ON THE STRETCHED EXPONENTIAL SURVIVAL PROBABILITY AND ITS RELATION TO RAJAGOPAL RELAXATION-TIME DISTRIBUTION*

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Two different frameworks of the relaxation phenomenon are considered to conduct a detailed analytical analysis on the relationship of the stretched exponential relaxation function and the relaxation function resulting from a certain log-relaxation-time density proposed by Rajagopal. In the first part the analytical comparison of these two functions in the purely heterogeneous picture is presented. The considerations are based on the interpretation of the relaxation function as a survival probability of the initial state of a relaxing system expressed by means of the weighted average of an exponential decay with respect to the distribution of the effective relaxation time. In the second part a certain degree of intrinsic nonexponentiality is assumed which allows to show the stochastic scheme leading directly from the Rajagopal density to the stretched exponential relaxation response. In both approaches the strict connection of Rajagopal function and the one-sided stable density is shown.

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1. Introduction

In the last two decades the nonexponential relaxation behaviour of complex systems became a problem of topical interest in nonequilibrium statistical physics. It is of importance in the study of a variety of problems from condensed matter physics [1,2], nuclear physics [3], spectroscopy [4,5], rheology [6], seismology [7], physical chemistry [8], molecular biophysics [9,10], cell and and population dynamics [11,12] *etc.*

The wide-ranging experimental information has led to the conclusion that the classical phenomenology of relaxation breaks down in complex systems. It has been found that the pure exponential (Debye) relaxation pattern is hardly ever seen in nature and that deviation from it may be relatively large [1,2,13,14]. Experimental evidence is given usually in terms of the relaxation function $\phi(t)$ that indicates the time evolution of an initial nonequilibrium state imposed on the system in time $t = 0$. The transition of the relaxing system at time $t > 0$ is defined by the change of some physical parameter that differs the initial and relaxed states. Typically, the experiment probes the ensemble average, in the sense that only the net effect of a large number of contributions from different sites within a sample is measured. Regarding the statistical average in the sample volume, one observes the nonexponential decay pattern of relaxation. Such an ensemble-averaged relaxation mechanism can be rationalized in two ways [15]. In the dynamically heterogeneous picture, the contributions are assumed to be purely exponential subject to the distribution of individual relaxation times. In the other extreme, *i.e.* in the homogeneous picture, the contributions to the effective relaxation response are not site specific. On the scale ranging from the heterogeneous to the homogeneous limit, one has to allow for the case of certain degree of intrinsic nonexponentiality combined with site-specific time scales.

Among different empirical relaxation functions suggested in literature the Kohlrausch–Williams–Watts (KWW) stretched exponential function

$$\phi_{\mathrm{KWW}}(t) = \exp\left[-(t/\tau_0)^{\alpha}\right] \tag{1}$$

plays an important role. The parameters $\tau_0 > 0$ and $0 < \alpha < 1$ are constants characteristic to the material. Although the stretched exponential decay law (1) is not universally valid [1], it appears frequently enough to call attention for the origins of its ubiquity. The wide occurrence of this relaxation pattern, independently of the particular system properties, has attracted (and still does) much theoretical attention for the underlying reason of this phenomenon. It has been commonly assumed that the empirical law (1) corresponds to a kind of universal behaviour which is independent of the details of examined systems. This idea has stimulated the proposal of several relaxation mechanisms (see *e.g.* [16–27]) that differ mainly in the mathematical interpretation of the relaxation function. In the framework of statistical models the fact that the large scale dynamical behaviour of the complex system is, to some extent, independent of its local nature, comes as no surprise. Intuitively, one expects averaging principles (like the law of large numbers) to be in force. It turns out, however, that it is very hard to make this intuition precise when one deals with stochastic systems. The crucial point is to find a mathematical language which allows one to relate the local random characteristics of the complex system (*i.e.* the site-specific properties of the time scales) to the deterministic empirical laws. Such a possibility follows from the general probabilistic formalism of limit theorems [19, 21–23, 28]. The language of limit theorems, in a natural way, gives an efficient and strict procedure of averaging the random contributions from different sites within the sample. It also formulates the statistical conditions leading to a particular response [22, 23].

The objective of this paper is to find the probabilistic scheme underlying the Rajagopal relaxation-time distribution [29]. This distribution has been recently proposed [30] as a function describing the effective-relaxation-time properties of the KWW relaxation pattern well enough in the picture of exponentially relaxing entities (relaxors). The Rajagopal function, because of its explicit analytical form, is of great importance from the experimental point of view, and hence, worth theoretical studies. The paper consists of two main parts. In the first, we study the relationship between the KWW and the relaxation function resulting from the Rajagopal distribution in the originally proposed framework [30]. We show the origins of similarities and differences in both responses. In the second part we discuss the heterogeneous picture within which the Rajagopal distribution leads directly to the KWW relaxation function.

2. Nonexponential relaxation. Probabilistic background

It is commonly accepted that the relaxation function $\phi(t)$, representing the effective nonexponential relaxation pattern, results from a superposition of exponential (Debye) processes with different relaxation times. Mathematically, this idea is usually expressed as the weighted average of an exponential decay $\mathrm{e}^{-t/\tau}$ with respect to the distribution of the relaxation time τ:

$$\phi(t) = \int_0^\infty g(\tau) \exp(-\frac{t}{\tau}) d\tau, \tag{2}$$

where $g(\tau)$ is the relaxation-time probability density function. The above formula has been used mainly as a formal mathematical tool [31, 32] convenient to describe, analyze and transform the data in order to compare them with the results obtained by different experimental methods. The notion of the complex system needs, however, understanding the distribution-function approach in the framework of probability theory which tools are capable of relating the local random properties to the effective representation of the system [21, 28].

It is well-known fact of probability theory [33] that formula (2), which concerns the case when the relaxation time cannot be limited to any finite nor countable set of values, or its discrete version $\phi(t) = \sum_i p_i \mathrm{e}^{-\frac{t}{\tau_i}}$ which concerns the case when the relaxation time can take values from the set $\{\tau_1, \tau_2, \ldots\}$ with the corresponding weights $p_1, p_2, \ldots$, denotes simply the expected value $\langle\cdot\rangle$ of the random function $\mathrm{e}^{-t/\tilde{T}}$ with respect to the distribution of the non-negative random variable $\tilde{T}$ representing the considered relaxation time:

$$\phi(t) = \langle \mathrm{e}^{-\frac{t}{\tilde{T}}} \rangle. \tag{3}$$

Introducing a new variable $\tilde{\beta} = \frac{1}{\tilde{T}}$, representing the corresponding relaxation rate, we get equivalent formula

$$\phi(t) = \langle \mathrm{e}^{-t\tilde{\beta}} \rangle \tag{4}$$

which assigns the relaxation function to the Laplace transform of the relaxation-rate probability density function. It follows from formula (3) that the relaxation function $\phi(t)$ defined in (2), is a function describing the relaxation process of the system as a whole and the formula (2) concerns, in fact, the effective behaviour of the macroscopic system represented by one (real or imaginary) object with the value of relaxation time τ randomly taken according to the distribution of the random variable $\tilde{T}$. Hence, neither formula (2) nor its discrete version directly contain information on local random properties of the relaxing system (*i.e.* on relaxation-time distributions of individual relaxors). This information is hidden in the explicit form of a particular relaxation function and can be brought to light by means of limit theorems of probability theory, using the fact that $\phi(t)$ has to fulfill the two-state master equation (see *e.g.* [24])

$$\frac{d\phi(t)}{dt} = -r(t)\phi(t), \quad \phi(0) = 1, \tag{5}$$

the solution of which has the meaning of the survival probability of the nonequilibrium (imposed) initial state of the relaxing system (*i.e.* the probability that the system as a whole will not make a transition out of its original state for at least time t after entering it at $t = 0$ [21, 25, 26]). The nonnegative, time-dependent quantity $r(t)$ is the transition rate of the relaxing system. The survival probability of the nonequilibrium state of the entire system can be related with the site specific properties of individual relaxors if one considers a system of N independent, exponentially relaxing species, each (i-th) characterized by its own random relaxation time T_i (or the corresponding relaxation rate $\beta_i = \frac{1}{T_i}$). Each relaxor is waiting for transition for a time θ_i; the waiting times of all N relaxors form a sequence $\{\theta_1, \ldots \theta_N\}$ of identically distributed, independent nonnegative random variables. The behaviour of each relaxor is conditioned only by the value taken by its characteristic relaxation constant. Namely, the probability that the i-th relaxor has not changed its initial state up to the moment t, under the condition that its relaxation-rate $\beta_i = \frac{1}{T_i}$ takes the value $b = \frac{1}{\tau}$, is

$$\Pr(\theta_i \geq t | \beta_i = b) = \exp\left[-\left(\frac{t}{\tau}\right)\right] = \exp(-bt) \qquad \text{for } t \geq 0,\ b > 0. \tag{6}$$

It has been shown by means of order statistics (see for details [21, 28]) that the above assumptions lead to the relaxation function of the form

$$\phi(t) = \prod_{i=1}^{N} \langle \mathrm{e}^{-t\beta_i/A_N} \rangle = \langle \mathrm{e}^{-t\tilde{\beta}_N} \rangle, \tag{7}$$

where

$$\tilde{\beta}_N = \frac{1}{A_N} \sum_{i=1}^{N} \beta_i \tag{8}$$

is the effective random relaxation rate of an "averaged" relaxor representing the properties of the system as a whole. A_N denotes a sequence of suitable normalizing (scaling) constants. On the basis of limit theorems of probability theory the distribution of $\tilde{\beta}_N$ in Eq. (8) can be satisfactorily approximated by the weak limit

$$\tilde{\beta} = \lim_{N\to\infty} \frac{1}{A_N} \sum_{i=1}^{N} \beta_i \tag{9}$$

even if the distribution of β_i is known in a relatively limited extent. It is a strict result that the only possible probability distributions for the effective relaxation rate $\tilde{\beta}$ are completely asymmetric Lévy-stable laws $\tilde{\beta} \overset{d}{\cong} \mathrm{const} S_\alpha$ (the relation $\overset{d}{\cong}$ denotes the equality in distribution) with the parameter $0 < \alpha < 1$ leading directly to the KWW relaxation function. As $\alpha \to 1$ we obtain the degenerate case, *i.e.* the density function of $\tilde{\beta}$ becomes the Dirac-delta $\varrho(b) = \delta(b - b_0)$, yielding the Debye decay. The necessary and sufficient condition for the convergence of the sum (9) is the self-similar property of the distribution of the individual relaxation rate β_i that may be expressed as:

$$\Pr(\beta_i \geq xb) \approx x^{-\alpha}\Pr(\beta_i \geq b) \quad \text{as} \quad b \to \infty \quad \text{and} \quad x > 0. \tag{10}$$

3. Relationship between the Rajagopal relaxation-time distribution and the KWW relaxation response

The relaxation function $\phi(t)$, represented by equivalent formulas (3) and (4) that connect the deterministic response of the relaxing system with its effective random representation, may be written in the following forms

$$\phi(t) = \int_0^\infty g(\tau)\mathrm{e}^{-\frac{t}{\tau}}d\tau = \int_0^\infty h(\log\tau)\mathrm{e}^{-\frac{t}{\tau}}d\log\tau = \int_0^\infty \varrho(b)\mathrm{e}^{-tb}db, \tag{11}$$

where $g(\tau) = C\frac{1}{\tau}h(\log\tau)$ and $\varrho(b) = \frac{1}{b^2}g\left(\frac{1}{b}\right) = \frac{C}{b}h(\log\frac{1}{b})$ is the probability density function of the random effective relaxation time $\tilde{T}$ and the probability density function of the random effective relaxation rate $\tilde{\beta}$, respectively. The constant C equals $(\ln 10)^{-1}$. As we shall show below, formula (11), that assigns the relaxation function to the Laplace transform of the relaxation-rate distribution $\phi(t) = \mathcal{L}(\varrho(b), t)$, is very useful in studying the probabilistic aspects of the empirical relaxation functions.

Let us consider the KWW relaxation function (1). As it was presented in the preceding section the effective relaxation rate $\tilde{\beta}$ yielding the KWW function is distributed according to the completely asymmetric stable law, more precisely, the effective relaxation rate $\tilde{\beta}$ is an α-stable (Lévy stable) non-negative random variable S_α; $\tilde{\beta} \overset{d}{\cong} \frac{1}{\tau_0}S_\alpha$ [19, 27]. The explicit form of the density of the variable S_α exists only in the case of $\alpha = \frac{1}{2}$ and is known as Lévy density [33, 34]. Therefore only in the case of $\alpha = \frac{1}{2}$ we can express $\varrho(b)$ explicitly obtaining the following:

$$\varrho(b) = \frac{1}{2\sqrt{\pi\tau_0}}\,b^{-3/2}\exp\left\{-\frac{1}{4\tau_0 b}\right\}. \tag{12}$$

The Rajagopal log-relaxation-time density discussed by Gomez and Alegria in [30] was obtained using the "method of the steepest descents" as an approximate function to infer the density yielding the KWW response [29]. It is given as an explicit analytical expression

$$h_{\mathrm{R}}(\log\tau) = \frac{\alpha}{(A\pi\,(1-\alpha))^{1/2}}\left(\alpha\frac{\tau}{\tau_0}\right)^{\frac{\alpha}{2(1-\alpha)}}\exp\left[-\,(1-\alpha)\left(\alpha\frac{\tau}{\tau_0}\right)^{\frac{\alpha}{(1-\alpha)}}\right] \tag{13}$$

which has the proper probabilistic sense only for $0 < \alpha < 1$. It yields the relaxation-rate probability function in the following form

$$\varrho_{\mathrm{R}}(b) = \frac{1}{b}\frac{\alpha}{\left(\pi\left(1-\alpha\right)\right)^{1/2}}\left(\frac{\alpha}{\tau_0 b}\right)^{\frac{\alpha}{2(1-\alpha)}}\exp\left[-\left(1-\alpha\right)\left(\frac{\alpha}{\tau_0 b}\right)^{\frac{\alpha}{(1-\alpha)}}\right] \tag{14}$$

indicating that (13) cannot (excluding the case of $\alpha = \frac{1}{2}$) yield the exact form of the KWW function. There exists, however, a strict connection between these two functions that allows to compare analytically the asymptotic behaviour of both the KWW and the corresponding Rajagopal relaxation function $\phi_{\mathrm{R}}(t)$.

For $\alpha = \frac{1}{2}$ there is an equality between the Lévy (Eq. (12)) and Rajagopal densities (Eqs (13), (14)), and thus the corresponding relaxation functions coincide so we exclude this case from the following considerations.

To point out the differences occurring in the remaining cases let us first focus on the asymptotic behaviour of the stable density function. As we mentioned above, there are no explicit formulas of the completely asymmetric stable densities $s(b,\alpha)$ (except for the case $\alpha = \frac{1}{2}$). Instead, the asymptotic approximations are well known [34].
We have for $b \to 0$

$$s(b;\alpha) = \frac{1}{b}\frac{\alpha}{\left(\pi\left(1-\alpha\right)\right)^{1/2}}\left(\frac{\alpha}{b}\right)^{\frac{\alpha}{2(1-\alpha)}}\exp\left[-\left(1-\alpha\right)\left(\frac{\alpha}{b}\right)^{\frac{\alpha}{(1-\alpha)}}\right] \tag{15}$$

and for $b \to \infty$

$$s(b;\alpha) \propto b^{-\alpha-1}. \tag{16}$$

Let us observe that the asymptotic formula (15) for small b is just the Rajagopal density (14). Moreover, formula (14) can be rewritten in the form (12) with $\tau_0 = 1$ using the substitution $z = \frac{1}{4}\frac{1}{1-\alpha}\left(\frac{\tau_0}{\alpha}b\right)^{\frac{\alpha}{1-\alpha}}$. This denotes that the $\tilde{\beta}_{\mathrm{R}}$ and effective relaxation time $\tilde{T}_{\mathrm{R}}$ in the Rajagopal case are related with the $\frac{1}{2}$- stable random variable $S_{\frac{1}{2}}$ by the following formulas

$$\tilde{\beta}_{\mathrm{R}} \overset{d}{\cong} \frac{\alpha}{\tau_0}\left(4(1-\alpha)S_{\frac{1}{2}}\right)^{\frac{1-\alpha}{\alpha}}, \tag{17}$$

$$\tilde{T}_{\mathrm{R}} \overset{d}{\cong} \frac{\tau_0}{\alpha}\left(4(1-\alpha)S_{\frac{1}{2}}\right)^{-\frac{1-\alpha}{\alpha}}. \tag{18}$$

Let us note that in the limiting case, when $\alpha \to 1$, the random variable $\tilde{\beta}_{\mathrm{R}}$ approaches to the degenerate one which is the deterministic case with $\tilde{\beta}_{\mathrm{R}} \overset{d}{\cong} \frac{1}{\tau_0} = \mathrm{const}$ generating the Debye response. The corresponding density function tends to δ-Dirac function $\varrho_{\mathrm{R}}(b) = \delta(b - \frac{1}{\tau_0})$ (see Fig. 1).

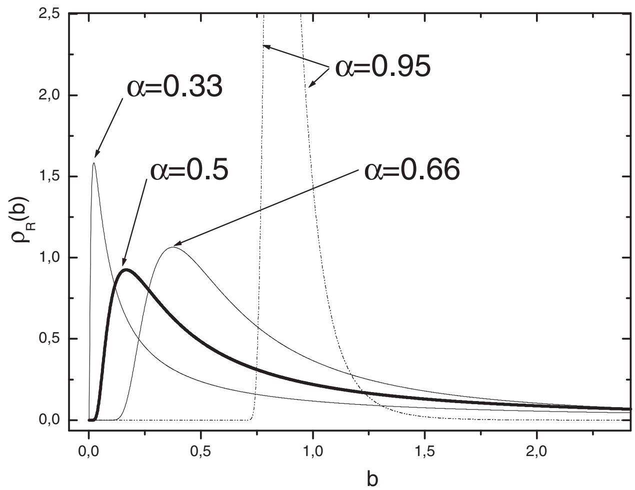

Fig. 1. Relaxation-rate densities $\varrho_{\mathrm{R}}(b)$ (Eq. (14)) for various values of $\alpha \in (0,1)$.

Both densities, $s(b,\alpha)$ and $\varrho_{\mathrm{R}}(b)$, coincide for small values b of the relaxation rate whereas for $b \to \infty$ the agreement is no longer maintained as the Rajagopal density $\varrho_{\mathrm{R}}(b)$ for $b \to \infty$ is proportional to $b^{-\frac{\alpha}{2(1-\alpha)}-1}$:

$$\varrho_{\mathrm{R}}(b) \propto b^{-\frac{\alpha}{2(1-\alpha)}-1} \quad \text{for } b \to \infty. \tag{19}$$

The asymptotic behaviour of the relaxation-rate density results in the asymptotic behaviour of its Laplace transform *i.e.* relaxation function. It is due to the well-known general fact that the properties for $x \to 0$ ($x \to \infty$) of a function $f(x)$ correspond to the properties of its Laplace transform $\mathcal{L}(f(x),t)$ for $t \to \infty$ ($t \to 0$). The Tauberian theorems [33] imply that the KWW response function exhibits for $t \to 0$ the following power law

$$f_{\mathrm{KWW}}(t) = -\frac{d\phi_{\mathrm{KWW}}(t)}{dt} \propto t^{\alpha-1} \tag{20}$$

which is the direct consequence of the power law $s(b;\alpha) \propto b^{-\alpha-1}$ fulfilled for $b \to \infty$ by the stable relaxation-rate density with $0 < \alpha < 1$. Analogous analysis of the Rajagopal density leads from

$$\varrho_{\mathrm{R}}(b) \propto b^{-\frac{\alpha}{2(1-\alpha)}-1} \quad \text{for} \quad b \to \infty \tag{21}$$

to

$$f_{\mathrm{R}}(t) = -\frac{d\phi_{\mathrm{R}}(t)}{dt} \propto t^{\frac{\alpha}{2(1-\alpha)}-1} \quad \text{for} \quad t \to 0. \tag{22}$$

In this case the Tauberian theorems imply the constraint $0 < \frac{\alpha}{2(1-\alpha)} < 1$ (*i.e.* $0 < \alpha < \frac{2}{3}$) on the power-law exponent in formula (21). This mathematical result is in agreement with the empirical data, *i.e.* with the general observation [13] that all types of the empirical functions used to fit the dielectric data exhibit a fractional-power dependence of the dielectric response $f(t)$ in the short-time limit:

$$f(t) \sim (\omega_p t)^{-n} \qquad \text{for} \quad t \ll {\omega_p}^{-1}, \tag{23}$$

where the parameter n falls in the range $(0, 1)$ and ω_p denotes the loss peak frequency. Thus, using the Rajagopal density function, only in the case of α falling in the range $(0, \frac{2}{3})$ we get the proper power law of the response function. In the limiting case of $\alpha = \frac{2}{3}$ we get $n = 0$ what denotes that the response function $f_{\mathrm{R}}(t)$ becomes a constant for $t \to 0$. Concluding, in the framework of the dynamically heterogeneous picture expressed by exponential integral kernel in the definition of $\phi(t)$, the Rajagopal relaxation function and the KWW function coincide in the region of long times $t \to \infty$ whereas they exhibit dissimilar power laws for $t \to 0$: $f_{\mathrm{R}}(t) > f_{\mathrm{KWW}}(t)$ for $\alpha \in (0, \frac{1}{2})$, $f_{\mathrm{R}}(t) < f_{\mathrm{KWW}}(t)$ for $\alpha \in (\frac{1}{2}, 1)$ and $f_{\mathrm{R}}(t) = f_{\mathrm{KWW}}(t)$ for $\alpha = \frac{1}{2}$ (see Fig. 2).

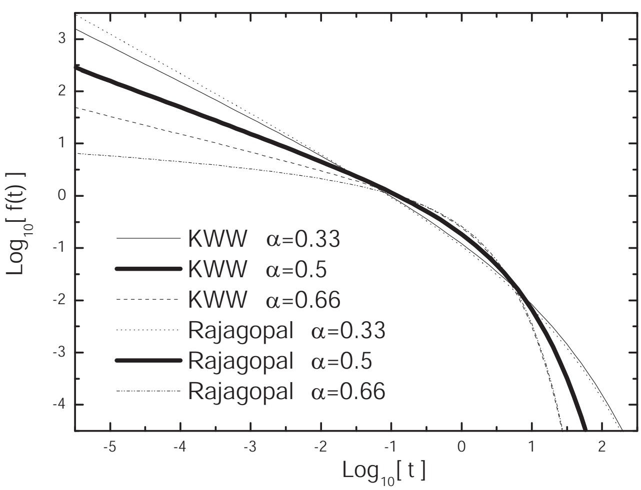

Fig. 2. Relationship between the Rajagopal and the KWW response functions. For long time t both functions coincide.

4. Nonexponential integral kernel. On the relationship with the Rajagopal density function

A more general approach to the response pattern [15, 35] can be obtained by changing the exponential integral kernel in (2) and rewriting the definition of the relaxation function as

$$\phi(t) = \int_0^\infty g(\tau) \exp\left[-(t/\tau)^{\alpha_{\rm intr}}\right] d\tau = \left\langle \exp[-(t/\tilde{T})^{\alpha_{\rm intr}}] \right\rangle . \tag{24}$$

The positive parameter $\alpha_{\rm intr} \neq 1$ in the integral kernel introduces various degrees of heterogeneity by changing the extent of the intrinsic nonexponentiality; $g(\tau)$, as in the former case, is the probability density function of the effective relaxation-time $\tilde{T}$ (not attributed to any particular object chosen from the entities forming the system).

To present the stochastic scheme leading to (24) we have to assume that each relaxor undergoes the nonexponential relaxation. Then the conditional site-dependent probability that the relaxor has not changed its initial state up to the moment t is

$$\Pr(\theta_i \geq t | \beta_i = b) = \exp(-(\frac{t}{\tau})^{\alpha_{\rm intr}}) = \exp(-b t^{\alpha_{\rm intr}}) \qquad \text{for } t \geq 0,\ b > 0. \tag{25}$$

The analogous analysis, as presented in Sect. 2, yields the relaxation function of the form

$$\phi(t) = \left\langle \exp(-t^{\alpha_{\rm intr}} \sum_{i=1}^{N} \beta_i / A_N^{\alpha_{\rm intr}}) \right\rangle = \left\langle \exp(-t^{\alpha_{\rm intr}} \tilde{\beta}_N) \right\rangle \tag{26}$$

with the effective relaxation rate

$$\tilde{\beta}_N = \sum_{i=1}^{N} \beta_i / A_N^{\alpha_{\rm intr}}; \qquad \alpha_{\rm intr} > 0,\ \alpha_{\rm intr} \neq 1 \tag{27}$$

The case $\alpha_{\rm intr} = 1$ corresponds to the classical approach (6) with each object relaxing exponentially.

As the same limit theory is in force, we conclude that asymptotically as $N \to \infty$ (for large number of individual relaxors) the only possible probability distribution for the effective relaxation rate $\tilde{\beta} = \lim_{N\to\infty} \tilde{\beta}_N$ is the completely asymmetric Lévy-stable law S_γ with $\gamma \in (0, 1)$. In this case $\phi(t)$ is of the form

$$\phi(t) = \exp\left[-(At^{\alpha_{\rm intr}})^{\gamma}\right], \tag{28}$$

and

$$\tilde{T} = \tilde{\beta}^{-1/\alpha_{\rm intr}} \stackrel{d}{\cong} A^{-1/\alpha_{\rm intr}}(S_\gamma)^{-1/\alpha_{\rm intr}}, \tag{29}$$

where A is a positive constant. To obtain the convergence the sequence of normalizing constants $A_N^{\alpha_{\rm intr}}$ has to be proportional to $N^{1/\gamma}$ and β_i have to belong to the domain of attraction of the Lévy-stable law. Again, it is not necessary to know the detailed nature of the relaxation time (rate) of an individual relaxor. The considered convergence is determined by the behaviour of the tail of $F_\beta(b)$ for large b, or equivalently by the behaviour of the distribution function of the individual relaxation time $F_T(\tau)$ for small τ. Namely, the necessary and sufficient condition for the relaxation time to obtain the limit in (27) is

$$F_{T_i}(ya) = \Pr(T_i < ya) \approx y^{\gamma\alpha_{\rm intr}} F_{T_i}(a) \qquad \text{for}\, a \to 0 \text{ and}\, y > 0. \tag{30}$$

Let us observe that as a result we obtain three types of the relaxation response (28). Provided that $\gamma\alpha_{\rm intr} < 1$ we get the stretched exponential decay *i.e.* $\exp(-t/\tau_0)^{\alpha_{\rm KWW}}$ with $\alpha_{\rm KWW} = \gamma\alpha_{\rm intr}$ and $\tau_0 = A^{-1/\alpha_{\rm intr}}$. For $\gamma\alpha_{\rm intr} = 1$ we have the Debye response, whereas $\gamma\alpha_{\rm intr} > 1$ yields the compressed exponential one.

The relationship (29) helps us to find the stochastic relaxation scheme hidden behind the Rajagopal effective-relaxation-time distribution. Let us recall that the corresponding effective relaxation time has been derived in Sect. 3 to be distributed as $\tilde{T}_{\rm R} \stackrel{d}{\cong} {\rm const}(S_{\frac{1}{2}})^{-\frac{1-\alpha}{\alpha}}$ (see (18)), which is exactly of the form (29). We conclude hence that the Rajagopal effective-relaxation-time distribution results in a heterogeneous system of relaxors exhibiting the intrinsically nonexponential decay (with the index $\alpha_{\rm intr} = \frac{\alpha}{1-\alpha}$) combined with the site-dependent rates $\beta_i = T_i^{-\frac{\alpha}{1-\alpha}}$ belonging to the domain of attraction of $\frac{1}{2}$-stable law. Due to (10) the distribution of individual T_i has to obey $\Pr(T_i < ya) \approx y^{\frac{1}{2}\frac{\alpha}{1-\alpha}} \Pr(T_i < a)$ for all $y > 0$ and $a \to 0$. For $\alpha_{\rm intr} < 2$ ($\alpha < \frac{2}{3}$) we obtain the stretched exponential response whereas for $\alpha_{\rm intr} = 2$ ($\alpha = \frac{2}{3}$) and $\alpha_{\rm intr} > 2$ ($\alpha > \frac{2}{3}$) respectively the Debye and compressed cases.

The response function $f(t)$ derived from (28) is for small t proportional to $t^{\frac{\alpha}{2(1-\alpha)}-1}$ therefore for $\alpha < \frac{2}{3}$ exhibits the same power law as $f_{\rm R}(t)$ (recall Eq. (22)). Both functions (with the constant A chosen properly) coincide for small values of t whereas as $t \to \infty$, $f_{\rm R}(t)$ differs from the KWW response function (see Fig. 3).

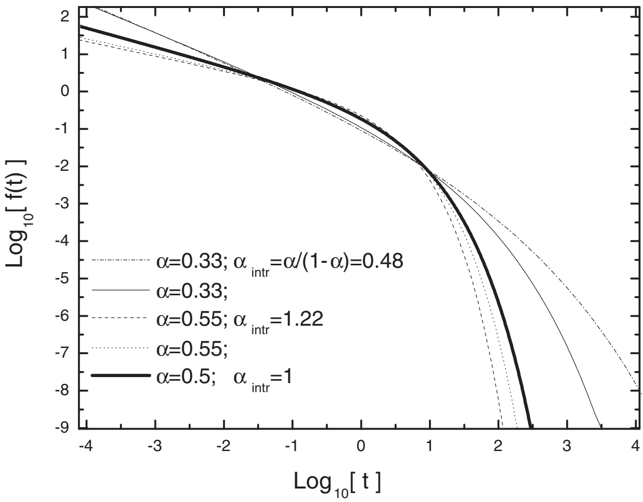

Fig. 3. Relationship between the Rajagopal response functions in the case of exponential (see (3)) and nonexponential integral kernel, characterized by the value of $\alpha_{\rm intr}$ (see (24)). For $\alpha = \frac{1}{2}$ both functions coincide. In the remaining cases the coincidence occurs only in the region of small t.

5. Conclusions

In this work, aiming to find a probabilistic scheme of relaxation, a detailed analysis of properties of the response function generated by the Rajagopal analytical relaxation-time distribution has been presented. This distribution has been proposed as an aproximate function to infer the relaxation-time density yielding the KWW-type relaxation response. In contrast, the well-known KWW empirical relaxation function is related to the relaxation-time density which cannot be represented by an analytical formula. Although the relaxation-time density can be given in the series representation only, the probabilistic scheme establishing the spatio-temporal scaling properties leading to the stretched exponential response has been already recognized.

In our approach, instead of representing the relaxation function as the weighted average of an exponential decay with respect to the distribution of the effective relaxation time (2), we have used the possibility of representing the relaxation function as the Laplace transform of the relaxation-rate density (4). This substitution allowed us to use the Tauberian theorems to find the condition under which the properties of both the KWW and Rajagopal responses can be compared. As a result we have obtained that both functions coincide for $t \to \infty$, whereas they exhibit dissimilar power laws for $t \to 0$. We have also shown that behind the KWW response there is hidden the effective relaxation rate with properties of the Lévy-stable non-negative random variable whereas behind the Rajagopal response the effective relaxation rate is a power function of the $\frac{1}{2}$ - stable non-negative random variable.

Using the idea of intrinsic nonexponentiality we have shown the stochastic scheme connecting directly the Rajagopal and the KWW functions. This approach allows one to obtain the broader class of nonexponential responses (the stretched, as well as, the compressed ones).

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