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RANDOM DYNAMICS, ENTROPY PRODUCTION AND FISHER INFORMATION*

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We analyze a specific role of probability density gradients in the theory of irreversible transport processes. The classic Fisher information and information entropy production concepts are found to be intrinsically entangled with the very notion of the Markovian diffusion process and that of the related (local) momentum conservation law.

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1. Motivations and associations

The main objective of the present paper is to analyze the role — origins, possible physical meaning and manifestations — of *two* analytical expressions which are omnipresent, directly or indirectly, in any theoretical framework addressing an issue of transport driven by Markovian diffusion processes. Both derive from the sole properties, and specifically the time evolution, of the probability density associated with the analyzed stochastic process (like *e.g.* the dynamics of tracer particles in a gas or fluid).

Let us specify the context by considering spatial Markov diffusion processes with a diffusion parameter (constant or time-dependent) D and generally space–time inhomogeneous probability density ρ.

One of the aforementioned expressions reads:

$$Q = 2D^2 \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = D^2 \left[\frac{1}{\rho}\Delta\rho - \frac{1}{2}\frac{1}{\rho^2}(\vec{\nabla}\rho)^2 \right] = \frac{1}{2}\vec{u}^{\,2} + D\vec{\nabla}\cdot\vec{u}\,, \qquad (1)$$

where $\vec{u} = D\vec{\nabla}\ln\rho$ is sometimes named an osmotic velocity field.

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Density gradients are here explicitly involved and it is useful to invoke at this point a vivid discussion, carried out recently, about the status of density gradient as a "real" (thermodynamic) force performing work on the particles and the related issue of the local irreversible entropy production, see *e.g.* [1–5], see however also [6–9].

Let us recall that the standard spatial Brownian motion involves $\overrightarrow{v} = -\overrightarrow{u}$, known as the diffusion current velocity and (up to a dimensional factor) identified with the "thermodynamic force of diffusion" [5] which drives the irreversible process of matter exchange at the macroscopic level. In terms of tracer particles, this irreversible process occurs even if they are so dilute that they never meet (nor interact with) each other.

On the other hand, even while the "thermodynamic force" is a concept of purely statistical origin associated with a collection of particles, in contrast to microscopic forces which have a direct impact on individual particles themselves, it is well known [5–7] that this force manifests itself as a Newtonian-type entry in local conservation laws describing the momentum balance: in fact that pertains to the average (local averages) momentum taken over by the "particle cloud", a statistical ensemble property quantified in terms of the probability distribution at hand. It is precisely the (negative) gradient of the above potential Q, Eq. (1), which plays the Newtonian force role in the momentum balance equations, [6, 7].

To elucidate the role of the *second* analytical expression of interest in our present considerations (Q was actually the *first)* let us observe that in one space dimension, for probability densities vanishing at spatial infinities, we have:

$$-\int Q\rho dx = \int \frac{u^2}{2}\rho dx \doteq \frac{1}{2}D^2\, F_X\,, \tag{2}$$

where F_X is the so-called Fisher information [10–12] of (encoded in) the probability density ρ which quantifies its "gradient content" (sharpness plus localization/disorder properties) and reads:

$$F_X = \int \frac{(\nabla\rho)^2}{\rho}dx\,. \tag{3}$$

An important property of the *Fisher information* (stemming from the Cramer–Rao inequality in the statistical inference theory, [10–14]) is that F_X^{-1} sets the lower bound for the variance of the random variable $X(t)$ with values in R^1, distributed according to $\rho(x,t)$:

$$\langle X^2\rangle - \langle X\rangle^2 \geq F_X^{-1}\,. \tag{4}$$

On the other hand, in direct correspondence with our previous discussion of $Q(x,t)$, let us point out that the integrand in Eq. (3), up to a dimensional factor, defines the so-called *local entropy production* inside the system sustaining an irreversible process of diffusion, [1,3]. Accordingly, [3,4]

$$\frac{dS}{dt} = D \int \frac{(\nabla \rho)^2}{\rho} dx = D\, F_X \geq 0 \tag{5}$$

stands for an entropy production rate when the Fick law-induced diffusion current (standard Brownian motion case) $j = -D\nabla\rho$, obeying $\partial_t \rho + \nabla j = 0$, enters the scene. Here $S = -\int \rho \ln \rho \, dx$ plays the role of the (time-dependent) *information entropy* in the nonequilibrium statistical mechanics framework for the thermodynamics of irreversible processes. It is rather clear that the high rate of the entropy increase corresponds to a rapid spreading (flattening down) of the probability density. That explicitly depends on the "sharpness" of density gradients.

The potential-type function $Q(x,t)$, the Fisher information $F_X(t)$, non-equilibrium measure of the entropy production dS/dt and the information entropy $S(t)$ are thus mutually entangled quantities, each being exclusively determined in terms of the probability density $\rho(x,t)$ and its spatial derivatives.

2. Hydrodynamical (local) momentum conservation laws — the zoo

As mentioned before, the function $Q(x,t)$ notoriously appears in various local conservation laws responsible for the momentum balance in suitable physical systems. Let us make a brief perusal of the respective partial differential equations.

In the standard statistical mechanics setting, the Euler equation does not refer to any Q, Eq. (1), but deserves reproduction for the obvious comparison purpose as a prototype momentum balance equation in the (local) mean:

$$(\partial_t + \vec{v} \cdot \vec{\nabla})\vec{v} = \frac{\vec{F}}{m} - \frac{\vec{\nabla} P}{\rho}\,, \tag{6}$$

where we generally assume $\vec{F} = -\vec{\nabla} V$ to represent the "normal" Newtonian force.

With regard to the manifest appearance of Q, we begin from an encounter with $\vec{\nabla} Q$ in an out-of-statistical mechanics example provided by the hydrodynamical formalism of quantum theory, [15]:

$$(\partial_t + \vec{v} \cdot \vec{\nabla})\vec{v} = \frac{1}{m}\vec{F} - \vec{\nabla} Q_q = \frac{1}{m}\vec{F} + \frac{\hbar^2}{2m^2}\vec{\nabla}\frac{\Delta \rho^{1/2}}{\rho^{1/2}}\,, \tag{7}$$

where $Q_q = -\frac{\hbar^2}{2m^2}\frac{\Delta\rho^{1/2}}{\rho^{1/2}}$ is the familiar de Broglie–Bohm quantum potential.

Another spectacular example pertains to the standard free Brownian motion in the strong friction (Smoluchowski diffusion) regime. Namely, we have, [6]:

$$(\partial_t + \vec{v} \cdot \vec{\nabla})\vec{v} = -2D^2\vec{\nabla}\frac{\Delta\rho^{1/2}}{\rho^{1/2}} \doteq -\vec{\nabla}Q\,, \tag{8}$$

where $\vec{v} = -D\frac{\vec{\nabla}\rho}{\rho}$; D is the diffusion constant (set formally $D \doteq \hbar/2m$ and notice the sign change in comparison with the previous quantum mechanical law).

The large friction (Smoluchowski again) limit of the driven phase-space random dynamics implies, [7]:

$$(\partial_t + \vec{v} \cdot \vec{\nabla})\vec{v} \;=\; \vec{\nabla}\,(\Omega - Q)\ , \tag{9}$$

where $\vec{v} \doteq \vec{v}(\vec{x}, t) = \frac{\vec{F}}{m\beta} - D\frac{\vec{\nabla}\rho}{\rho}$, the volume force (notice the positive sign) reads $+\vec{\nabla}\Omega$ instead of the previous $-\vec{\nabla}V$. Here $Q = 2D^2\frac{\Delta\rho^{1/2}}{\rho^{1/2}}$ and (recall the spectral analysis of Fokker–Planck operators, *cf.* [7])

$$\Omega = \frac{1}{2}\left(\frac{\vec{F}}{m\beta}\right)^2 + D\vec{\nabla}\cdot\left(\frac{\vec{F}}{m\beta}\right)\,. \tag{10}$$

For a class of "perverse" diffusion processes (respecting the so-called "Brownian recoil principle", [7]), we deal with Markovian diffusion processes with the inverted sign of $\vec{\nabla}(\Omega - Q)$ in the local momentum conservation law, so that the previous Eq. (9) takes the form:

$$(\partial_t + \vec{v} \cdot \vec{\nabla})\vec{v} \;=\; \vec{\nabla}\,(Q - \Omega)\ . \tag{11}$$

By introducing $\psi = \rho^{1/2}\exp(iS)$ and $\vec{v} = 2D\vec{\nabla}S$, we set a link with the "true" (notice an imaginary unit i) Schrödinger-type dynamics:

$$i\partial_t\psi = -D\Delta\psi + \frac{\Omega}{2mD}\,\psi\,.$$

Useful observation: the total energy $\int_{R^3}(\frac{\vec{v}^2}{2} - Q + \Omega)\rho d^3x = \int_{R^3}(\frac{\vec{v}^2}{2} + \frac{\vec{u}^2}{2} + \Omega)\rho d^3x$ of the system is a conserved finite quantity. Here $\vec{u}(\vec{x}, t) \doteq D\vec{\nabla}\ln\rho(\vec{x}, t)$. Notice that $(D/2)dS/dt$ of Eq. (5), makes an explicit contribution to an overall energy of the system.

The conservation of the total energy tells that the entropy production and the kinetic energy due to diffusion currents stay in competition.

For a special case of the frictionless random phase-space dynamics, [9], we arrive at:

$$\left[\partial_t + \overrightarrow{v} \cdot \overrightarrow{\nabla}\right] \overrightarrow{v} = \frac{\overrightarrow{F}}{m} + 2d^2(t) \overrightarrow{\nabla} \left[\frac{\Delta \rho^{1/2}}{\rho^{1/2}}\right], \tag{12}$$

where $\overrightarrow{F}$ denotes the external force acting on the particle, and $d(t)$ is the time-dependent diffusion parameter. This form of the law has been derived by explicitly solving the Fokker–Kramers equation with properly adjusted (Gaussian densities) initial data, for the following cases:

1. free particle: $F \equiv 0$, $n = 1$,
2. charged particle in a constant magnetic field: $\overrightarrow{F} = e\overrightarrow{v} \times \overrightarrow{B}$, $n = 2$,
3. harmonically bound particle: $F = -m\omega^2 x$, $n = 1$.

Presumably this form is universal (no general proof at the moment). The coefficient $d^2(t)$ in all those cases can be represented as a product of variances $(n = 1)$ evaluated with respect to conditioned phase-space $w_x(u,t) = \frac{f(x,u,t)}{\rho(x,t)}$ and configuration space (marginal) $\rho\left(x,t\right) = \int f\left(x,u,t\right) du$ densities respectively:

$$d^2\left(t\right) \doteq \left(\left\langle u^2\right\rangle_x - \left\langle u\right\rangle_x^2\right)\left(\left\langle x^2\right\rangle - \left\langle x\right\rangle^2\right) . \tag{13}$$

One may prove that $d^2(t)$ is bounded from below which results in the Heisenberg-type inequality for variances: of $U(t)$ with respect to the conditioned phase-space density $w_x(u,t)$, and $X(t)$ with respect to the marginal density $\rho(x,t)$.

3. Diffusion processes and differential equations — pedestrian reasoning

Let us sketch how the previous observations come out within the traditional setting of phase-space stochastic processes.

3.1. Standard Brownian motion

Let us consider the competition between deterministic/random driving and friction in the standard Brownian motion:

$$\frac{d\overrightarrow{x}}{dt} = \overrightarrow{u}\,, \tag{14}$$

$$\frac{d\overrightarrow{u}}{dt} = -\beta\overrightarrow{u} + \frac{\overrightarrow{F}}{m} + \overrightarrow{A}\left(t\right)\,, \tag{15}$$

where $\left\langle A_i\left(s\right)\right\rangle = 0$ and $\left\langle A_i\left(s\right) A_j\left(s'\right)\right\rangle = 2q\delta\left(s - s'\right)\delta_{ij}$; $\overrightarrow{F} = -\overrightarrow{\nabla} V$.

For the case of the *standard* Brownian motion, we know a priori, in view of the fluctuation–dissipation theorem, that $q = D\beta^2$ where $D = \frac{kT}{m\beta}$, while β is given by the Stokes formula $m\beta = 6\pi\eta a$.

The resulting (Markov) phase-space diffusion process is determined by solutions of the Kramers equation: an initially given $f(\vec{x}_0, \vec{u}_0, t_0)$ is propagated according to:

$$\left(\partial_t + \vec{u} \cdot \vec{\nabla}_{\vec{x}} + \frac{\vec{F}}{m} \cdot \vec{\nabla}_{\vec{u}}\right) f = C(f) = \left(q\nabla^2_{\vec{u}} + \beta \vec{u} \cdot \vec{\nabla}_{\vec{u}}\right) f . \tag{16}$$

Here we adopt the kinetic theory notation for a substitute of collision term, where $\int C(f) d^3u = 0$, while $\frac{1}{\rho}\int \vec{u} C(f) d^3u = -\beta \vec{v}(\vec{x}, t)$.

Accordingly, the continuity equation holds true for the marginal (spatial) probability density $\rho = \int f d^3u$ and $\vec{v} \doteq \frac{1}{\rho}\int \vec{u} f d^3u$. That has a devastating effect on the form of the corresponding momentum conservation law in the large friction regime.

The associated Smoluchowski process with a forward drift $\vec{b}(\vec{x}) = \frac{\vec{F}}{m\beta}$ is analyzed in terms of the normalized Wiener process $\vec{W}(t)$: the infinitesimal increment of the configuration (position) random variable $\vec{X}(t)$ reads: $d\vec{X}(t) = \frac{\vec{F}}{m\beta}dt + \sqrt{2D}d\vec{W}(t) \quad \longrightarrow \ \partial_t \rho = D\triangle\rho - \vec{\nabla}\cdot(\rho \vec{b})$.
In the hydrodynamical picture, we infer the closed system of two (special to Markovian diffusions!) local conservation laws in the form appropriate for the Smoluchowski process, (remember about specific functional forms of Ω and Q):

$$\partial_t \rho + \vec{\nabla} \cdot (\vec{v}\rho) \ = \ 0\,, \tag{17}$$

$$(\partial_t + \vec{v} \cdot \vec{\nabla})\vec{v} \ = \ \vec{\nabla}(\Omega - Q)\ . \tag{18}$$

3.2. Free random dynamics with no friction

Now, $\frac{dx}{dt} = u$ and $\frac{du}{dt} = A(t)$, hence:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = C(f) = q\frac{\partial^2 f}{\partial u^2}\,. \tag{19}$$

We know [9] the transition density:

$$p(x, u, t|\, x_0, u_0, t_0 = 0) = \frac{1}{2\pi}\frac{\sqrt{12}}{2qt^2}\exp\left[-\frac{(u-u_0)^2}{4qt} - \frac{3\left(x - x_0 - \frac{u+u_0}{2}t\right)^2}{qt^3}\right]. \tag{20}$$

By choosing an initial phase space density:

$$f_0\left(x,u\right)=\left(\frac{1}{2\pi a^2}\right)^{\frac{1}{2}}\exp\left(-\frac{\left(x-x_{\mathrm{ini}}\right)^2}{2a^2}\right)\left(\frac{1}{2\pi b^2}\right)^{\frac{1}{2}}\exp\left(-\frac{\left(u-u_{\mathrm{ini}}\right)^2}{2b^2}\right), \tag{21}$$

so that $f\left(x,u,t\right)=\int p\left(x,u,t\right|x_0,u_0,t_0=0)\,f_0\left(x_0,u_0\right)dx_0du_0$ and passing to the hydrodynamical picture (unpleasant steps), we observe that $\int C(f)du=0$ and $\int uC(f)du=0$ which yields the following outcomes, [9]:

$$\rho\left(x,t\right)=\left(\frac{1}{2\pi\left(a^2+b^2t^2+\frac{2}{3}qt^3\right)}\right)^{\frac{1}{2}}\exp\left(-\frac{\left(x-x_{\mathrm{ini}}-u_{\mathrm{ini}}t\right)^2}{2\left(a^2+b^2t^2+\frac{2}{3}qt^3\right)}\right), \tag{22}$$

$$\rho\left(u,t\right)=\left(\frac{1}{2\pi\left(b^2+2qt\right)}\right)^{\frac{1}{2}}\exp\left(-\frac{\left(u-u_{\mathrm{ini}}\right)^2}{2\left(b^2+2qt\right)}\right), \tag{23}$$

$$\langle u\rangle_x=u_{\mathrm{ini}}+\frac{b^2t+qt^2}{a^2+b^2t^2+\frac{2}{3}qt^3}\left[x-x_{\mathrm{ini}}-u_{\mathrm{ini}}t\right]\doteq v\,, \tag{24}$$

$$\left\langle u^2\right\rangle_x-\langle u\rangle_x^2=\frac{q\,t^3\,\left(2\,b^2+q\,t\right)+3\,a^2\,\left(b^2+2\,q\,t\right)}{3\,a^2+t^2\,\left(3\,b^2+2\,q\,t\right)}\doteq\frac{P_{\mathrm{kin}}}{\rho}\,, \tag{25}$$

This implies the local momentum conservation law:

$$\left(\frac{\partial}{\partial t}+v\cdot\nabla\right)v=-\frac{\nabla P_{\mathrm{kin}}}{\rho}=+2\left(d^2\right)\nabla\left[\frac{\Delta\rho^{1/2}}{\rho^{1/2}}\right]\doteq+\nabla Q \tag{26}$$

with

$$d^2(t)=a^2b^2+2a^2qt+\frac{2}{3}b^2qt^3+\frac{1}{3}q^2t^4\doteq D^2(t)\,. \tag{27}$$

Remember about: $d^2\left(t\right)\doteq\left(\left\langle u^2\right\rangle_x-\langle u\rangle_x^2\right)\left(\left\langle x^2\right\rangle-\langle x\rangle^2\right)$ and notice that $d^2(t)\geq a^2b^2$.

3.3. Noiseless limit, $C(f)=0$ for all f

Upon disregarding random forcing (set $q\to 0$ in Eq. (19)), we arrive at:

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+\frac{F}{m}\frac{\partial f}{\partial u}=0\,, \tag{28}$$

where clearly $\int C(f)du=0=\int uC(f)du=\int u^2C(f)du$.

Things now look classical and there is good reason for that, since Eq. (28) is the familiar Liouville equation. However this "classical look" appears slightly deceiving.

Indeed, the $q \to 0$ limit of the frictionless free dynamics gives rise to:

$$f(x,u,t) = \frac{1}{2\pi\sqrt{a^2b^2}} \exp\left(-\frac{(u-u_{\mathrm{ini}})^2}{2b^2} - \frac{(x-x_{\mathrm{ini}}-tu)^2}{2a^2}\right) \tag{29}$$

with marginals:

$$\rho(u,t) = \left(\frac{1}{2\pi b^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(u-u_{\mathrm{ini}})^2}{2b^2}\right) \tag{30}$$

and

$$\rho(x,t) = \left(\frac{1}{2\pi\left(a^2+b^2t^2\right)}\right)^{\frac{1}{2}} \exp\left(-\frac{(x-x_{\mathrm{ini}}-u_{\mathrm{ini}}t)^2}{2\left(a^2+b^2t^2\right)}\right). \tag{31}$$

The local moments read:

$$\langle u\rangle_x = u_{\mathrm{ini}} + \frac{b^2t}{a^2+b^2t^2}\left(x-x_{\mathrm{ini}}-u_{\mathrm{ini}}t\right) \tag{32}$$

and

$$\left\langle u^2\right\rangle_x - \langle u\rangle_x^2 = \frac{a^2b^2}{a^2+b^2t^2} \tag{33}$$

which yields the (local) momentum conservation law in the fairly nonclassical form:

$$\left(\frac{\partial}{\partial t} + \langle u\rangle_x \nabla\right)\langle u\rangle_x = 2a^2b^2\nabla\left[\frac{\Delta\rho(x,t)^{1/2}}{\rho(x,t)^{1/2}}\right] \doteq \nabla Q\,. \tag{34}$$

By setting $a\,b = \frac{\hbar}{2m}$ we recover the standard quantum mechanical "hydrodynamics", Eq. (7), to be compared with the Brownian variant of the law, Eq. (8). Notice that $\langle x^2\rangle - \langle x\rangle^2 = a^2 + b^2t^2$ and:

$$a^2b^2 = \left(\left\langle u^2\right\rangle_x - \langle u\rangle_x^2\right)\left(\left\langle x^2\right\rangle - \langle x\rangle^2\right) = \frac{\hbar^2}{4m^2} \tag{35}$$

for all times. That is another expression for the standard quantum mechanical Heisenberg indeterminacy relation, see *e.g.* [16–19].

We recall that $Q_q = -\frac{\hbar^2}{2m^2}\frac{\Delta\rho^{1/2}}{\rho^{1/2}}$ is the de Broglie–Bohm quantum potential. Is there anything specific or mysterious in its origin and physical meaning?

4. Miscellaneous contexts: Hamilton–Jacobi, Liouville, Kramers equations, calculus of variations

Let us recall so-called *wave equations* of classical mechanics:

$$\partial_t \rho = -\nabla \cdot \left(\rho \frac{\nabla S}{m}\right) \tag{36}$$

with

$$\partial_t S + \frac{(\nabla S)^2}{2m} + V = 0\,. \tag{37}$$

Rename: $\frac{S}{m} \rightarrow S$, set $v = \nabla S$, eventually take a gradient of the above Hamilton–Jacobi equation. Then, we have:

$$\partial_t \rho = -\nabla \cdot (v\rho) \tag{38}$$

and

$$\partial_t v + (v \cdot \nabla)v = -\nabla V\,. \tag{39}$$

Clearly, in the above there is *nothing alike the* ∇Q contribution, so characteristic to our previous dynamical examples, *cf.* Eqs. (7)–(9), (11), (12), (18), (26). What is the primary reason of so conspicuous absence of that term in Eq. (39)?

To set a connection with the Liouville equation we follow a standard assumption, [15]: assign a *unique* momentum value at each space point and consider phase-space densities as generalized functions

$$f_0(x,p) = \rho_0(x)\delta(p - \nabla S_0(x)) \;\longrightarrow\; f(x,p.t) = \rho(x,t)\delta(p - \nabla S(x,t)) \tag{40}$$

which (weakly) solve

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + \frac{F}{m}\frac{\partial f}{\partial u} = 0\,. \tag{41}$$

In view of the fact that the Liouville equation preserves in time the *precise* knowledge of initial data, we have:

$$f_0(x,p) = \delta(x-x_0)\delta(p-p_0) \rightarrow f(x,p,t) = \delta(x-x(t,x_0,p_0))\delta(p-p(t,x_0,p_0)) \tag{42}$$

to be compared with the density function, Eq. (29).

As a side remark, let us notice that the Hamilton–Jacobi equation can be derived via the least action principle by employing the Lagrangian density

$$\mathcal{L} = \rho\left[\partial_t S + \frac{1}{2}(\nabla S)^2 + \frac{V}{m}\right] \tag{43}$$

with ρ and S considered as canonically conjugate variables, [15].

Where has gone our ∇Q (and Q itself)?

Let us come back to the previous $q \to 0$ free motion case, Eqs. (21) and (29). For all times $t \geq 0$ both spatial and velocity parts of the phase-space density are well behaved functions (not Dirac deltas). Hence, and indispensable, crucial step has been there to admit from the beginning both the spatial and momentum (velocity) indeterminacy (spreading, unsharpness). At time $t = 0$, we assign to each point x a "bunch" of possible (to be picked up at random from a given probability law) momenta — a Gaussian distribution of momenta at each spatial point and in addition we adopt a definite probability law for the position variable. As an immediate outcome, we get Eq. (34) *i.e.*:

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla\right) v = \nabla Q \,, \tag{44}$$

where: $+2a^2b^2 \left[\frac{\Delta\rho(x,t)^{1/2}}{\rho(x,t)^{1/2}}\right] \doteq Q$. This is the gradient form of:

$$\partial_t S + \frac{1}{2}(\nabla S)^2 - Q = 0 \tag{45}$$

which derives (via the standard variational calculus) from the Lagrangian density:

$$\mathcal{L} = \rho \left[\partial_t S + \frac{1}{2}(\nabla S)^2 + \frac{u^2}{2}\right] \tag{46}$$

to be compared with the previous "precise" (sharp) momentum variant in the absence of conservative forces:

$$\mathcal{L} = \rho \left[\partial_t S + \frac{1}{2}(\nabla S)^2\right] \,. \tag{47}$$

Now we need to come back to Eqs. (1)–(5), where entangled relationships among Q, Fisher information and local (information) entropy production were established for the Brownian motion.

In direct affinity with Eq. (46), we can develop the Hamiltonian formalism (ρ and S are canonically conjugate, $D = ab$), [20] which employs:

$$H = \int \mathcal{H} dx = \int dx\, \rho \left[\frac{1}{2}(\nabla S)^2 + \frac{u^2}{2}\right] = \int dx\, \rho \left[\frac{1}{2}(\nabla S)^2 + 2D^2\, F_X\right] \,. \tag{48}$$

We can now devise convincing arguments which relate the emergence of F_X and Q in the above with the a priori introduced and *simultaneously valid* spatial and momentum indeterminacy. To this end we shall discuss the position and velocity unsharpness issue from the two, looking diverse, perspectives.

Concerning the spatial (position) indeterminacy, we need the existence of the F_X term in Eq. (48). From the properties of the Fisher information, [10, 19], there follows that F_X goes to infinity when the spatial probability density approaches the delta function (sharp localization) limit. (The same happens when the probability density is discontinuous or vanishes over certain interval.) Hence, the Hamiltonian (48) is properly defined only in case of the nonsingular, unsharp spatial localization.

With regard to the velocity (momentum) unsharpness, let us invoke classic observations in the so-called quantum theory of motion (Bohm theory, Holland (1993)), where one argues as follows (notice a "subtle" difference if compared to our probabilistic arguments).

Equations: $\partial_t \rho = -\nabla(v\rho)$ where $v = \frac{1}{m}\nabla S$ and

$$\left(\partial_t + \frac{1}{m} v \cdot \nabla\right) v = -\nabla(V + Q_q)\,, \tag{49}$$

where $Q_q \simeq -Q$ *imply* that the distribution function:

$$f(x,p,t) = \rho(x,t)\delta\left[p - \nabla S(x,t)\right] \tag{50}$$

obeys the law of evolution:

$$\partial_t f + \frac{p}{m} \cdot \nabla_x f + \nabla_x (V + Q_q) \cdot \nabla_p f = 0\,. \tag{51}$$

This equation reduces to the classical Liouville one only when $Q_q = 0$, while the whole body of our previous discussion has explicitly referred to the Liouville equation as a primary building block of the theory. Consequently, in this context, a possibility that p (respectively u) can be sharply defined at each spatial point is definitely excluded, even if the spatial contribution is *a priori* assumed to be unsharp.

Comment: For comparison with the previous random motion discussion, one should realize that the continuity (and thus Fokker–Planck) equation plus the Hamilton–Jacobi type equation of the general form (we formally use V/m instead of more correct Ω):

$$\partial_t S + \frac{1}{2}(\nabla S)^2 \pm (\Omega - Q) = 0 \tag{52}$$

referring to the local conservation law:

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla\right) v = \mp \nabla(\Omega - Q) \tag{53}$$

both derive via the calculus of variations from, [21, 22]:

$$L = \int \mathcal{L} dx = \int \rho \left[\partial_t S + \frac{1}{2}(\nabla S)^2 \pm \left(\frac{u^2}{2} + \Omega\right)\right] dx\,. \tag{54}$$

The related Hamiltonian reads:

$$H = \int \mathcal{H} dx = \int dx\, \rho \left[\frac{1}{2} (\nabla S)^2 \pm \left(\frac{u^2}{2} + \Omega \right) \right] . \tag{55}$$

All that refers exclusively to the general phase-space equation:

$$\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + \frac{F}{m} \frac{\partial f}{\partial u} = C\,(f) \tag{56}$$

and *not* to any equation of the form (51).

5. Supplement: variational arguments in the theory of the Brownian motion

In connection with formulas (1)–(5) it is instructive to recall that in the Lagrangian formulation of the theory of random motion [23] the maximum rate of the information entropy increase has been found to maximize the Fisher information. After suitable notation adjustments, we realize that in Ref. [23] it is exactly $-F_X$ that is minimized to yield the free spatial Brownian motion.

In the very same case, [24], the relationship (*cf.* Eq. (4) for comparison):

$$\langle X^2 \rangle \langle u(X,t)^2 \rangle = D \tag{57}$$

was established for the heat kernel solution of $\partial_t \rho = D \Delta \rho$, provided there holds $\langle X \rangle = 0 = \langle u(X,t) \rangle$. For more general solutions of the heat equation for which the mean values of $X(t)$ and $u(X(t),t)$ do vanish, one arrives at a general indeterminacy relationship with an obvious affinity to the previously mentioned Cramer–Rao inequality:

$$\langle X^2 \rangle \langle u(X,t)^2 \rangle = \left[D^2 \int dx\, \frac{1}{\rho} \cdot (\nabla \rho)^2 \right]^{1/2} \left[\int dx\, x^2\, \rho \right]^{1/2} \geq D\, . \tag{58}$$

Quite in parallel, a bit more general case was addressed in Ref. [18], where a general (nonvanishing forward drift) one dimensional diffusion process with (time-dependent) diffusion coefficient $D(t)$ was considered, under slightly weaker restrictions: $\langle u(X,t) \rangle = 0$ while $\langle X \rangle \neq 0$. The problem addressed, has been an issue of when the product of variances $[\langle X^2 \rangle - \langle X \rangle^2]\, [\langle u(X,t)^2 \rangle]$ is minimized.

The outcome is that a minimum is reached for a concrete product value equal $D^2(t)$ and that a necessary and sufficient condition for the probability density $\rho(x,t)$ to yield that minimum, is that it has a Gaussian form:

$$\rho(x,t) = \frac{1}{(2\pi)^{1/2} [\langle X^2 \rangle - \langle X \rangle^2]^{1/2}} \exp \left[- \frac{(x - \langle X \rangle)^2}{2[\langle X^2 \rangle - \langle X \rangle^2]} \right] \tag{59}$$

in agreement with our previous discussion. For non-Gaussian probability densities, an inequality of the type (58) holds true.

Indeed, it is a classic observation, [14], that for a generic probability density $\rho_\alpha(x,t)$ with the first moment $\int x\,\rho(x,t)dx = f(\alpha,t)$ and finite second moment, for which there exist both partial derivatives $\frac{\partial\rho_\alpha(x,t)}{d\alpha}$ and $\frac{\partial f(\alpha,t)}{\partial\alpha}$ (for all α in an interval in R^1 or generally in R^1, and for almost all $x \in R^1$), then we arrive at an inequality:

$$\int (x-\alpha)^2\rho_\alpha(x,t)dx \int \left(\frac{\partial ln\rho_\alpha}{\partial\alpha}\right)^2 \rho_\alpha(x,t)dx \geq \left(\frac{df(\alpha,t)}{d\alpha}\right)^2 . \qquad (60)$$

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