Vol. 34 (2003) ACTA PHYSICA POLONICA B No 7

ON QUANTUM CORRELATIONS FOR STOCHASTIC DYNAMICS OF XXZ TYPE*

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(Received December 3, 2002)

The evolution of quantum correlations for XXZ model is studied. It is shown that the simplified entanglement measure which follows from entanglement of formation can be used as an efficient tool in investigating the properties of the dynamics in question. In particular, the behavior of this measure for pure states gives us information about decoherence or entanglement that can occur during the time evolution for the system. We present some numerical results which confirm that the generalized conditional expectation defining the stochastic dynamics for XXZ model contains the proper (*i.e.* genuine quantum) interactions between subsystem and its environment.

PACS numbers: 05.30.–d, 03.65.Yz

1. Introduction

It is widely accepted that quantum entanglement is the essential property of states of composite quantum system. It manifests itself in correlations that are stronger than those attainable in any classical systems. This leads to surmise that genuine quantum maps should produce true quantum correlations [8] besides being well defined on non-commutative structures. In other words, quantum dynamics should imply the production and evolution of entanglement.

Quite recently, it has been realized that there is a general scheme for a quantization of stochastic dynamics which describe interacting classical particles [3–6]. In particular, in that scheme a general recipe for quantum stochastic dynamics of jump type was given. In our recent paper [2] we

* Presented at the XV Marian Smoluchowski Symposium on Statistical Physics, Zakopane, Poland, September 7–12, 2002.

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have studied two particular models of quantum stochastic dynamics which can be considered as examples of quantum generalizations of Glauber dynamics. In the first example, based on one dimensional Ising model with nearest neighbor interactions, our analysis showed that this particular case of quantum stochastic dynamics does not exhibit quantum features in the sense that there is no production of quantum correlations. The second example of dynamics studied in [2] was based on quantum XXZ model and we have shown that there are signatures of quantum correlations in this case.

In the present paper we continue our analysis of the jump type quantum stochastic dynamics based on XXZ model. Namely, to get more cogent evidence in support of production of quantum correlations for that example of quantum maps we will examine a (simplified) measure of entanglement. More precisely, although there are meaningful measures of entanglement (*cf.* [7] and references therein), efficient criteria for entanglement of complex quantum systems are still lacking (complex systems are understood as systems more involved than a pair of qubits).

Therefore, to carry out our analysis of time evolution of entanglement we derive an interesting formula for time evolution of simplified version of entanglement of formation. Then, numerical results that concern the evolution of entanglement are given. Finally, we would like to emphasize that, contrary to the analysis given in [2], we shall not use a high-temperature expansion, *i.e.* we will not use any approximation for description of considered dynamical maps.

2. Jump-type dynamics for quantum XXZ model

Consider a composite system $I+II$ associated with a region $\Lambda = \Lambda_I \cup \Lambda_{II}$, where $\Lambda_I, \Lambda_{II} \subset \boldsymbol{Z}^d$. The system Λ is described by $\mathcal{H}_1 \otimes \mathcal{H}_2$, $S_1 \otimes S_2$ and $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \cong \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, where $\mathcal{H}_1$ ($\mathcal{H}_2$) are the finite dimensional Hilbert spaces associated with Λ_I (Λ_{II}), S_1 (S_2) — respective sets of density matrices - are the spaces of mixed states, $\mathcal{B}(\mathcal{H}_1)$ ($\mathcal{B}(\mathcal{H}_2)$) — the sets of all bounded linear operators — are the algebras of observables. Interacting system are described by interaction potentials associated with region Λ (Λ_I, Λ_{II}, respectively). This leads to the corresponding Hamiltonians H_Λ (H_{Λ_I}, $H_{\Lambda_{II}}$) and to Gibbs state $\rho_\Lambda = \dfrac{\mathrm{e}^{-\beta H_\Lambda}}{\mathrm{Tr}\,(\mathrm{e}^{-\beta H_\Lambda})} \equiv \rho$ (ρ is an invertible operator, *i.e.* ρ^{-1} exists). In this work we study a concrete kind of the jump process, *i.e.* exchange type dynamics (for general description see [6]). This kind of dynamics is induced by a local symmetry. Consider a symmetry transformation (local automorphism) ψ on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that

$$\psi(A) = A \quad \text{for} \quad A \in \mathcal{B}(\mathcal{H}_1)\,, \quad \psi^2 = \mathbf{1}\,.$$

Note that if $\dim(\mathcal{H}_1), \dim(\mathcal{H}_2) < \infty$, the above properties imply $\mathrm{Tr}\,(\psi(\cdot)) = \mathrm{Tr}\,(\cdot)$. We shall consider a particular type of symmetries, which are implemented by exchanges of observables between sites of the spin chain. Using transformation ψ one can define a projection τ on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ as follows

$$\tau(\cdot) \equiv \tfrac{1}{2}(\mathbf{1} + \psi)(\cdot)\,.$$

We observe that τ is not a morphism. According to the general theory of semigroups [1] the dynamic T_t induced by the local transformation ψ on the set of observables is of the form

$$T_t(\cdot) = \exp(t\mathcal{L}(\cdot))\,, \tag{1}$$

with

$$\mathcal{L} = \mathcal{E} - \mathbf{1}\,,$$

where $\mathcal{E} : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a generalized conditional expectation in the Accardi–Cechini sense. Performing calculations similar as in the appendix of [6] one can show that for the considered dynamics operator $\mathcal{E}$ takes the following form

$$\mathcal{E}(A) = \tau(\gamma^* A \gamma)\,, \tag{2}$$

where

$$\gamma = \rho^{1/2}(\tau\rho)^{1/2}\,. \tag{3}$$

An application of the relation between Schrödinger and Heisenberg picture leads to the evolution T_t^d of a state σ:

$$\mathrm{Tr}\,(T_t^d(\sigma))A = \mathrm{Tr}\,\sigma T_t(A)$$

for any state σ and any observable A. Moreover, one has

$$\mathrm{Tr}\,(\mathcal{E}^d(\sigma)A) = \mathrm{Tr}\,\sigma\mathcal{E}(A)\,.$$

Consequently

$$\begin{aligned}\mathrm{Tr}\,(\mathcal{E}^d(\sigma)A) &= \mathrm{Tr}\,(\sigma\tau(\gamma^* A\gamma))\\ &= \mathrm{Tr}\,\tau(\sigma)\gamma^* A\gamma = \mathrm{Tr}\,\gamma\tau(\sigma)\gamma^* A\,.\end{aligned}$$

This results in

$$\mathcal{E}^d(\sigma) = \gamma\tau(\sigma)\gamma^*\,.$$

Let us add, that the presented construction has a straightforward generalization to the infinite dimensional case, thus, thermodynamic limit can be performed [3, 5]. Again, as a result we get uniformly continuous semigroup T_t.

In this work we analyze a one dimensional quantum XXZ model. In particular we will consider a one-dimensional finite $\frac{1}{2}$-spin chain with $N+1$ sites indexed from 0 to N and the corresponding algebra of observables generated by

$$\sigma^{i_0} \otimes \sigma^{i_1} \otimes \ldots \otimes \sigma^{i_N} ,$$

where $i_k \in \{0, 1, 2, 3\}$, $k = 0, \ldots N$, and σ^j, $j = 0, 1, 2, 3$ are Pauli matrices.

Here, $\Lambda = \{0, 1, \ldots, N\}$ and $\Lambda_I, \Lambda_{II} \subset \Lambda$, $\Lambda_I \cup \Lambda_{II} = \Lambda$, $\Lambda_I \cap \Lambda_{II} = \phi$, while $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ are $2^{|\Lambda_I|}$ — and $2^{|\Lambda_{II}|}$ — dimensional Hilbert spaces, respectively. As an example consider a local transformation ψ_{kl} defined as follows

$$\psi_{kl}(A_1 \otimes \ldots \otimes A_k \otimes \ldots \otimes A_l \otimes \ldots \otimes A_N) = A_1 \otimes \ldots \otimes A_l \otimes \ldots \otimes A_k \otimes \ldots \otimes A_N \quad (4)$$

which describes the exchange between the sites. In particular, one can choose $l = k+1$ which is related to a description of transport properties in the considered model.

The Hamiltonian of the XXZ system has the form:

$$H = -\sum_{n=1}^{N} (\sigma^1_{n-1}\sigma^1_n + \sigma^2_{n-1}\sigma^2_n + \Delta\sigma^3_{n-1}\sigma^3_n) ,$$

where σ^j, $j = 1, 2, 3$ are Pauli matrices. Recall that $\Delta \neq 1$ is responsible for anisotropy of the model. The corresponding Gibbs state is represented by the density matrix

$$\rho = Z^{-1} \exp(-\beta H) ,$$

where $Z = \mathrm{Tr}\,(\mathrm{e}^{-\beta H})$.

3. Entanglement measure

It was shown in [7] that EoF can serve as a well defined measure of entanglement. Moreover, it was indicated that to get other well defined measure of entanglement it is enough to replace the von Neumann entropy function in the original definition of EoF by a concave, positive continuous function vanishing only on pure states. It is an easy observation that the so called linear entropy $S_{\mathrm{L}}(\rho) = -\mathrm{Tr}\,(\rho(\rho - 1))$ satisfies the above conditions. Therefore

$$M(\sigma) = \inf_{\sigma = \sum_i \lambda_i \sigma_i} \sum_i \lambda_i \mathrm{Tr}\,_2 \left[(\mathrm{Tr}\,_1 \sigma) - (\mathrm{Tr}\,_1 \sigma)^2 \right] , \quad (5)$$

where infinum is taken over all decompositions of σ, provides a well defined measure of entanglement. Further, we note (*cf.* [7]) that in (5) it is enough to restrict oneself to decomposition of σ into a convex combination of pure states. The main difficulty in calculation of (5) is to carry out inf over all prescribed decompositions.

To overcome that problem we begin with the simplified version of (5):

$$M^a(\sigma) = \mathrm{Tr}_2\left[(\mathrm{Tr}_1\sigma) - (\mathrm{Tr}_1\sigma)^2\right], \tag{6}$$

where again, σ is the density matrix determining the considered state. Clearly

$$M(\sigma) \leq M^a(\sigma)\,. \tag{7}$$

We shall use $M^a(\sigma)$ for the following definition of entanglement production:

$$E_a(T_t^d\rho) = M^a(T_t^d\rho) - M^a(\rho)\,. \tag{8}$$

This measure, applied to small t and a pure state ρ, leads to

$$E_a(T_t^d\rho) = M^a((1-t)\rho + t\mathcal{E}^d(\rho)) - M(\rho)\,, \tag{9}$$

where we have used: $T_t^d\rho = (1-t)\rho + t\mathcal{E}^d(\rho) + ao(t^2)$ ($o(t^k) = a_1t^k + a_2t^{k+1} + \ldots$) and the obvious fact that $M(\rho) = M^a(\rho)$ for a pure state.

Equation (9) can be rewritten as

$$E_a(T_t^d\rho) = -t\left[\mathrm{Tr}_2\left(\{\mathrm{Tr}_1\rho, \mathrm{Tr}_1\mathcal{E}^d(\rho)\}\right) + 2\mathrm{Tr}_2(\mathrm{Tr}_1\rho)^2\right] + o(t^2)\,, \tag{10}$$

where $\{\cdot,\cdot\}$ denotes anticommutator. Obviously, one has

$$E(T_t^d\rho) = M(T_t^d\rho) - M(\rho) \leq E_a(T_t^d\rho)\,. \tag{11}$$

Let us consider the following case: ρ is a pure nonseparable state such that, for small t,

$$\frac{E_a(T_t^d\rho)}{t} \leq 0\,. \tag{12}$$

Therefore, $E(T_t^d\rho) \leq 0$ and $M(T_t^d\rho) \leq M(\rho)$. Consequently, the negative sign of $\frac{E_a(T_t^d\rho)}{t}$ implies a decrease of entanglement (so there is a signature of decoherence).

As a next case let us consider again pure nonseparable state ρ such that, for small t, $\frac{E_a(T_t^d\rho)}{t} \geq 0$. To state something about $E(T_t^d\rho)$ for that case we should discuss the difference between $M(T_t^d\rho)$ and $M^a((1-t)\rho + t\mathcal{E}^d(\rho))$, again for small t.

To this end, let us consider a (convex) decomposition $\sum \lambda_i P_i$ of $T_t^d \rho$ into pure states. Clearly

$$\left|\left|\sum \lambda_i P_i - ((1-t)\rho + t\mathcal{E}^d(\rho))\right|\right| \leq Ao(t^2)\,,$$

where $A \geq 0$ is a constant. As Tr_1 is a linear projection, the linear entropy S_{L} is a continuous function then

$$\left|M^a\left(\sum \lambda_i P_i\right) - M^a((1-t)\rho + t\mathcal{E}^d(\rho))\right| < A' o(t^2)$$

for some positive constant A'. On the other hand, as S_{L} is continuously (strongly) differentiable and S''_{L} exists then

$$M^a\left(\sum \lambda_i P_i\right) = M^a(\rho + t(\mathcal{E}^d(\rho) - \rho)) = M^a(\rho) + E_a(T_t^d \rho)\,.$$

Hence

$$E_a(T_t^d \rho) \geq \sum \lambda_i M^a(P_i) - M^a(\rho)\,.$$

Furthermore

$$E_a(T_t^d \rho) \geq M(T_t^d \rho) - M^a(\rho) \equiv M(T_t^d \rho) - M(\rho)$$

and

$$M^a\left(\sum \lambda_i P_i\right) = M^a(\rho) + o(t)\,.$$

Thus, positivity of $E_a(T_t^d \rho)$ allows a production of entanglement.

Finally, let us consider the case of ρ being pure separable state. We recall that $\mathcal{E}^d(\rho) = \gamma\tau(\rho)\gamma^*$ where $\tau = \frac{1}{2}(\mathbf{1} + \psi)$, while ψ is an automorphism of order 2. As we consider finite dimensional case, each automorphism is implemented by a unitary operator. Thus, there are such pure states — projectors that τ(projector) is a projector. Then

$$M^a(\mathcal{E}^d(\rho)) = M(\mathcal{E}^d(\rho)) = \mathrm{Tr}_2\left(\mathrm{Tr}_1(\mathcal{E}^d(\rho))(1 - \mathrm{Tr}_1(\mathcal{E}^d(\rho)))\right)\,.$$

Clearly $M^a(\mathcal{E}^d(\rho)) > 0$ shows that $\mathcal{E}^d(\rho)$ is entangled, hence $(1-t)\rho + t\mathcal{E}^d(\rho)$ is entangled. Consequently, for the map $\rho \to T_t^d(\rho)$ (for small t) one has entanglement production.

4. Numerical results

In this section we show the results of numerical evaluation of entanglement coefficient $E_0 = \lim_{t\to 0} E_a(T_t^d \rho)/t$. All the results have been obtained assuming the length of spin chain equal to 5 and local automorphism ψ

defined by (4) with $k = 1$ and $l = 2$. Fig. 1 shows the value of E_0 *versus* inverse temperature β and anisotropy parameter Δ for four randomly chosen pure states. In fact, we have examined a lot of pure states and the qualitative behavior for most of them is similar, so we can treat the presented ones as good representatives. It is seen that clearly distinguished domains of positive and negative value of E_0 coefficient occur on the (β, Δ) plane.

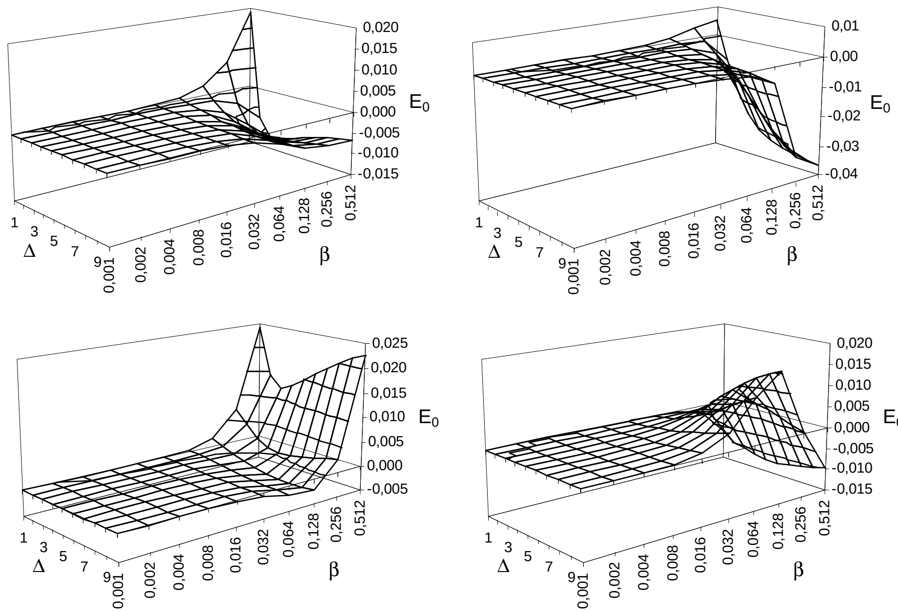

Fig. 1. Entanglement coefficient $E_0 = \lim_{t\to 0} E_a(T_t^d \rho)/t$ *versus* β and Δ for four randomly chosen pure states

On the other hand, Fig. 2 shows the value of E_0 *versus* inverse temperature β and anisotropy parameter Δ for two examples of separable pure states. Similarly as before the presented results provide us with a good exemplification of the general rule. We can observe that, in contrast to the previous case, there is only one stable domain of positive value of E_0 coefficient (stability is understood here as a strict monotonicity with respect to both variables β and Δ). Note that the logarithmic scale has been used in Fig. 2.

5. Conclusions

Our analysis of the model of quantum stochastic dynamics of XXZ type strongly supports the conjecture (*cf.* [9–12]) that entanglement, as a "non-local" property, is fragile under the influence of the environment, here

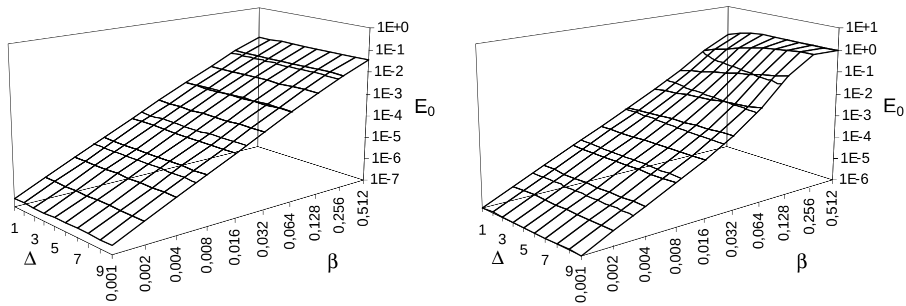

Fig. 2. Entanglement coefficient $E_0 = \lim_{t\to 0} E_a(T_t^d \rho)/t$ *versus* β and Δ for two randomly chosen separable pure states

with respect to the change of inverse temperature β and the "measure" of anisotropy (*cf.* Fig. 1).

As the case of positivity of E_0 has not a decisive interpretation, we present supplementary results concerning the entanglement of $\mathcal{E}^d(\rho)$ for pure separable state. Again, the main difficulty in carrying out this analysis follows from the operation inf over all convex decompositions into pure states. However, for the considered type of dynamics (*cf.* the end of Section 3), $\tau(\rho)$ is a separable state provided that ρ has that property. On the other hand, the XXZ Hamiltonian implies "entangled" form of γ. Consequently, one can expect that entanglement coefficient E_0 of $\mathcal{E}^d(\rho)$ is positive apart from these states which are fixed points of evolution. Our numerical results support that conjecture and clearly show the production of entanglement for pure separable states. Finally, we would like to point out that non-negativity of E_0 for that case follows from the fact that the measure of entanglement has that property.

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