NO-CHAOS CRITERIA FOR CERTAIN CLASSES OF DRIVEN NONLINEAR OSCILLATORS*

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(Received December 3, 2002)

Besides three-dimensional autonomous nonlinear dynamical systems, periodically driven nonlinear oscillators constitute elementary classes of systems that can potentially exhibit chaotic behavior. In this contribution we investigate conditions on the shape of the potential and the functional form of the periodic driving that are necessary for the occurrence of chaotic behavior in these systems by deriving analytical criteria that exclude chaotic long-time solutions.

PACS numbers: 05.45.Ac, 02.30.Hq, 05.90.+m

1. Introduction

Since its outset about three decades ago the systematic investigation of nonlinear dynamical systems with particular emphasis on chaotic dynamics has been developed in a persistently active research area of interdisciplinary interest (for reviews *cf.* [1–6]). Although there is still an ongoing debate about its rigorous mathematical definition (*cf.* [7]), the characterization of *chaotic behavior* as recurrent, bounded, aperiodic long-time dynamics appearing in nonlinear dynamical systems seems to be commonly accepted in physics [6]. Tightly annexed to such a behavior is the sensitive dependence of the corresponding long-time evolution on the initial conditions that manifests itself in at least one positive Lyapunov exponent. In spite of the substantial progress that has been accomplished in the last three decades, our current understanding of the appearance of chaotic behavior in dynamical systems is far from being complete. An unresolved challenge of particular importance is the following problem: Given an autonomous or non-autonomous dynamical system, $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$ or $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}, t)$ with $\boldsymbol{x}(t) = (x_1(t), ..., x_n(t))$ and

* Presented at the XV Marian Smoluchowski Symposium on Statistical Physics, Zakopane, Poland, September 7–12, 2002.

n characterizing the dimension of the corresponding phase space, can we decide only on the basis of the functional form of the vector field $\boldsymbol{v}(\boldsymbol{x})$ or $\boldsymbol{v}(\boldsymbol{x}, t)$ and without invoking numerical analysis whether and in what control parameter ranges chaotic behavior might exist or not? The celebrated Poincaré–Bendixson theorem provides a partial answer: Basically, the long-time evolution of a two-dimensional autonomous dynamical system can only attain fixed points, periodic solutions, divergent solutions or heteroclinic cycles [5] and, therefore, chaotic behavior is excluded. As a consequence, the eventual existence of chaotic behavior necessitates, besides some kind of nonlinearity in the vector field, at least a phase space dimension of *(i)* three or larger in the autonomous case or *(ii)* two or larger in the non-autonomous case. As far as case *(i)* is concerned, this constitutes one underlying reason why the famous Lorenz model [8], Rössler model [9] and the minimal chaotic models by Sprott [10] can be chaotic at all. Recently, systematic investigations by Zhang Fu and Heidel [11] for of three-dimensional dynamical systems with one quadratic nonlinearity and Linz [12] for third-order scalar differential equations $\dddot{x} = J(x, \dot{x}, \ddot{x})$ with an arbitrary nonlinearity in x have revealed that further exclusion conditions for the occurrence of chaotic behavior exist. In combination with previous systematic numerical searches for elementary chaotic systems [10, 13, 14] these works have lead to a quite satisfactory identification of chaotic minimal three-dimensional dynamical systems (for a recent review, *cf.* [15]).

Due to their ubiquity in nature and technology [3], non-autonomous nonlinear oscillators being generically driven by an external time-periodic function

$$\ddot{x} = F(x, \dot{x}, t) = -G(x, \dot{x})\dot{x} - \partial_x W(x, t) \tag{1}$$

are, from the physical point of view, even more appealing elementary classes of dynamical systems with potential chaotic behavior. Mechanically speaking, such systems represent the Newtonian motion of a particle excited by an applied force $F(x, \dot{x}, t)$ (reduced by the mass of the particle) that consists of a friction term $G(x, \dot{x})\dot{x}$ and a term resulting from a time-periodic potential $W(x, t) = W(x, t + T)$. Since the seminal works of Crutchfield and Huberman [16] and Ueda [17] in the late 1970's, many specific functional forms of the rich class of dynamical systems (1) are known to exhibit a plethora of complicated dynamics (including chaos) as function of the entering control parameters. For a general survey, we refer to Ref. [3].

The focus of this investigation is *(i)* to determine some general exclusion criteria for chaotic behavior in certain functional forms of periodically driven oscillators (1) and, by that, *(ii)* to narrow the subset of potentially chaotic forms of (1). As we shall see, the exclusion of chaotic behavior by proving that the long-time solution $x(t)$ is either bounded, $|x(t)| < \infty$, and approaching a fixed point or a periodic solution as $t \to \infty$ or unbounded

$|x(t)| \to \infty$ as $t \to \infty$ can, to some extent, be achieved without the explicit knowledge of the system's solution. Since a general discussion for Eq. (1) does not seem to be feasible, we restrict our investigation to the two physically most important cases where *(i)* the time-dependent driving $f(t)$ and the position dependent part of potential $V(x)$ in $W(x,t)$ can be separated, commonly known as additive and parametric driving [1] and *(ii)* the friction term is linear in $\dot{x}$.

2. Basic assumptions

To keep the discussion as general and the substantiation of the results as rigorous as possible, we demand, the following, from the physical point of view very weak requirements on the driving, the friction and the potential: *(i)* The periodic driving $f(t) = \langle f \rangle + p(t)$ consists of a periodic, bounded and at least piecewise continuous function $p(t) = p(t+T)$ with a minimal period $T \neq 0$ and zero mean, $\langle p \rangle = (1/T) \int_t^{t+T} p(t)\, dt = 0$, and has a mean value $\langle f \rangle = (1/T) \int_t^{t+T} f(t)\, dt$ that might be non-zero. *(ii)* The friction coefficient $\gamma(x)$ entering in the friction term $\gamma(x)\dot{x}$ might generally depend nonlinearly on the position x, but not on $\dot{x}$. Moreover, $\gamma(x)$ should be at least continuous in x and bounded except for $x \to \pm\infty$. *(iii)* The potential $V(x)$ is supposed to be at least continuous. Then, the solutions $x(t)$ of the subsequently considered driven oscillators (provided they exist) should be at least differentiable once and their derivatives $\dot{x}(t)$ at least continuous.

3. Additive driving

The term additive driving commonly refers to situations where the applied force $F(x, \dot{x}, t)$ (and not the potential $V(x)$) is subject to external, generically periodic perturbations that couple additively into the oscillator equation (1). Specifically for these systems, an exclusion criterion for chaos reads:

Theorem 1: An additively driven nonlinear oscillator determined by

$$\ddot{x} + \gamma(x)\dot{x} + \partial_x V(x) = f(t) \tag{2}$$

with a periodic driving $f(t)$ and a friction coefficient $\gamma(x)$ fulfilling the afore-mentioned conditions cannot exhibit chaotic behavior if the difference of the mean value of the driving and the *slope* of the potential $V(x)$ is either positive semidefinite, $\langle f \rangle - \partial_x V(x) \geq 0$, for all x or negative semidefinite, $\langle f \rangle - \partial_x V(x) \leq 0$, for all x.

Proof: To demonstrate the statement, we insert the afore-mentioned decomposition of the driving $f(t) = \langle f \rangle + p(t)$ into Eq. (2) and obtain, after rearrangement,

$$\ddot{x} + \gamma(x)\dot{x} - p(t) = \langle f \rangle - \partial_x V(x). \tag{3}$$

Next, we use the elementary fact that the friction term can be interpreted as the time derivative of a x-dependent function $\Gamma(x)$, i.e. $(d/dt)\Gamma(x) = \gamma(x)\dot{x}$ with $\partial_x \Gamma(x) = \gamma(x)$ or, equivalently, $\Gamma(x) = \int^x \gamma(x)\,dx$. Similarly, the periodic part of the driving $p(t)$ can, under the afore-mentioned assumptions, always be recast as the time derivative of a function $P(t)$, $p(t) = (d/dt)P(t)$. Since $p(t)$ is supposed to be periodic with zero mean, it follows from straightforward application of Fourier series theory that $P(t)$ is also periodic, albeit with a generally non-zero mean that depends on the initial conditions. Consequently, we can recast (3) in the form

$$\frac{d}{dt}\left[\dot{x} + \Gamma(x) - P(t)\right] = \langle f \rangle - \partial_x V(x). \tag{4}$$

Integrating Eq. (4) once with respect to time t yields

$$\dot{x} + \Gamma(x) - P(t) = \int^t \left[\langle f \rangle - \partial_x V(x)\right]\, dt + C\,, \tag{5}$$

where the finite integration constant C comprises the initial values of left-hand side of (5). If now, as we suppose, the integrand on the right-hand side of (5) is either positive semidefinite, $\langle f \rangle - \partial_x V(x) \geq 0$, for all x or negative semidefinite, $\langle f \rangle - \partial_x V(x) \geq 0$, for all x, the integral on the right-hand side of (5) must monotonically grow with time t or saturate. As a consequence, the long-time limit of the right-hand side of (5) can only attain a constant value (including zero) or diverge to plus or minus infinity,

$$\dot{x} + \Gamma(x) - P(t) \sim K \quad \text{as } t \to \infty \tag{6}$$

with $K = \kappa + C$, κ being the asymptotic limit of the integral of $\langle f \rangle - \partial_x V(x)$ for large t, and K being either a finite constant or $\pm\infty$. For the latter option, one obtains by balancing the long-time limit of the left and right-hand side of (6) that also the left-hand side must diverge. Since, however, $P(t)$ and $\Gamma(x)$ (at least for finite x) are bounded this is can only happen if either x or $\dot{x}$ or both together diverge. In any of these cases the long-time solution of (2) becomes unbounded and, therefore, cannot be chaotic. The other feasible option is that the right-hand side saturates to the finite constant K in the long time limit. Then, however, the long time limit of (6) reduces to a periodically driven nonlinear relaxator $\dot{x} \sim K - \Gamma(x) + P(t) = h(x,t)$. The final ingredient is the application of an analogue of the Poincaré–Bendixson theorem for non-autonomous dynamical systems as, for example, provided by Hale and Koçak [Ref. [5], p.118]: A non-autonomous first order differential equation $\dot{x} = h(x,t)$, with $h(x,t)$ being periodic with a period T, continuous in t, and at least differentiable once in x can only have diverging or periodic solutions. This results from the following simple arguments: Consider the stroboscopic sequence $x_n = x(t+nT, t_0, x_0)$ that labels the successive values of the solution $x(t,t_0,x_0)$ [started with the initial condition $x(t_0) = x_0$] at times $t+nT$ with $n = 0,1,2,3....$ As a consequence of the uniqueness and the continuity of the solutions $x(t,t_0,x_0)$, this sequence must be monotonic, i.e. either approach a limit function (corresponding to a limit cycle for $x(t)$) or becoming unbounded ($x(t) \to \pm\infty$). Since $\gamma(x) = \partial_x \Gamma(x) = -\partial_x h(x,t)$ is assumed to be at least continuous and $p(t) = \partial_t P(t) = \partial_t h(x,t)$ at least piecewise continuous, $\Gamma(x)$ and $P(t)$ in Eq. (6) fulfill the necessary requirements on $h(x,t)$, and, therefore, aperiodic or chaotic solutions in (2) are excluded. This concludes the proof.

Several remarks are in order. *(i)* Obviously, a potential $V(x)$ being linear in x, or equivalently, possessing a constant slope cannot lead to chaotic behavior in Eq. (2) independent of the specific nonlinearity entering in the friction term $\gamma(x)\dot{x}$. *(ii)* The no-chaos criterion is independent of the specific time-dependence of the forcing $f(t)$ and the sign and the magnitude of the friction coefficient $\gamma(x)$ and, therefore, applies to $\gamma(x)$ =const. or zero as well. *(iii)* For additive driving with zero mean, $\langle f \rangle = 0$, theorem 1 states that chaotic behavior in Eq. (2) cannot occur if the potential $V(x)$ is either strictly monotonic in x or monotonic with some possible saddle points or plateaus. This implies that potentials such as $V(x) = \alpha x^m$ or $V(x) = \max(0; \alpha x^m)$ (m positive and odd), $V(x) = \alpha\Theta(x)x^m$ (m positive, odd or even and $\Theta(x)$ Heaviside's unit step function), or $V(x) = \alpha\tanh(x+l)$ and $V(x) = \alpha\sinh(x+l)$ with α and l being constants cannot lead to chaotic behavior. *(iv)* In turn, one can also read off a necessary condition for the potential appearance of chaotic behavior in Eq. (2): If $\langle f \rangle = 0$ the potential must have at least one maximum or minimum. The latter is commonly used as an intuitive explanation why chaotic behavior might appear: An undriven oscillator in a potential with at least one well can typically exhibit (damped) periodic oscillations. Additional external driving might then trigger chaotic motion. For a highly interesting theoretical investigation of this point, we refer to the work by Eilenberger and Schmidt [18]. *(v)* For additive driving with non-zero mean $\langle f \rangle \neq 0$, chaotic behavior is excluded if $\partial_x[\langle f \rangle x - V(x)]$ has no zero-crossings for any x and t implying that the boundedness of $\partial_x V(x)$ is essential for ruling out chaotic behavior. Assuming that $-B_1 \leq \partial_x V(x) \leq B_2$ for all x ($B_1, B_2 > 0$), then the conditions $\langle f \rangle \geq B_2$ or $\langle f \rangle \leq -B_1$ enforce non-chaoticity. As an example, consider the multi-well potential $V(x) = A\sin(x)$ with $A > 0$: Chaotic behavior in (2) can be ruled out if $\langle f \rangle \geq A$ or $\langle f \rangle \leq -A$. *(vi)* A recent interesting numerical search by Gottlieb and Sprott [19] for the most elementary, chaotic, conservative, additively driven oscillators with *one single* control parameter has revealed that the functional forms $\ddot{x} + g(x) = \sin(\omega t)$ with $g(x) \in \{\sin(x), \sinh(x), \tanh(\mathrm{x}), x^n (n = 3, 5, 7, 9, 11), x^3 - x, x|x|, x|x|^3, x|x|^{-1/2}\}$ all exhibit chaotic behavior for certain ranges of ω. Here, it is interesting to observe that the violation of our no-chaos criterion (all potentials $V(x)$ have a minimum) directly enforces chaotic behavior for certain parameters ω.

4. Parametric driving

A distinct way of forcing nonlinear oscillators is determined by a multiplicative (or parametric) coupling of the periodic forcing $f(t)$ to the potential $V(x)$ or, equivalently, to the part of the applied force $F(x, \dot{x}, t)$ that does not depend on the velocity $\dot{x}$. For these systems, a no-chaos criterion is given by

Theorem 2: A parametrically driven nonlinear oscillator determined by

$$\ddot{x} + \gamma(x)\dot{x} + f(t)\partial_x V(x) = 0 \tag{7}$$

with a periodic driving $f(t)$ and a friction coefficient $\gamma(x)$ fulfilling the aforementioned conditions cannot exhibit chaotic behavior if the product of the driving and the *slope* of the potential $V(x)$ is either positive semidefinite, $f(t)\partial_x V(x) \geq 0$, for all x and t or negative semidefinite, $f(t)\partial_x V(x) \leq 0$, for all x and t.

Proof: The demonstration of the statement follows very closely the arguments in the proof of theorem 1. Using again $(d/dt)\Gamma(x) = \gamma(x)\dot{x}$, or equivalently, $\Gamma(x) = \int^x \gamma(x)\, dx$, (7) can be recast in the form

$$\frac{d}{dt}\left[\dot{x} + \Gamma(x)\right] = -f(t)\partial_x V(x) \tag{8}$$

and a subsequent integration of (8) with respect to time yields

$$\dot{x} + \Gamma(x) = -\int^t f(t)\partial_x V(x) dt + C \tag{9}$$

with initial value terms of the left-hand side absorbed in the constant C. If the integrand $f(t)\partial_x V(x)$ is either positive semidefinite or negative semidefinite for all x and t the integral on the right-hand side can only diverge or saturate into a constant in the long-time limit. If the right-hand side of (9) diverges, balancing both sides of (9) necessitates that x and/or $\dot{x}$ also diverge in the long-time limit. On the other hand, if the right-hand side of (9) approaches a constant, one obtains asymptotically a first order autonomous differential equation, $\dot{x} + \Gamma(x) \sim$const., that can only approach a fixed point or diverge to $\pm\infty$. This concludes the proof.

Several remarks are in order. *(i)* As major difference between additive and parametric forcing, ruling out chaotic behavior here requires that neither $f(t)$ nor the slope of $V(x)$ have zero crossings as function of t or x, respectively. *(ii)* In turn, chaotic behavior can already appear if the potential $V(x)$ is monotonic. A striking example for this case is the radial motion $(x(t) > 0)$ of an ion in the so-called dynamic Kingdon trap invented by Blümel [20], $\ddot{x} + \gamma\dot{x} + f(t)/x = 0$, where the reverse Feigenbaum route to chaos appears as function of the increasing amplitude Δ in $f(t) = \langle f \rangle - \Delta\cos(t)$. *(iii)* In principle, also the opposite case, a periodic driving $f(t)$ that is either positive or negative for all times and a derivative of the potential $\partial_x V(x)$ that appropriately oscillates about zero might potentially lead to a chaotic dynamics. To our knowledge, however, no such example has been identified so far. *(iv)* As can be observed from the proof of theorem 2, the no-chaos criterion also excludes any periodic long time solution.

As a straightforward generalization, the afore-mentioned argument can also be applied if the potential $V(x)$ consists of two parts

$$V(x) = V_1(x) + f(t)V_2(x)\,, \tag{10}$$

where only one part is parametrically modulated. In this case, one obtains as no-chaos condition that

$$\partial_x V_1(x) + f(t)\partial_x V_2(x) = \partial_x[V_1(x) + \langle f \rangle V_2(x)] + p(t)\partial_x V_2(x) \tag{11}$$

must be either positive or negative semidefinite for x and t. As an example, consider $V(x) = A_0 x + A_1\cos(l_1 x) - A_2\cos(2\pi t)\sin(l_2 x)$ with A_0, A_1, A_2 and l_1, l_2 positive. Chaotic behavior is excluded if $A_0 > l_1 A_1 + l_2 A_2$.

5. Additive and parametric driving

To discuss the combined action of additive and parametric driving, f_a and f_p, in a nonlinear oscillator, we suppose that $f_a(t)$ and $f_p(t)$ might be functionally different and might even possess different periods; both driving, however, should obey the afore-mentioned conditions. Combining the strategies being used to substantiate theorem 1 and 2, an additively and parametrically driven oscillator

$$\ddot{x} + \gamma(x)\dot{x} + f_p(t)\partial_x V(x) = f_a(t) \tag{12}$$

can be directly recast in the form

$$(d/dt)\,[\dot{x} + \Gamma(x) - P_a(t)] = \langle f_a \rangle - f_p(t)\partial_x V(x) \tag{13}$$

or, after carrying out one integration with respect to time in (13), as

$$\dot{x} + \Gamma(x) - P_a(t) = \int^t [\langle f_a \rangle - f_p(t)\partial_x V(x)]\, dt + C\,. \tag{14}$$

As a consequence, chaotic behavior in Eq. (12) can be ruled out if the integrand on the right-hand side of (14) $\langle f_a \rangle - f_p(t)\partial_x V(x)$ is either positive semidefinite for all x and t or negative semidefinite for all x and t. If $\langle f_a \rangle = 0$, then chaotic behavior can only be excluded if the no-chaos condition for *parametric* driving holds. If $\langle f_a \rangle \neq 0$, however, suppression of chaotic behavior in a system that can be chaotic for $f_a(t) = 0$ can be achieved if $f_p(t)\partial_x V(x)$ is bounded. As an example, consider the magnetic oscillator by Kim [21] with $\gamma(x)$ =const. and $f_p(t)\partial_x V(x) = -A\cos(2\pi t)\sin(2\pi x)$. If one couples an additional additive driving to this system with $\langle f_a \rangle \geq |A|$, the system can no longer be chaotic for any parameter range.

6. Discussion and conclusion

We have shown by comparatively elementary and nevertheless rigorous arguments that chaotic behavior can be ruled out in certain functional forms of additively and/or parametrically driven nonlinear oscillators. Our approach is based on recasting these evolution equations in the form of integro-differential equations that allow for formal manipulations in the integrand without the need of explicitly knowing the solution. In turn, functional forms of driven nonlinear oscillators that do not fulfill any of the presented no-chaos criteria are potentially chaotic, at least in some ranges of the entering parameters and specific initial conditions. A violation of the no-chaos criteria derived above can be considered as a necessary, albeit not sufficient condition for the appearance of chaos. In fact, considering the proofs of both theorems, a far stronger, albeit very implicite necessary condition for appearance of chaos can be conjectured: the integrands on the rhs of Eqs. (5), (9), or (14) have to oscillate in time about zero with a zero average in order to guarantee that the corresponding integrals conspire to an effective self feedback. Since, however, the actual dynamics $x(t)$ entering into the potential $V(x) = V(x(t))$ needs to be known, it seems unlikely to achieve further refinements for chaotic dynamics in the considered systems on the general level discussed here.

Speculatively, one might ask what the most elementary functional form of a quasi-linear oscillator is that might be chaotic for some control parameter ranges. Based on our necessary conditions, the equation

$$\ddot{x} + \gamma\dot{x} + \beta\,\mathrm{sgn}(x) = \alpha\,\mathrm{sgn}[\sin(\omega t)] \tag{15}$$

describing the motion in a piecewise linear (and, therefore, nonlinear) potential $V(x) = \beta|x|$ subject to a piecewise constant switching between $+\alpha$ and $-\alpha$ at fixed times $T = \pi n/\omega$ $(n \in N)$ has the necessary ingredients to be potentially chaotic in some ranges of the parameters α, β, γ and ω and, therefore, deserves further investigations.

I am grateful to J.C. Sprott for useful discussions and making his results (Ref. [19]) available to me prior publication.

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