FLUCTUATIONS IN NONLINEAR SYSTEMS: A SHORT REVIEW*

F.J. de la Rubia

Depart. de Física Fundamental, Universidad Nacional de Educación a Distancia
Apdo. de Correos 60141, 28080 Madrid, Spain

J. Buceta

Department of Chemistry and Biochemistry and Inst. for Nonlinear Science
University of California
San Diego, 9500 Gilman Drive, La Jolla, California 92093-0340, USA

J.L. Cabrera

Laboratorio de Física Estadística, Centro de Física, IVIC
Apartado 21827, Caracas 1020A, Venezuela

J. Olarrea

Depart. de Matemática Aplicada y Estadística, Escuela Técnica Superior de Ingenieros Aeronáticos, Univers. Politécnica de Madrid, 28040 Madrid, Spain

and J.M.R. Parrondo

Departamento de Física Atómica, Molecular y Nuclear
Facultad de Ciencias Físicas, Universidad Complutense de Madrid
Ciudad Universitaria s/n, 28040 Madrid, Spain

(Received November 29, 2002)

We review some results that illustrate the constructive role of noise in nonlinear systems. Several phenomena are briefly discussed: optimal localization of orbits in a system with limit cycle behavior and perturbed by colored noise; stochastic branch selection at secondary bifurcations; noise-induced order/disorder transitions and pattern formation in spatially extended systems. In all cases the presence of noise is crucial, and the results reinforce the modern view of the importance of noise in the evolution of nonlinear systems.

PACS numbers: 05.45.–a, 02.50.Ey, 47.54.+r

* Presented at the XV Marian Smoluchowski Symposium on Statistical Physics, Zakopane, Poland, September 7–12, 2002.

1. Introduction

The effect of noise on the behavior of a system has been an important topic of interest for many years. However, the perception of the role of noise has changed in the last decades, and it has evolved from the traditional idea that the noise is a nuisance, something to be avoided and distorting the desirable regular behavior of the system, to the modern view of noise being an important ingredient in its evolution, allowing the system to explore and choose among many possibilities. Moreover, nowadays it is recognized that the noise may play an ordering role, enhancing the response to an external signal, leading the system to new phases, or creating and maintaining spatial patterns, for example. This new and somehow unexpected noise-induced order may have important implications in several branches of Physics, Chemistry and Biology, and is a subject of a very productive activity.

In this paper we review some results that clearly show several of the noise-induced effects mentioned above. Our aim is just to present a few relevant examples of the constructive role of noise, without intending to discuss all the important results in this field or to include an exhaustive bibliography. For the sake of conciseness, we will skip many technical details in our presentation, and direct the interested reader to the references.

2. Localization enhanced by colored noise

One of the most popular topics in the last decade is that of Stochastic Resonance (SR). In its most popular assertion, SR is normally understood to be the phenomenon by which an additive noise (usually considered uncorrelated) can enhance the coherent response of a periodically driven subthreshold nonlinear system. First proposed in climate model studies [1], SR has been predicted and observed in many different theoretical and experimental systems (see [2] for an extensive review, and [3] for some recent experimental results). Besides the indicated classical combination "nonlinear system, additive white noise and periodic external force", SR can also occur in systems with very different characteristics, *e.g.*, systems with aperiodic forcing, autonomous systems without external periodic forcing but with intrinsic periodicity (limit cycle), systems with multiplicative noise or perturbed by colored noise (correlated noise), arrays of oscillators, or systems with time delay. Also recently the case of periodically driven systems with multiplicative colored noise has been considered, analyzing not only the effect of the noise intensity (the only parameter when considering white noise), but also that of the correlation time of the noise on the SR phenomenon.

In this section we consider a system without periodic external forcing, but with an intrinsic oscillatory behavior (limit cycle) and perturbed by a multiplicative colored noise. We will focus on the effect of the correlation time of the noise on the behavior of the system [4].

The system is the well known Sel'kov model for glycolysis [5]

$$\begin{aligned}\dot{x} &= -x + \lambda_t y + x^2 y\,,\\ \dot{y} &= b - \lambda_t y - x^2 y\,.\end{aligned} \tag{1}$$

We will consider the control parameter as a random variable $\lambda_t = \lambda + \zeta_t$, *i.e.*, as a deterministic part λ, plus an stochastic perturbation ζ_t, which is assumed to be an OU process, *i.e.*, a stationary Gaussian Markov noise with zero mean, $\langle \zeta_t \rangle = 0$, and exponential correlation

$$\langle \zeta_t \zeta_{t'} \rangle = (D/\tau) \exp\left(-\left|t - t'\right| / \tau\right),$$

where τ is the correlation time and $D/\tau = \sigma^2$ is the variance of the noise. We will refer to the square root of the variance, σ, as the intensity of the noise. It is a simple exercise to show that for a certain range of values of the parameter b, the deterministic counterpart of (1) undergoes a supercritical Hopf bifurcation at $\lambda \equiv \lambda_{\mathrm{H}}$, and, therefore, the system shows sustained oscillations.

In order to analyze the evolution of the system we numerically integrate (1) with λ in the limit cycle parameter domain. A good indicator of the behavior of the system is the Residence Times Distribution Function (RTDF) over the phase space (*i.e.*, the distribution of the system on the different available attractors) that tells you where and how long has been the system [2].

To do that, we consider a deterministic attractor $\Lambda(\lambda)$, *i.e.*, the attractor obtained with the deterministic counterpart of the stochastic system, evaluated at a particular value of the control parameter, λ. Next, we divide the system phase space in $N+1$ attractors associated with $N+1$ values of the parameter separated a distance $\Delta\lambda$. In this way, a mesh is composed by concentric deterministic attractors centered around the stationary equilibrium state $(x^*, y^*)\mid_{\lambda\sim\lambda_{\mathrm{H}}}$, with λ in the fixed point domain. With this construction, we have a series of $N+1$ attractors

$$\{\Lambda(\lambda_{N/2_-}) \ldots \Lambda(\lambda_{1_-}), \Lambda(\lambda_0), \Lambda(\lambda_{1_+}) \ldots \Lambda(\lambda_{N/2_+})\},$$

where we use the definition $\lambda_{k\pm} \equiv \lambda \pm k\Delta\lambda$. This series divides the phase space in N rings, each one denoted by $\Gamma(\gamma_k) \equiv (\Lambda(\lambda_k), \Lambda(\lambda_{k+1}))$, where $\gamma_k \equiv (\lambda_{k+1} + \lambda_k)/2$ is the mean control parameter obtained with the control parameters that define the boundary of the ring. The stochastic system is integrated on this mesh, and its evolution describes random trajectories, visiting during a finite time each ring of the mesh. During the integration process we measure the residence time in the rings as follows: let t_1^k and t_2^k be the entrance and exit times to the ring $\Gamma(\gamma_k)$, respectively. The residence time in this ring is $t(\gamma_k) = t_2^k - t_1^k$, and we denote the residence time of the n visit event to the ring $\Gamma(\gamma_k)$ by $t_n(\gamma_k)$. Then, if during an integration time I, which is achieved by integrating R realizations of M time steps, there have been V_k visit events to the ring $\Gamma(\gamma_k)$, the mean residence time of the system in this ring is given by the mean of the residence events, that is, $T(\Gamma(\gamma_k)) \equiv \sum_{n=1}^{V_k} \frac{t_n(\gamma_k)}{I}$. Therefore, given a pair (σ, τ), the function defined by the histogram $P(T) \equiv P(T(\gamma_k)) \equiv P(\frac{T(\Gamma(\gamma_k))}{\Delta\lambda})$ is a measure of the probability density for the system state to be in the region defined by the ring $\Gamma(\gamma_k)$. This numerical procedure shows that the system mostly visits the attractors surrounding the ring $\Gamma(\langle\lambda_t\rangle)$. We remark that for our study we have carefully selected the simulation parameters to ensure that the phase space partition does not contain overlapped attractors such that this has a well defined meaning. In particular $\langle\lambda_t\rangle$ and σ are selected in such a way that under a fluctuation of 3σ the system trajectories remain in the region of nonoverlapped attractors. An illustrative example of the residence times density function (RTDF) as a function of the correlation time is depicted in Fig. 1.

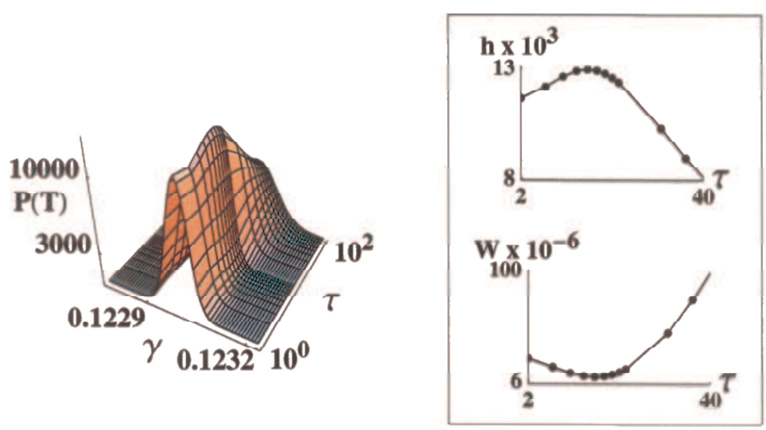

Fig. 1. The RTDF for Eq. (1) with $\lambda = 0.123$ and $\sigma = 5 \times 10^{-4}$. The inset plots show the height and width behavior.

As can be seen from Fig. 1, the localization of the system trajectories depends strongly on τ. The RTDF height shows a nonmonotonous behavior reaching a maximum at a particular value of $\tau \sim \tau^*$ and, at the same value, the width W calculated at the height $h/\sqrt{e}$ shows a remarkable minimum, as represented in the inset curves. As a consequence *the correlation time of the parametric random perturbation acts as a tuner which controls (in a statistical sense) the behavior of the system, maximizing its localization on the region of the phase space surrounding* $\langle\lambda_t\rangle$. Furthermore, the relation h/W (which is essentially the quality factor of the RTDF) has a maximum for a particular value of τ, and this optimal value depends on $\lambda = \langle\lambda_t\rangle$, as can be appreciated in Fig. 2. The behavior shown is a clear indication of a τ-induced SR phenomenon closely related to the localization enhancement.

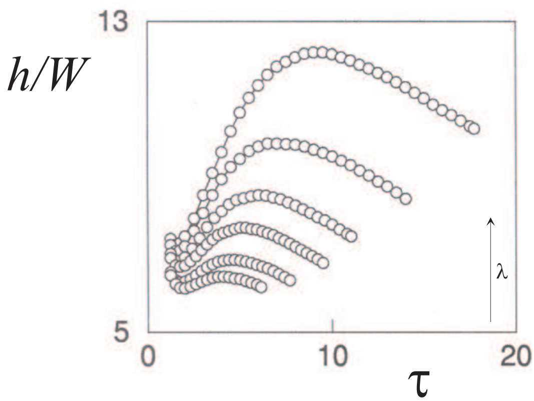

Fig. 2. h/W (the quality factor of the RTDF) *versus* τ. The different curves are (from bottom to top) for increasing values of λ, as indicated by the vertical arrow. The resonance-like behavior increases when moving into the oscillatory region.

To relate the optimal correlation time for maximal localization, τ^*, with the temporal scales of the deterministic counterparts we first study the behavior of the postponement of the bifurcation point because of the multiplicative noise in order to obtain the postponed bifurcation point $\lambda^*_{\mathrm{H}}(\sigma,\tau)$ [6].We next calculate the effective distance to the bifurcation point $\Delta\lambda^* = |\lambda - \lambda^*_{\mathrm{H}}|$, and measure from the deterministic temporal series the period, T^*, of the oscillations when the system is evaluated at a distance $\Delta\lambda^*$ from the deterministic bifurcation point. With this information, we calculate the quantity $\Delta T^* \equiv |T^* - T(\lambda_{\mathrm{H}})|$, where $T(\lambda_{\mathrm{H}})$ is the period of the deterministic system at precisely the Hopf bifurcation point and obtain the following relation between τ^* and ΔT^*

$$\tau^* \sim (\Delta T^*)^{\alpha} \tag{2}$$

with $\alpha = -0.58$.

It is worth pointing out that, from the behavior of the quantity h/W depicted in Fig. 2 for the RTDF, it is clear that a concentration of orbits around a narrow range of bands in the phase space implies a bigger weight of those particular frequencies in the power spectrum of the system, and, as a consequence, a nonmonotonous behavior qualitatively similar to that of Fig. 2 should be expected for the quality factor, β, of the power spectrum, indicating an increase of the coherence in the system response [7]. This is indeed the case for our model (with a power spectrum quality factor showing a maximum for a value of the correlation time close to τ^*).

It is important to mention that a qualitatively similar result can also be obtained for other systems with different nonlinearities and characteristics [4, 8], all of them showing the same effect of enhanced localization of orbits mediated by the correlation time of the multiplicative noise. In all cases the effect is characterized by a power law with exponent close to $-1/2$ indicating the possible universal character of this phenomenon.

3. Stochastic branch selection at secondary bifurcations

The evolution of an actual nonlinear system occurs through bifurcations that take place at certain values of the control parameter. In many situations, a deterministic analysis can not give a complete picture of the behavior of the system. The assumption of deterministic, fixed values of the control parameter is difficult to maintain, and it seems more reasonable to consider it as a random parameter.

In that case, the fact that the stochastic system can explore more dynamical situations (as the control parameter changes in time) may introduce significative changes in its temporal evolution. Descriptions based only on static properties (like stationary distributions [9]) can not give a complete picture of the behavior, and a dynamical description is necessary. In quite general cases, that more complete description can be obtained by the qualitative theory of stochastic systems [10], based on control theory, and the study of Lyapunov exponents [11] or, when it is possible, by the knowledge of the first passage times statistics (as it happens when the stochastic perturbation is dichotomous [12]).

In the study of the behavior of a stochastic system close to a bifurcation point, two assumptions are usually made: the random control parameter is Gaussian (therefore taking values in the whole interval $[-\infty, +\infty]$), and the system is considered close to its first bifurcation. However, in important situations neither of those assumptions are correct or interesting. First, in some cases the control parameter must be bounded, *i.e.*, taking values in a finite interval, by definition (a particularly clarifying example is the quality factor in prebiotic evolution [13], that must take values in the interval $[0, 1]$) or by physical arguments (small variations of temperature, for instance). It is known that the behavior of systems perturbed by bounded noise is quite different, not only from that of the unperturbed, deterministic situation, but also from the stochastic case with Gaussian perturbations. Effects like disappearance of stationary solutions, slowing down, bistability and random symmetry breaking are known to occur [14], and produce substantial changes in the asymptotic behavior (see [12] for a complete analytical description of one dimensional systems perturbed by dichotomous noise).

On the other hand, the behavior of the system at secondary and higher bifurcations is fundamental in the evolution towards more complex structures. From the point of view of the geometry of the bifurcation diagram, there is an apparently trivial, but with important implications, difference between primary and higher order bifurcations. While for the former the system approaches the bifurcation point through an horizontal straight line of steady states, in the latter case it passes the bifurcation point following a tilted line, as shown qualitatively in Fig. 3. There is, therefore, a lack of symmetry at higher order bifurcations. As we will see, this will be a crucial factor to understand the asymptotic behavior of the system.

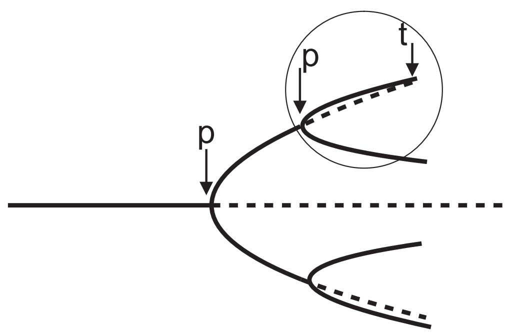

Fig. 3. Schematic view of a deterministic bifurcation diagram, showing one primary and two secondary bifurcations (within the circle). The symbols p and t indicate pitchfork and transcritical, respectively.

In this section we will discuss the temporal evolution of one-dimensional simple systems with bounded stochastic control parameter, passing through two paradigmatic bifurcations that usually come together in high order bifurcations: pitchfork and transcritical (see Fig. 3). To be precise, for the analytical discussion the bounded noise will be a dichotomous process, while the numerical simulations that will illustrate the behavior will be made with a continuous bounded noise.

Let us first consider the deterministic equation

$$\dot{x} = -(\alpha_t - x^2)(\alpha_t - kx) \tag{3}$$

as an example of a system undergoing bifurcations without horizontal branches. To fix ideas we choose $k > 0$, and with this, the straight branch has a positive slope. Two local bifurcations occur at the values $\alpha_1 = 0$ (pitchfork) and $\alpha_2 = k^2$ (transcritical).

We now suppose that the bifurcation parameter α is perturbed by a symmetric dichotomous noise, ξ_t, around its mean value

$$\alpha_t = \alpha + \xi_t \,. \tag{4}$$

In this way, α_t can only take two possible values: $\alpha \pm \Delta$ alternatively, with the time between switches governed by the distribution $\phi(t) = \lambda \exp(-\lambda t)$. The average residence time in each of the two states is $1/\lambda$, whereas the correlation time of the noise is given by $\tau_c = 1/2\lambda$.

The dynamics of this model is the stochastic mixture of two deterministic autonomous systems governed by the equation (3) when the control parameter takes one of its two possible values $\alpha \pm \Delta$. We will call $F_\pm$ the two forces acting on the system, *i.e.*, the right hand side of (3) for $\xi_t = +\Delta$ and $\xi_t = -\Delta$, respectively. The asymptotic analysis may be carried out by looking at the first passage and sojourn times statistics for intervals between two successive zeroes of the corresponding forces $F_\pm$ (see [12] for the technical details). If a and b are the extremes of such an interval, the Laplace transform of the first passage times distributions are [12]

$$\tilde{f}_b^{+}(s|x_0) \;=\; \mathrm{e}^{-(\lambda+s)T_+(x_0\to b)} + \lambda \int\limits_{x_0}^{b} \frac{dx_1}{F_+(x_1)} \mathrm{e}^{-(\lambda+s)T_+(x_0\to x_1)} \tilde{f}_b^{-}(s|x_1)\,, \tag{5}$$

$$\tilde{f}_b^{-}(s|x_0) \;=\; -\lambda \int\limits_{a}^{x_0} \frac{dx_1}{F_-(x_1)} \mathrm{e}^{-(\lambda+s)T_-(x_0\to x_1)} \tilde{f}_b^{+}(s|x_1)\,, \tag{6}$$

where $f_{a,b}^{\pm}(t|x_0)dt$ are the probabilities of first reaching, in the time interval $(t, t+dt)$, the boundary a or b starting at $x_0 \in [a,b]$, and taking into account the initial value of the noise $\pm\Delta$. The quantities

$$T_\pm(x \to y) = \int\limits_{x}^{y} \frac{dx'}{F_\pm(x')} \tag{7}$$

are the times to go from x to y under the corresponding force.

Equations for the other two distributions can be obtained from (5) and (6) by switching simultaneously b to a and $+$ to $-$. The qualitative behavior of the system can be deduced from the zeroth order moments (the escape probabilities) of the distributions, while the first moments give us an indication of the characteristic times in which the system evolves.

The result of the analysis allows us to draw the stochastic bifurcation diagram (that depends on the interval of variation of the control parameter) and discuss the asymptotic behavior of the system [12]. For our simple model this can be qualitatively visualized in the stochastic bifurcation diagram of Fig. 4, corresponding to a perturbation of small amplitude, *i.e.*, $\Delta < \Delta_c = \frac{1}{2}(\alpha_2 - \alpha_1)$.

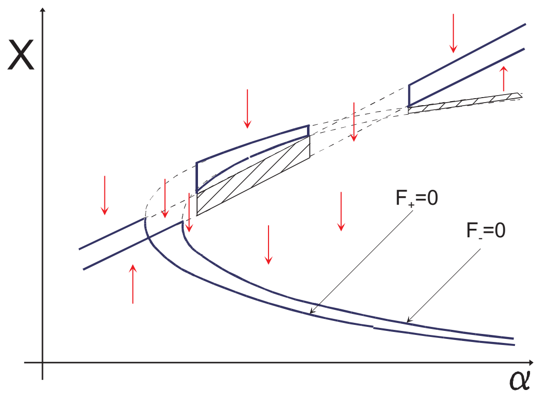

Fig. 4. Stochastic bifurcation diagram, corresponding to Eq. (3) with bounded noise. The arrows indicate the direction of the flow (see the text for an explanation of the meaning of the different regions).

The interpretation of this bifurcation diagram is as follows: for a fixed value of the mean of the bifurcation parameter α, and in the limit $t \to \infty$, the trajectories will be confined between the thick lines (which are portions of the curves $F_+ = 0$ and $F_- = 0$). Projection of the intervals between these thick lines onto the y axis gives the exact support region of the corresponding stationary probability distribution. The arrows indicate the behavior depending on the initial conditions (the direction of the stochastic flow), and the filled regions denote bistability areas, in the sense that a trajectory starting in any of these areas will leave it, with probability one, by crossing one of the boundaries. If the upper boundary is crossed, the trajectory will reach the upper stationary region (which is an invariant set). If the other boundary is crossed, the trajectory will reach the lower stationary (invariant) region. Notice that, due to the bounded character of the noise, the two invariant sets are not connected, and jumps between the two regions are not permitted. This last behavior, that is recovered when considering a Gaussian noise, would be the consequence of the existence of a unique, bimodal, stationary probability distribution, whereas in the present case there are actually two different stationary probability distributions with unconnected supports. Finally, dotted lines are not part of the bifurcation diagram, but are included to indicate regions in which some delay occurs, *i.e.*, regions in which trajectories move back and forth for some time but they end up escaping, with probability one, in the direction indicated by the arrows.

The main conclusions that can be deduced from this qualitative picture under these conditions, are: *(i)* the noise forces the system to choose, for any initial condition and with probability one, only one of the branches in a pitchfork bifurcation, depending on the slope of the line of stationary states crossing the bifurcation point; *(ii)* the noise, again for any initial condition and with probability one, drives the system away from the region around a transcritical bifurcation. In other words, *the noise drives the system deterministically through successive bifurcations.*

It should be remarked that the qualitative behavior discussed above does not change if instead of a dichotomous noise, one considers a continuous diffusion-like process taking values in a bounded interval, although the quantitative properties may be impossible to calculate analytically. Moreover, other situations can be observed if the interval of variation of the control parameter changes, and, for example, the bistability regions may disappear if the interval becomes big enough to cover the two local bifurcation points. We will not discuss these possibilities any further.

To end up this section, and to illustrate the temporal evolution of the system, in Fig. 5 we show the result of a numerical simulation of our model with the control parameter perturbed by a bounded, symmetric and continuous noise. In Fig. 5(a) the system passes through two successive pitchfork bifurcations, whereas Fig. 5(b) depicts the very fast escape of the system from the region around a transcritical bifurcation.

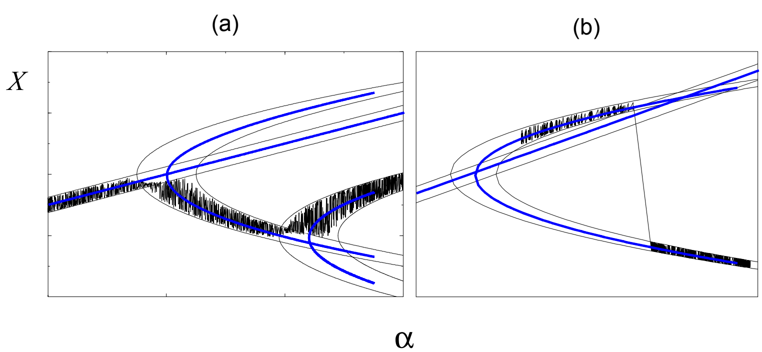

Fig. 5. Numerical simulation of the system passing through two pitchfork bifurcations (a), and a transcritical bifurcation (b).

4. Noise in spatially extended systems

The systems considered in the preceding sections had in common that they were homogeneous. When considering spatially extended systems, many more possibilities may appear due to the spatial couplings in the system. In particular, the combination of spatial coupling and noise may amplify and extend in time a short time noise-induced instability, leading to a diversity of phenomena, such as, for instance, first and second order phase transitions, noise-induced ordered phases, and ordered spatial structures [15, 16]. In the context of the present paper, we will concentrate on the appearance of order/disorder phase transitions and on spatial pattern formation.

4.1. Noise-induced phase transitions

To begin with, we consider the time-dependent Ginzburg–Landau model (GL), well known in the theory of equilibrium critical phenomena. To mimic random quenched impurities, which play an important role in the physical properties of many systems, and , at the same time, to have a system amenable to analytical study, we perturb the control parameter of the original GL model by a dichotomous stochastic process with infinite correlation time. If $\{\psi_i\}$ is a scalar field defined on a d-dimensional square lattice, the time dependent GL model with dichotomous quenched impurities is given by the following dimensionless Langevin equation [17]

$$\partial_t \psi_i = (\alpha + \zeta_i)\psi_i - \psi_i^3 + \frac{D}{2d} \sum_{\langle ji \rangle} (\psi_j - \psi_i) + \eta_i \,, \tag{8}$$

where the sum runs over the $2d$ nearest neighbors of site i, and η_i are Gaussian white noises with zero mean and correlation

$$\left\langle \eta_i(t) \eta_j(t') \right\rangle = \delta_{ij} \delta(t - t') \,. \tag{9}$$

The quenched multiplicative noises ζ_i, that represent impurities in the system, are Markovian dichotomous processes spatially uncorrelated and with infinite correlation time. The probability density of the impurities is

$$P(\zeta_i) = p_+ \delta(\zeta_i - \Delta) + p_- \delta(\zeta_i + \Delta) \,, \tag{10}$$

where Δ denotes the intensity of the multiplicative noise. Note that, when referring to the impurities, the terms *probability* and *proportion* can be exchanged in the thermodynamic limit, since then $p_\pm$ is equal to the proportion of $\pm\Delta$ impurities.

The Eq. (8) can also be described in terms of a local potential at each site given by

$$V\left(\psi_i;\zeta_i\right) = -\frac{(\alpha+\zeta_i)}{2}\psi_i^2 + \frac{\psi_i^4}{4}\,. \tag{11}$$

The sign of the quadratic term of the potential determines the local dynamics at each site. If $(\alpha+\zeta_i)<0$, we have a single well potential with a single equilibrium point at $\psi_i=0$. On the other hand, if $(\alpha+\zeta_i)>0$, the state $\psi_i=0$ becomes unstable and $V\left(\psi_i;\zeta_i\right)$ is a double well potential with two symmetric stable points at $\psi_i=\pm\sqrt{\alpha+\zeta_i}$.

To have an idea of the phase diagram of the model, we use a Weiss mean field approximation for spatially extended systems [16]. Replacing, in the diffusive term, the field at the nearest neighbors by the mean value, $\langle\psi\rangle$, we can drop the lattice index and write down the following equation for the temporal evolution of the field at a generic site

$$\partial_t\psi = (\alpha+\zeta)\psi - \psi^3 + D\left(\langle\psi\rangle - \psi\right) + \eta\,, \tag{12}$$

where ζ is a random variable with a probability distribution given by (10). This equation is equivalent to the system

$$\partial_t\psi_\pm = (\alpha\pm\Delta)\psi_\pm - \psi_\pm^3 + D\left(\langle\psi\rangle - \psi_\pm\right) + \eta\,, \tag{13}$$

where $\psi_\pm$ is the field at a site where $\zeta=\pm\Delta$.

Equation (12) is not a closed evolution equation for the stochastic process ψ, but it can be easily solved in the stationary regime with $\langle\psi\rangle$ as a parameter. The stationary solution reads

$$P_{\rm st}\left(\psi;\langle\psi\rangle\right) = p_+P_{\rm st}^+\left(\psi;\langle\psi\rangle\right) + p_-P_{\rm st}^-\left(\psi;\langle\psi\rangle\right)\,, \tag{14}$$

where $P_{\rm st}^\pm\left(\psi;\langle\psi\rangle\right)$ are the stationary probability densities for the two dynamics defined by (13). These probability densities are

$$P_{\rm st}^\pm\left(\psi;\langle\psi\rangle\right) = N^\pm{\rm e}^{-2\left[V(\psi;\pm\Delta)+D\psi(\psi-\langle\psi\rangle)\right]}\,, \tag{15}$$

where the potential $V\left(\psi;\zeta\right)$ is defined in (11) and $N^\pm$ are normalization constants.

Finally, the following self-consistent condition must be fulfilled

$$\langle\psi\rangle = \int_{\Re}\psi P_{\rm st}\left(\psi;\langle\psi\rangle\right){\rm d}\psi\,. \tag{16}$$

This equation has $\langle\psi\rangle=0$ as a solution for any value of the parameters. This solution is called the *disordered phase.* However, non symmetric solutions exist in some regions of the space of parameters. These solutions with $\langle\psi\rangle \neq 0$ are called *ordered phases.* A phase transition occurs when the system is driven from a region with only the symmetric solution to a region with ordered phases.

According to the mean field theory, phase transitions occur at those values of the parameters satisfying the condition

$$\int_{\Re} \psi \left. \frac{\partial P_{\rm st}\left(\psi;\langle\psi\rangle\right)}{\partial\langle\psi\rangle}\right|_{\langle\psi\rangle=0} {\rm d}\psi = 2D\int_{\Re} \psi^2 P_{\rm st}\left(\psi;0\right){\rm d}\psi = 1\,. \qquad (17)$$

In order to discuss the phase diagram in the plane (Δ, D) given by Eq. (17), we must distinguish two cases: $\alpha < 0$ and $\alpha > 0$.

For α negative and $\Delta < |\alpha|$, it is obvious that no ordered phase can exist, since $\psi = 0$ is stable in the two possible local potentials (11). On the other hand, if $\Delta > |\alpha|$ a fraction p_+ of sites feels a double well potential and then, for strong enough coupling D, an ordered phase may appear. In Fig. 6 we plot the phase diagram for $\alpha = -0.75$ and several values of p_+. Note that,

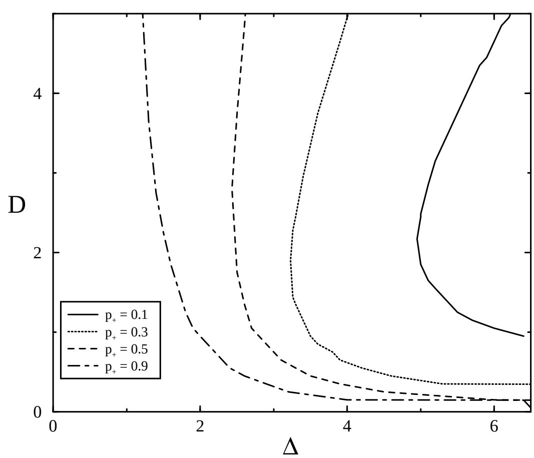

Fig. 6. Phase diagram for $\alpha = -0.75$ and different values of p_+. The ordered regions are located to the right of the phase boundaries. The possibility of DOD reentrant phase transitions depends on the values of p_+.

below a certain value of p_+, a reentrant transition disorder-order-disorder (DOD) with the coupling appears. That is, by continuously increasing the coupling we can first drive the system from a disordered to an ordered state and then back to a disordered state. This reentrant phenomenon is always present in the system below that critical value of p_+. Note that by decreasing p_+ the ordered phase shifts to the right, due to the fact that the fraction of double well local potentials decreases and then these potentials have to be deeper, *i.e.*, Δ must be larger, in order to keep stable the ordered phase.

For $\alpha > 0$, we have two possibilities. If $\Delta < \alpha$, every site feels a double well potential and therefore an ordered phase appears for a given value of the coupling. On the other hand, if $\Delta > \alpha$ the competition between the two dynamics produces new transitions depending on the value of p_+. Fig. 7 shows the phase boundaries for $\alpha = 0.75$. There is a topological change around $p_+ = 0.22$: below this value, the region of disordered states is connected and the region of ordered states is disconnected (see for instance the curve for $p_+ = 0.2$, whereas above $p_+ = 0.22$ it is the other way around (see $p_+ = 0.225$). Note that in this case there are two kinds of reentrant phase transitions: the one described previously (DOD) and a new one order-disorder-order (ODO) increasing the noise intensity Δ. This new reentrant phase transition appears above a given value of the coupling.

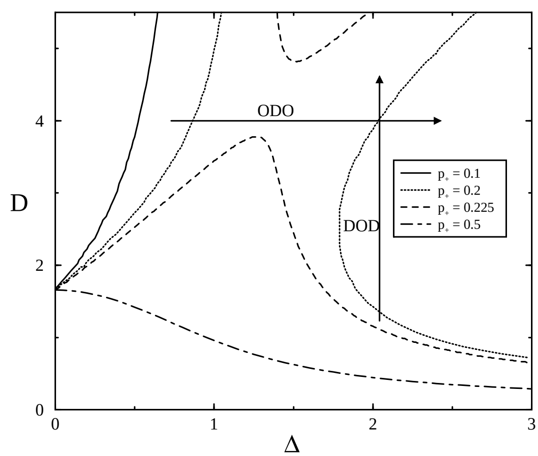

Fig. 7. Phase diagram for $\alpha = -0.75$ and different values of p_+. Two distinct reentrant phase transitions, DOD and ODO, are indicated by the arrows.

The behavior of the order parameter, $m = |\langle\psi\rangle|$, is depicted in Fig. 8 as a function of D for the DOD transition (a), and as a function of Δ for the ODO transition (b).

It is important to stress that the approximate mean field analysis presented above is qualitatively confirmed by precise computer simulations of a two-dimensional version of the original model (8), which, on the other hand, allow us to locate the position of the critical points and study the dependence of the order parameter and the susceptibility with the size of the system [17]. Fig. 9 depicts the result of a numerical simulation of the field, clearly showing the ODO by increasing the noise strength Δ.

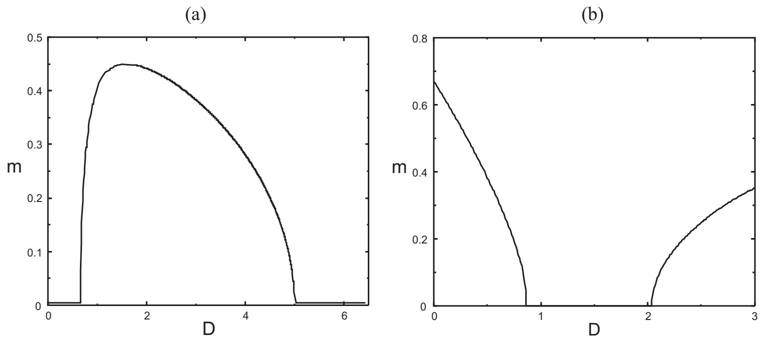

Fig. 8. Behavior of the order parameter: (a) as a function of $D(p_+=0.3, \alpha=-0.75, \Delta=4)$ for a DOD reentrant transition; (b) as a function of Δ ($p_+=0.2$, $\alpha=0.75$, $D=4$) for an ODO reentrant transition.

4.2. Pattern formation

The ordering or disordering effect of additive and multiplicative noise on the formation and stability of spatially ordered structures in different systems has attracted much interest in the last decade [15]. In particular, for the Swift–Hohenberg equation (a model to describe the onset of Rayleigh–Bénard convection), it has been shown that the presence of a multiplicative noise in the control parameter may advance the appearance of patterns. However, in this and other models the existence of patterns is already in the deterministic version of the equations, and it is interesting to ask for the possibility of having *pure noise-induced patterns*, *i.e.*, a stochastic system with patterns that are not present in the deterministic situation.

To study that possibility we take the following equation [18, 19]

$$\frac{\partial \varphi}{\partial t} = -\varphi\left(1+\varphi^2\right)^2 - D\left(\nabla^2 + k_0^2\right)\varphi + \left(1+\varphi^2\right)\xi\,, \tag{18}$$

where $\varphi(\boldsymbol{r}, t)$ is a scalar field taking values in a d dimensional space, the spatial coupling is of the Swift–Hohenberg type, and ξ is a noise, white in space and time, with the properties

$$\langle \xi_i(t)\rangle = 0 \quad \langle \xi_i(t)\xi_j(t')\rangle = \sigma^2 \delta_{i,j}\delta(t-t')\,. \tag{19}$$

The dispersion relation related to the coupling operator $\mathcal{L} = -D\left(\nabla^2 + k_0^2\right)$, when applied to a plane wave, is

$$\mathcal{L}\mathrm{e}^{i\mathbf{kr}} = \omega\left(\boldsymbol{k}\right)\mathrm{e}^{i\boldsymbol{kr}} = \left[-D\left(k_0^2 - \boldsymbol{k}^2\right)^2\right]\mathrm{e}^{i\boldsymbol{kr}} \tag{20}$$

with k_0 as the most unstable mode. It is easy to show that if we set $\xi = a$, a constant, non stochastic value, the homogeneous solution is the unique steady state of (18).

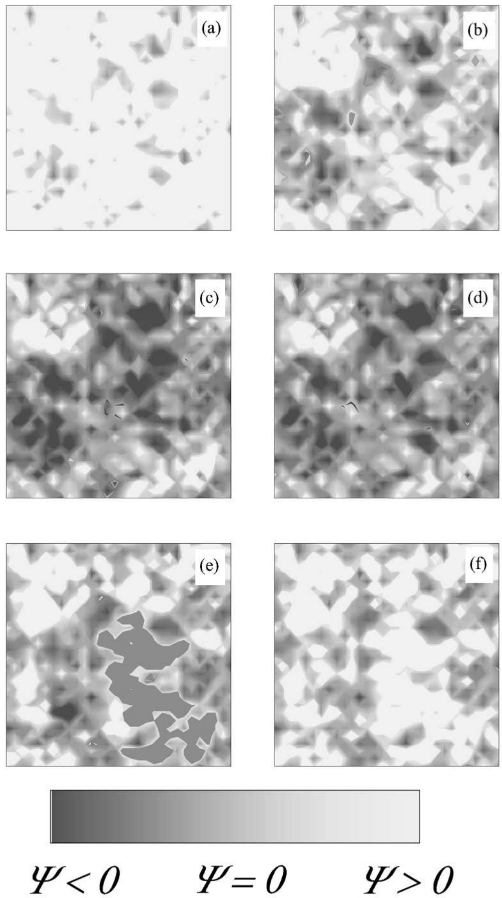

Fig. 9. Stationary values of the field for increasing values of Δ, from (a) to (f), showing an ODO phase transition.

To obtain information on the behavior of the stochastic system, a mean field analysis can be performed on a discretized version of (18) on a d-dimensional lattice with lattice spacing Δx [18]. Considering a specific site $\varphi_i = \varphi$ (and $\xi_i = \xi$), and replacing the value of the scalar variable φ_j at the sites coupled to φ_i by a non-uniform average field

$$\varphi_j = \langle\varphi\rangle \cos\left[\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')\right] . \tag{21}$$

The equation for φ_i takes on the following simple form

$$\dot{\varphi}_i = -\varphi_i\left(1+\varphi_i^2\right)^2 + \left(1+\varphi_i^2\right)\xi + D\omega(\boldsymbol{k})\varphi_i - D_{\text{eff}}(\varphi_i - \langle\varphi\rangle) \tag{22}$$

with

$$\omega\left(\boldsymbol{k}\right)=-D\left(k_0^2-\frac{2d}{\Delta x^2}+\frac{2}{\Delta x^2}\sum_{i=1}^{d}\cos\left(\Delta x k_i\right)\right)^2 \tag{23}$$

and

$$D_{\text{eff}}=D\left[\left(\frac{2d}{\Delta x^2}-k_0^2\right)^2+\frac{2d}{\Delta x^4}+\omega(\boldsymbol{k})\right]. \tag{24}$$

Notice that $\langle\varphi\rangle$ now plays the role of the *amplitude* of the spatial patterns, and that if $\Delta x \ll 1$ the dispersion relation (23) reduces to the continuous case $\omega\left(\boldsymbol{k}\right)=-D\left(k_0^2-\boldsymbol{k}^2\right)^2$.

Applying now the standard techniques of the mean field theory [16], we may obtain the points in the D *versus* σ^2 plane at which a particular $\boldsymbol{k}$-vector becomes unstable. Fig. 10 depicts the phase diagram for a two-dimensional system and for some particular choices of the parameters.

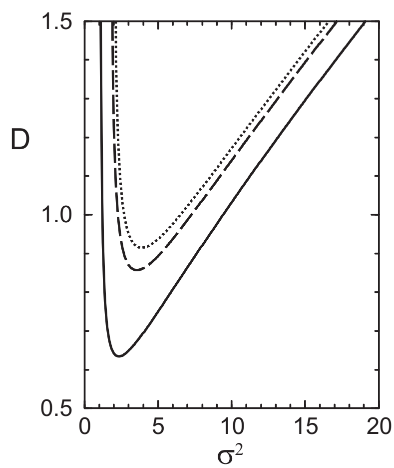

Fig. 10. Phase diagram obtained by a mean field analysis with $k_0=\Delta x=1$. The different curves are for distinct values of the wave vector $\boldsymbol{k}$ (increasing from top to bottom). The solid line corresponds to the most unstable mode, such that $\omega\left(\boldsymbol{k}\right)=0$.

The system shows patterns in the regions above the curves, and it can be seen that, for a fixed value of D, and increasing the intensity of the noise the system goes from an homogeneous state to a spatial pattern situation. This spatially ordered state is destroyed by another transition that happens when the noise intensity is big enough, although this final state is different from the one corresponding to small noise, since now many modes are destabilized. Fig. 11 shows the result of a numerical simulation of (18) on a two-dimensional lattice, illustrating the noise-induced patterns.

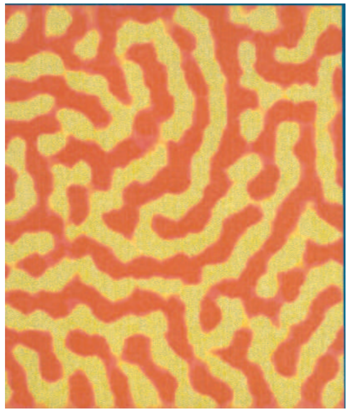

Fig. 11. Numerical simulation of a two-dimensional version of Eq. (18), showing the noise-induced patterns. The values of the parameters are $k_0 = \Delta x = 1$, $D = 15$, and $\sigma^2 = 10$.

At this point it is worth pointing out that the nonlinearities of the model are not crucial for the existence of a phase transition in our model, as it can be shown that in the limit $D \to \infty$, the phase transition occurs at the point where the coefficient of the linear term, including the noise contribution resulting from the Stratonovich interpretation, vanishes [16].

Finally, it is interesting to mention that a novel mechanism for pattern formation, in systems with alternating dynamics, has been recently proposed [20], opening new possibilities in this important field.

5. Conclusions

In this paper we have presented several examples showing convincing evidence of the constructive role of noise. As illustrated in the quite different cases considered, the noise may lead to a better localization of orbits in periodic systems, drive the system almost deterministically through successive bifurcations, and promote ordered phases and spatial patterns not present in the deterministic counterpart. It seems clear that one should get rid of the idea that the presence of noise in nonlinear systems is merely a disturbance over a convenient regular (deterministic) behavior. The evolution of actual nonlinear systems should be considered as a combination of deterministic laws and fluctuations.

This work has been supported by Dirección General de Investigación (Spain) Project No. BFM2001-0291. JB also acknowledges financial support by MECD (Spain) Grant No. EX2001-02880680.

REFERENCES

[1] R. Benzi, A. Sutera, A. Vulpiani, *J. Phys.* **A14**, L453 (1981); C. Nicolis, G. Nicolis, *Tellus* **33**, 225 (1981).

[2] L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni, *Rev. Mod. Phys.* **70**, 223 (1998).

[3] I. Hidaka, D. Nozaki, Y. Yamamoto, *Phys. Rev. Lett.* **85**, 3740 (2000); F. Marino, M. Giudici, S. Barland, S. Balle, *Phys. Rev. Lett.* **88**, 040601 (2002); T. Mori, S. Kai, *Phys. Rev. Lett.* **88**, 218101 (2002).

[4] J.L. Cabrera, J. Gorroñogoitia, F.J. de la Rubia, *Phys. Rev. Lett.*, **82**, 2816 (1999).

[5] E.E. Sel'kov, *Eur. J. Biochem.* **4**, 79 (1968).

[6] L. Fronzoni, R. Mannella, P.V.E. McClintock, F. Moss, *Phys. Rev.* **A36**, 834 (1987).

[7] H. Gang, T. Ditzinger, C.Z. Ning, H. Haken, *Phys. Rev. Lett.* **71**, 807 (1993).

[8] M.N. Lorenzo, V. Pérez-Muñuzuri, R. Deza, J.L. Cabrera, *Int. J. Bif. and Chaos* **11**, 2663 (2001).

[9] W. Horsthemke, R. Lefever, *Noise-Induced Transitions*, Springer-Verlag, Berlin 1984.

[10] L. Arnold, W. Kliemann, *Qualitative Theory of Stochastic Systems*, in *Probabilistic Analysis and Related Topics*, edited by A.T. Bharucha-Reid, Academic Press, New York 1983, Vol.3.

[11] L. Arnold, *Lyapunov Exponents of Nonlinear Stochastic Systems*, in *Nonlinear Stochastic Dynamic Engineering Systems*, edited by F. Ziegler and G.I. Schueller, Springer-Verlag, Berlin 1988.

[12] J. Olarrea, J.M.R. Parrondo, F.J. de la Rubia, *J. Stat. Phys.* **79**, 669 and 683 (1995).

[13] J.C. Nuño, F. Montero, F.J. de la Rubia, *J. Theor. Biol.* **165**, 553 (1993).

[14] F. Colonius, W. Kliemann, *Nonl. Dyn.* **5**, 353 (1994).

[15] J. García-Ojalvo, J.M. Sancho, *Noise in Spatially Extended Systems*, Springer, New York 1999.

[16] C. Van den Broeck, J.M.R. Parrondo, R. Toral, R. Kawai, *Phys. Rev.* **E55**, 4084 (1997).

[17] J. Buceta, J.M.R. Parrondo, F.J. de la Rubia, *Phys. Rev.* **E63**, 031103 (2001).

[18] J.M.R. Parrondo, C. Van den Broeck, J. Buceta, F.J. de la Rubia, *Physica* **A224**, 153 (1996).

[19] This particular choice is dictated by the fact that its simple diffusive version is a paradigmatic example of pure noise-induced phase transitions (see [16] for details).

[20] J. Buceta, K. Lindenberg, J.M.R. Parrondo, *Phys. Rev. Lett.* **88**, 024103 (2002).