in the Ising spin glasses where the percolation cluster can be defined in the a similar way as that of ferromagnet [24, 25], the research on the relation between geometrical clusters and physical clusters is still examined to elucidate the mechanism underlying a thermodynamic transition. However, the geometrical interpretation of thermal phenomena is still not fully understood [26]. Study critical phenomena with probabilistic cellular automata offers a new possibility to get an insight into the problem. At first the search should be undertaken to define a cluster structure which will lead to the coincidence of the stochastic dynamics critical point and the percolation threshold.

This work is supported by Gdańsk University grant number BW 5400-5-0235-2. Simulations were done on computers of TASK — Academic Computer Center in Gdansk, Poland.

REFERENCES

[1] R. Bidaux, N. Boccara, H. Chate, *Phys. Rev.* **A39**, 3094 (1989).

[2] G. Odor, A. Szolnoki, *Phys. Rev.* **E53**, 2231 (1996).

[3] W. Hwang, S. Kwon, He. Park, Hy. Park, *Phys. Rev.* **E57**, 6438 (1998).

[4] E. Domany, *Phys. Rev. Lett.* **52**, 871 (1984); E. Domany, W. Kinzel, *Phys. Rev. Lett.* **53**, 311 (1984).

[5] Ch.H. Bennett, G. Grinstein, *Phys. Rev. Lett.* **55**, 657 (1985).

[6] J.L. Lebowitz, Ch. Maes, E. Speer, *J. Stat. Phys.* **59**, 117 (1990).

[7] S. Bigelis, E.N.M. Cirillo, J.L. Lebowitz, E.R. Speer, *Phys. Rev.* **E59**, 3935 (1999).

[8] A.L. Toom, N.B. Vasilyev, O.N. Stavskaya, L.G. Mityushin, G.L. Kurdyumov, S.A. Prigorov, *Stochastic Cellular Systems: Ergodicity, Memory, Morphogenesis*, Ed. R.L. Dobrushin, V.I. Kryukov, A.L. Toom, Manchester University Press, Manchester 1990.

[9] R. Fernandez, *Physica A* **263**, 117 (1999).

[10] D. Makowiec, *Phys. Rev.* **E60**, 5195 (1997).

[11] D. Makowiec, *Phys. Rev.* **E60**, 3787 (1999).

[12] D. Makowiec, P. Gnacinski, LNCS 2493, p. 104–114, Springer-Verlag, 2002.

[13] P. Marcq, H. Chate, P. Manneville, *Phys. Rev. Lett.* **77**, 4003 (1996).

[14] P. Marcq, H. Chate, P. Manneville, *Phys. Rev.* **E55**, 2606 (1997).

[15] C.M. Fortuin, P.W. Kasteleyn, *Physica* **57**, 536 (1972).

[16] A. Coniglio, Ch.R. Nappi, F. Peruggi, L. Russo, *Commun. Math. Phys.* **51**, 315 (1976).

[17] A. Coniglio, W. Klein, *J. Phys.* **A13**, 2275 (1980).

[18] D. Stauffer, *Phys. Rep.* **54**, 1 (1979).

[19] D. Stauffer, *Physica A* **242**, 1 (1997).

[20] J.P. Hovi, A. Aharony, *Phys. Rev.* **E53**, 235 (1996).

[21] J. Hoshen, R. Kopelman, *Phys. Rev.* **B14**, 3438 (1976).

[22] M. De Meo, D.W. Heermann, K. Binder, *J. Stat. Phys.* **60**, 585 (1990).

[23] Ch. Munkel, D.W. Heermann, J. Adler, M. Gofman, D. Stauffer, *Physica A* **193**, 540 (1993).

[24] Y. Kasai, A. Okiji, *Prog. Theor. Phys.* **79**, 1080 (1988).

[25] A. Coniglio, *Physica A* **266**, 379 (1999).

[26] H.E. Stanley, J.S. Andrade Jr, S. Havlin, H.A. Makse, B. Suki, *Physica A* **266**, 5 (1999).