CONTRACTIONS OF LIE ALGEBRAS AND GENERALIZED CASIMIR INVARIANTS

Rutwig Campoamor-Stursberg

Laboratoire de Mathématiques et Applications
Faculté de Sciences et Techniques
Université de Haute Alsace
4, rue des Frères Lumière, 68093 Mulhouse, France
e-mail: R.Campoamor@uha.fr

(Received December 10, 2002)

We prove that if $\mathfrak{g}'$ is a contraction of a Lie algebra $\mathfrak{g}$ then the number of functionally independent invariants of $\mathfrak{g}'$ is at least that of $\mathfrak{g}$. This allows to obtain some criteria to ensure the non-existence of non-trivial invariants for Lie algebras, as well as to deduce some results on the number of derivations of a Lie algebra. In particular, it is shown that almost any even dimensional solvable complete Lie algebra has only trivial invariants. Moreover, with the contraction formula we determine explicitly the number of invariants of Lie algebras carrying a supplementary structure, such as linear contact or linear forms whose differential is symplectic, without having explicit knowledge on the structure of the contracting algebra. This in particular enables us to construct Lie algebras with non-trivial Levi decomposition and none invariants for the coadjoint representation as deformations of frobeniusian model Lie algebras.

PACS numbers: 02.20.Sv

1. Introduction

Invariants of the coadjoint representation of Lie algebras, also called generalized Casimir invariants, are rather important in physics, since they are related to some preserved quantities in any theory described in terms of symmetry algebras [1]. In any case, these invariants provide various information on the irreducible representations of the considered Lie algebras [1], and characterize specific properties of a physical system, like mass formulae for dynamical groups. For the classical Lie algebras the problem was solved long ago by Racah, and by the Levi decomposition the class which must be analyzed is that of solvable Lie algebras, joint with the representations of the Levi subalgebra on the radical. Various authors have approached the problem in recent years, and the difficulty of finding or even characterizing the invariants of solvable Lie algebras has been pointed out [2, 3].

Another crucial feature in the study of a physical system is the determination of its symmetries, more specifically, of the symmetry group of the system [4,5]. This leads naturally to the comparison of systems having similar groups. Contractions of Lie algebras were introduced as a tool to explain formally why some theories arise as limiting cases of other theories, like the passage from the de Sitter algebra to the Poincaré algebra or from the latter to the Galilei algebra, and were further developed by various authors [6–8]. Endowed with these methods, it is natural to ask which is the relation of the invariants of a Lie algebra and the invariants of its contractions. For the special case of classical Casimir invariants, the question reduces to relate the corresponding universal enveloping algebras. This question was analyzed in detail in [9] for the special case of the simple Lie algebra $\mathfrak{sl}(3,\mathbb{C})$. This problem is definitively interesting for physical applications, since contractions can be interpreted in some sense as approximations, and the understanding of the behaviour of the invariants could provide useful interpretations of physical phenomena. One could argue that, since any Lie algebra contracts to the Abelian Lie algebra, a contraction will obviously have more invariants than the contracted algebra. This pattern coincides also with the (few) Lie algebras analyzed with respect to this observation, thus could be taken as an "experimental proof". However, such a conclusion is formally non rigorous, and can easily lead to mistakes. Until 1999, basing on the theory of semisimple Lie algebras and all constructions known, it was implicitly accepted that any rigid Lie algebra was rational (*i.e.*, defined over the field of rational numbers), which also seemed to be a quite natural consequence of this theory. However, so obvious the fact appeared, it was pointed out that this conclusion is false, since there exist rigid Lie algebras which are not only non-rational, but even non-real [10]. Therefore, if the observation on the number of invariants of contractions is to be sustained, it must be proven formally for nonpolynomial invariants (for polynomials the assertion follows from the properties of the universal enveloping algebras).

Among other interesting problems related to the invariants of Lie algebras, a characterization of solvable Lie algebras with Abelian nilradical admitting only trivial invariants was presented in [11]. In the same paper, the author commented the importance of finding a corresponding characterization for solvable Lie algebras with non-Abelian nilradical. In view of the cases treated and other examples, such a characterization probably does not exist, as it should comprise simultaneously the solvable and the non-solvable Lie algebras with nonzero Levi subalgebra. However, sufficiency conditions to ensure the non-existence of non-trivial invariants can be found [2,3].

In this paper we approach the problem of analyzing the relation between the number of invariants of a Lie algebra and the number of invariants of its contractions. In Section 2 we recall the general results on contractions of Lie algebras and the generalized Casimir invariants. In Section 3 we prove formally, making use of the Beltrametti–Blasi formula [12], that a contraction of a Lie algebras has at least as many functionally independent invariants as the algebra it comes from. This generalizes the pattern observed for some special types of algebras to arbitrary Lie algebras. As consequences of this result, we deduce criteria to ensure that a Lie algebra has no non-trivial invariants. The formula can also be used to determine upper bounds for the number of invariants in the case where a direct computation becomes too difficult. In Section 4 we apply the result to the study of derivations of Lie algebras. In particular, we prove that any Lie algebra with less derivations than its dimension necessarily has a fundamental set of invariants formed by rational functions, of which at least one corresponds to a degree one polynomial. This also tells that in absence of a fundamental set of invariants formed by rational invariants, a Lie algebra cannot be complete [13]. The latter case is of special interest, since most of the even-dimensional solvable complete Lie algebras in dimensions $n \leq 9$ have only trivial invariants, showing that this class is, in even dimension, an adequate starting point to search for criteria on the non-existence of non-trivial invariants. Even some of their contractions, which are not complete any more, are strong candidates for admitting only trivial invariants.

Finally, in Section 5, we analyze the invariants for some Lie algebras carrying an additional structure, such as linear contact forms or symplectic forms. The interest of such properties is out of discussion in view of the importance of symplectic structures in physics [14] The advantage of Lie algebras having supplementary structures like these lies in the fact that they can always be classified up to contraction [15, 16], which enables us to determine the number of invariants without any information about the precise structure of the contracting algebra. This method is also of interest for deformation theory, which, under certain restrictions, is deeply related with contraction theory [17, 18].

Unless otherwise stated, any Lie algebra $\mathfrak{g}$ considered here is defined over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and nonsplit, *i.e.*, $\mathfrak{g}$ does not decompose as a direct sum of ideals.

2. Contractions of Lie algebras. Generalized Casimir invariants

2.1. Contractions of Lie algebras

Traditionally contractions of Lie algebras are presented as limits [7, 8], although other authors have approached the contraction problem from the point of view of group actions [17, 18].

A Lie algebra $\mathfrak{g} = (\mathbb{K}^n, \mu)$ may be considered as an element μ of the variety $\mathrm{Hom}\left(\bigwedge^2 \mathbb{K}^n, \mathbb{K}^n\right)$ via the skew-symmetric bilinear map $\mu : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ defining the Lie bracket on $\mathfrak{g}$. Thus we can identify the Lie algebra $\mathfrak{g}$ with its law μ. The set $\mathcal{L}^n$ of Lie algebras is then a subset of the variety $\mathrm{Hom}\left(\bigwedge^2 \mathbb{K}^n, \mathbb{K}^n\right)$ on which the general linear group $GL(n, \mathbb{K})$ acts by :

$$(g \circ \mu)(x, y) = g^{-1}(\mu(gx, gy)), \quad g \in GL(n, \mathbb{K}); x, y \in \mathbb{K}^n .$$

Clearly the orbit under this action are the isomorphism classes of μ. Now a Lie algebra μ_∞ is called a contraction of a Lie algebra μ_0 if $\mu_\infty \in \overline{\mathcal{O}(\mu_0)}$, the Zariski closure of the orbit. The contraction is called nontrivial if μ lies in the boundary of the orbit. This geometrical definition is nothing more than a topological reformulation of the classical concept of Inönü–Wigner contractions and its variations [7, 8]. As known, these contractions can be viewed as singular changes of basis, starting from a fixed basis $\{X_1, \ldots, X_n\}$ of a Lie algebra $\mathfrak{g}$. That is, considering a sequence of endomorphisms $\{f_p(\varepsilon_1, \ldots, \varepsilon_r)\}_{p \in \mathbb{N} \cup \{0\}}$ (where f_0 can be taken as the identity and ε_i designate the parameters), for any p we have:

$$\mu_p := f_p^{-1} \circ \mu_0(f_p, f_p) \ . \tag{1}$$

Thus, if the limit exists, it also represents a Lie algebra, and the law of the contraction is given by

$$\mu_\infty = \lim_{p \to \infty} \mu_p \, . \tag{2}$$

Therefore, if $\left\{C_{ij}^k\right\}$ are the structure constants of $\mathfrak{g}_0 = (\mathbb{K}^n, \mu_0)$ over a basis $\{X_1, \ldots, X_n\}$ and $\left\{C_{ij}^k(p)\right\}$ the structure constants of $\mathfrak{g}_p = (\mathbb{K}^n, \mu_p)$, the law of μ_∞ is given by

$$\widetilde{C_{ij}^k} = \lim_{p \to \infty} C_{ij}^k(p) \, . \tag{3}$$

The most elementary example is the well known fact that any Lie algebra contracts to the Abelian algebra of the same dimension. The contraction is easily seen to be realized by the endomorphisms $\left\{f_t = t^{-1}\mathrm{id}\right\}$, where id denotes the identity matrix. This special kind of contraction, depending on an unique parameter, is called one-parameter subgroup contraction [18].

2.2. Generalized Casimir invariants

The standard method to obtain the Casimir operators and its generalizations of a Lie algebra is their interpretation as invariants of the coadjoint representation of the corresponding Lie group [1, 11, 19].

The problem of finding its invariants is indeed reduced to that of solving a system of linear first order partial differential equations. If $B = \{X_1, \ldots, X_n\}$ is a basis of the n-dimensional Lie algebra $\mathfrak{g}$ and $\{x_1, \ldots, x_n\}$ a coordinate system on the dual space, then the infinitesimal generators of the action are denoted by $\widetilde{X}_i$. If moreover the structure constants of $\mathfrak{g}$ are given by $[X_i, X_j] = C_{i,j}^k X_k$ over the basis B, a function $F \in C^\infty(\mathfrak{g}^*)$ is an invariant of the coadjoint representation if and only if it satisfies the two following conditions:

1. $\widetilde{X}_i = \sum_j (-C_{i,j}^k) x_k \frac{\partial}{\partial x_j}$ and $\left[\widetilde{X}_i, \widetilde{X}_j\right] = C_{i,j}^k \widetilde{X}_k$,
2. F is a solution of the system $\left\{\widetilde{X}_i F = 0\right\}_{1 \leq i \leq n}$.

Solutions to this system are usually found by integration of the corresponding system of characteristic equations or other standard integration procedures [4]. If the solutions are polynomials, then they correspond to the classical Casimir operators. If the solutions are rational or transcendental functions, we call them generalized Casimir operators. The latter type of solutions has shown its importance in the theory of integrable Hamiltonian systems, as pointed out in [20].

A maximal set of functionally independent solutions will be called a fundamental set of invariants. Since a Lie algebra law is an alternated tensor of type $(2,1)$, this rank of the algebra does not depend on the basis chosen. By antisymmetry, this rank must be even, and from the analysis undertaken in [12] it follows that the number of invariants N satisfies the congruence $N \equiv \dim(\mathfrak{g}) \pmod 2$. Polynomial solutions will naturally correspond to the classical Casimir operators (possibly after symmetrizing). In particular, an odd dimensional Lie algebra has at least one nontrivial invariant.

3. The contraction formula

As told in the introduction, in [9] the authors studied the contractions of the simple Lie algebra $\mathfrak{sl}(3, \mathbb{C})$, and observed that invariants of the contractions can be obtained as limits of the invariants of $\mathfrak{sl}(3, \mathbb{C})$, at least in the case of Inönü–Wigner contractions. In particular, from this analysis we get that contractions are expected to have more invariants than the algebra they come from. The known cases seem to agree with this observation, which however does not constitute an evidence for its correctness. In this section we prove that this important observation generalizes indeed to contractions of any Lie algebra (the orbit closure argument allowing to deal with all particular types of contractions simultaneously).

As follows from the work of Beltrametti and Blasi [12], the number of functionally independent invariants of the coadjoint representation ad^* of a

Lie algebra $\mathfrak{g}$ is given by $\mathcal{N} = \dim(\mathfrak{g}) - r(\mathfrak{g})$, where $r(\mathfrak{g})$ is the maximum rank of the commutator table considered as a $(n \times n)$-matrix, where $n = \dim(\mathfrak{g})$. That is, the matrix is $A_{ijk} := \left(C_{ij}^{k} x_k\right)_{1 \leq i < j, k \leq \dim(\mathfrak{g})}$ over the basis $\{X_1, \ldots, X_n\}$, $\left\{C_{ij}^{k}\right\}$ being the structure constants over this basis. It is clear that on the transformed basis $\{f_p X_1, \ldots, f_p X_n\}$ we obtain the matrix $A_{ijk}^{p} = \left(C_{ij}^{k}(p)\, x_k\right)$.

Theorem 1 *If* $\mathfrak{g}_1 = (\mathbb{K}^n, \mu_1)$ *is a contraction of* $\mathfrak{g}_0 = (\mathbb{K}^n, \mu_0)$, *then* $\mathcal{N}(\mathfrak{g}_1) \geq \mathcal{N}(\mathfrak{g}_0)$.

Proof

Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}_0$ and $f_p(\varepsilon_1, \ldots, \varepsilon_r)$ be the sequence of endomorphisms such that

$$\mu_1 = \lim_{p \to \infty} \mu_p$$

By application of elementary properties of matrix analysis algebra we obtain that:

$$\operatorname{rank}(A_{ijk}) \geq \operatorname{rank}\left(\lim_{p \to \infty}\left(A_{ijk}^{p}\right)\right). \tag{4}$$

This inequality holds for any representative μ_0 of the Lie algebra $\mathfrak{g}_0$ and any family $f_p \in GL(n, \mathbb{K})$ realizing the contraction. This proves that starting from the orbit of $\mathfrak{g}_0$ we preserve of decrease the rank of the matrices in (4).

As a consequence of the fact that contractions of Lie algebras over a field k can be characterized in terms of discrete valuation k-algebras whose quotient field has transcendence degree one over k, and that formal deformations can be described using inverse limits and the completion of valuation algebras [18], we deduce that a contraction can be realized as a deformation [17], which ensures that the maximal rank of commutation matrices A_{ijk} of representatives μ_1 of $\mathfrak{g}_1$ is lower or equal to the rank of some commutator matrix of a representative of $\mathfrak{g}_0$. This shows that no representative of the orbit of $\mathfrak{g}_1$ can reverse the inequality (4). Therefore we obtain that:

$$r(\mathfrak{g}_0) \geq r(\mathfrak{g}_1) \tag{5}$$

and from the formula for the number of invariants:

$$\mathcal{N}(\mathfrak{g}_0) \leq \mathcal{N}(\mathfrak{g}_1), \tag{6}$$

that is, the contraction $\mathfrak{g}_1$ has at least $\mathcal{N}(\mathfrak{g}_0)$ invariants. ∎

This result constitutes a complete proof of the intuition that contractions have "less brackets" than the Lie algebra they come from, and it is independent of any experimental observation made on particular cases. Geometrically this is more or less clear, as the dimension of the orbit of contracted algebras is lower than the orbit dimension of the starting algebra, and therefore one should expect that the contraction has more invariants. Observe further that this result cannot be formulated in terms of deformations, since there exist deformations which are not related to a contraction:

Example 1 *Let $\mathfrak{r}\oplus\mathbb{K}$ be the solvable Lie algebra given by the brackets*

$$[X_1, X_2] = X_1,\ [X_1, X_3] = [X_2, X_3] = 0$$

over the basis $\{X_1, X_2, X_3\}$. The family of algebras $L(\alpha)$ given by

$$[X_1, X_2] = X_1,\ [X_2, X_3] = -\alpha X_3,\ [X_1, X_3] = 0$$

with $\alpha \neq 1$ is a deformation of $\mathfrak{r}\oplus\mathbb{K}$, but none element of this family contracts to $\mathfrak{r}\oplus\mathbb{K}$ [17].

From Theorem 1 we deduce a result which was anticipated in [9] for the special case of $\mathfrak{sl}(3,\mathbb{C})$:

Corollary 1 *If $\mathfrak{g}$ is a contraction of a simple Lie algebra of rank p, then $\mathfrak{g}$ has at least p functionally independent invariants.*

Thus Theorem 1 establishes a necessary condition on the number of invariants of contractions. This is specially of interest when we search for Lie algebras admitting only trivial invariants [*i.e.*, all whose invariants are constant functions]:

Lemma 1 *If $\mathfrak{g}$ has a contraction without any non-trivial invariants, then $\mathfrak{g}$ itself has only trivial invariants.*

This lemma gives an alternative procedure to ensure the non-existence of non-trivial invariants: to find a contraction which has only trivial invariants. On the other hand, it also provides us a geometrical consequence: the orbits of such Lie algebras are not included in the orbit closure of simple Lie algebras, since they cannot be contractions of them. However, its interest is the possibility of applying it to Lie algebras for which we know contractions whose invariants have already been determined. An elementary example illustrates this fact: For $m \geq 2$ let $\mathfrak{r}$ be a $(2m+2)$-dimensional solvable

Lie algebra whose nilradical is isomorphic to the parametrized nilpotent Lie algebra $\mathfrak{n}_{\alpha_1,\ldots,\alpha_{[\frac{m}{2}]+2}}$ given by:

$$\left.\begin{array}{l}[X_{2+j},X_{2m+1-j}]=X_1, \quad 0\geq j\geq m-1\\ [X_{2+j},X_{2m+1}]=\alpha_j X_j, \quad 1\leq j\leq m-2\\ \left(\alpha_1,\ldots,\alpha_{[\frac{m}{2}]+2}\right)\in\mathbb{K}^{[\frac{m}{2}]+2}\end{array}\right\}, \tag{7}$$

where $\{X_1,\ldots,X_{2m+1},X_{2m+2}\}$ is a basis of $\mathfrak{r}$ and $\alpha_j+\alpha_{2m-j}=0$ for $1\leq j\leq[\frac{m}{2}]+2$.

Observe that X_1 belongs to the centre of the nilradical, and therefore $\mathfrak{r}$ has trivial centre whenever $[X_{2m+2},X_1]\neq 0$. Let $f_t\in GL(2m+2,\mathbb{K})$ be the sequence defined by

$$f_t\left(X_j\right)=t^{2m+2-j}X_j, \quad 1\leq j\leq 2m+2\,.$$

Clearly these endomorphisms define a one-parameter subgroup contraction, at it can easily be seen that $\lim_{t\rightarrow\infty}\left(f_t^{-1}\mathfrak{r}(f_t,f_t)\right)$ is a solvable Lie algebra whose nilradical is isomorphic to the Heisenberg Lie algebra

$$\mathfrak{h}_m=\langle X_1,\ldots,X_{2m+1}|[X_{2+j},X_{2m+1-j}]=X_1\rangle\,.$$

If we suppose, moreover, that $[X_{2m+2},X_1]\neq 0$, then the contracted algebra is of type $L\left(m,1\right)$ [21], and either from a direct computation or by application of the formulae given there, since the centre is trivial, we obtain that $\mathcal{N}\left(L(m,1)\right)=0$, and therefore $\mathfrak{r}$ has only trivial invariants. A direct computation of the invariants of $\mathfrak{r}$ is much more complicated, due to the presence of the $[\frac{m}{2}]+2$ parameters depending on the dimension. Although in general the preceding result cannot be announced using deformations, it applies in particular to deformations which are related to contractions. Recall that a jump deformation μ_t of a Lie algebra (V,μ_0) is a formal deformation $\mu_t=\mu_0+t\phi_1+t^2\phi_2+\ldots\left(\phi_i\in\mathrm{Hom}\left(\wedge^2V,V\right)\right)$ which remains constant for generic $t\neq 0$. That is, if u is an additional variable and coefficient are extended to $\mathbb{K}\left(\left(t\right)\right)\left[\left[u\right]\right]$, we have an isomorphism $\mu_t=\mu_{(1+u)t}$.

Proposition 1 *Let $\mathfrak{g}$ be a Lie algebra satisfying $\mathcal{N}\left(\mathfrak{g}\right)=0$. Then any jump deformation $\mathfrak{g}'$ also satisfies $\mathcal{N}\left(\mathfrak{g}'\right)=0$.*

The applicability of this consequence is constrained by the necessity conditions for the existence of jump deformations, such as the non-nullity of the cohomology group $H^1\left(\mathfrak{g},\mathfrak{g}\right)$.

4. Invariants and derivations of Lie algebras

In this paragraph we analyze some questions relating the invariants of a Lie algebra $\mathfrak{g}$ with the structure of its Lie algebra of derivations $\mathrm{Der}(\mathfrak{g})$. In particular, we are interested on the invariants of Lie algebras satisfying the inequality $\dim \mathrm{Der}(\mathfrak{g}) \leq \dim \mathfrak{g}$.

Among the multiple properties of contractions of Lie algebras existing, we emphasize the following, which will be used in this section and whose proof is straightforward:

Lemma 2 *Let $\mathfrak{g}_1$ be a contraction of the Lie algebra $\mathfrak{g}_0$. Then the following conditions are satisfied:*

1. $\dim[\mathfrak{g}_0, \mathfrak{g}_0] \geq \dim[\mathfrak{g}_1, \mathfrak{g}_1]$,
2. $\dim Z(\mathfrak{g}_0) \leq \dim Z(\mathfrak{g}_1)$,
3. $\dim \mathrm{Der}(\mathfrak{g}_0) < \dim \mathrm{Der}(\mathfrak{g}_1)$,

where $Z(\mathfrak{g})$ denotes the centre of $\mathfrak{g}$.

Recall that a Lie algebra $\mathfrak{g}$ is said algebraic if it is isomorphic to the Lie algebra of a linear algebraic group. Among the Lie algebras satisfying the condition presented at the beginning of this section, we obtain the complete Lie algebras [centerless Lie algebras all whose derivations are inner], which in particular cover semisimple Lie algebras, whose invariants are perfectly known. Now the existence of Lie algebras satisfying $\dim \mathrm{Der}(\mathfrak{g}) < \dim \mathfrak{g}$ has been a conjecture for long time, until the first examples were found in 1971 [22]. The central structural result was obtained by Carles in 1984 [23]:

Theorem 2 *Let $\dim \mathrm{Der}(\mathfrak{g}) \leq \dim \mathfrak{g}$. Then $\mathfrak{g}$ is an algebraic Lie algebra (with non-trivial centre if $\dim \mathrm{Der}(\mathfrak{g}) < \dim \mathfrak{g}$). Moreover, in this case $\mathfrak{g}$ is perfect, i.e., $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$).*

From this Theorem we immediately deduce an interesting result on the invariants of such Lie algebras:

Proposition 2 *Let $\mathfrak{g}$ be a Lie algebra such that $\dim \mathrm{Der}(\mathfrak{g}) \leq \dim \mathfrak{g}$. Then $\mathfrak{g}$ admits a fundamental set of invariants formed by rational functions, of which at least one can be taken as a polynomial of degree one.*

This tells in particular that a Lie algebra with less derivations than its dimension can never contract to a complete Lie algebra. More precisely, a complete Lie algebra can never arise as a contraction of a Lie algebra. This makes these algebras more difficult to localize, since the orbit closure method fails. However, using the invariants we can deduce the following property:

Proposition 3 *If the Lie algebra* $\mathfrak{g}$ *does not admit a fundamental set of invariants formed by rational functions, then* $\dim \mathrm{Der}(\mathfrak{g}) > \dim \mathfrak{g}$. *In particular* $\mathfrak{g}$ *cannot be complete.*

The proof follows immediately from Theorem 2. For the special case of Lie algebras with trivial centre, the proposition ensures the existence of an outer derivation (*i.e.*, a derivation which is not of the form $\mathrm{ad}(X)$ for some $X \in \mathfrak{g}$). The result is remarkable since it relates the structure of the invariants with the number of derivations of $\mathfrak{g}$.

Taking together the above results, it follows that for Lie algebras satisfying $\dim \mathrm{Der}(\mathfrak{g}) \leq \dim \mathfrak{g}$, only the complete case can provide us with Lie algebras having no non-trivial invariants (in even dimension). Since all known examples are solvable, one can ask whether any algebra satisfying $\mathcal{N}(\mathfrak{g}) = 0$ and the conditions above must be solvable.

4.1. Solvable complete Lie algebras in dimension $n \leq 8$

Solvable complete Lie algebras have been completely classified up to dimension 9, while non-solvable complete Lie algebras are classified up to dimension 7 [13]. Since solvable complete Lie algebras $\mathfrak{r}$ decompose as a semidirect product $\mathfrak{r} = \mathfrak{n} \oplus \mathfrak{t}$ of its nilradical $\mathfrak{n}$ and a maximal toral subalgebra $\mathfrak{t}$ [*i.e.*, an Abelian subalgebra formed by ad-semisimple elements] [13], the classification reduces to the case where $\mathfrak{n}$ is nonsplit. Following [13], the solvable complete Lie algebras with nonsplit nilradical are called simple complete. The distribution of isomorphism N classes by dimension is given in Table I.

TABLE I

Number N of isomorphism classes.

Dimension	2	3	4	5	6	7	8	9
N	1	0	0	1	2	6	11	9

From Table I we see that in low dimension the even dimensional case has more isomorphism classes. Table II presents the isomorphism class, labelled like in [13], and the number of invariants of these algebras for the even dimensions:

The remarkable fact from this table is that most of these algebras have only non-trivial invariants, which leads to the question whether for higher dimensions this pattern is preserved. In any case, this shows that solvable complete Lie algebras in even dimension is a class which is worthy to be analyzed, as well as its contractions (see also the solvable rigid Lie algebras in even dimensions [2, 3]).

TABLE II

Number of invariants of solvable complete Lie algebras in even dimension ≤ 8.

Algebra	Brackets		dim	$\mathcal{N}(\mathfrak{r})$
$\mathfrak{r}_4^1$	$[X_5,X_i]=X_i,\quad i=1,3$		6	0
	$[X_5,X_4]=2X_4,$	$[X_6,X_i]=X_i,\ i=2,3,4$		
	$[X_1,X_i]=X_{i+1},\ i=2,3$			
$\mathfrak{r}_5^6$	$[X_6,X_i]=iX_i,\quad 1\leq i\leq 5$		6	0
	$[X_1,X_i]=X_{i+1}, i=2,3,4$	$[X_2,X_3]=X_5,$		
$\mathfrak{r}_5^1$	$[X_6,X_i]=iX_i,\quad 1\leq i\leq 5$		8	2
	$[X_7,X_i]=X_i,\quad i=2,4,5,$	$[X_8,X_i]=X_i,\quad i=3,4,5,$		
	$[X_1,X_4]=[X_2,X_3]=X_5$			
$\mathfrak{r}_5^2$	$[X_6,X_i]=X_i,\quad i=1,3,5$		8	0
	$[X_7,X_i]=X_i,\ i=2,4$	$[X_8,X_i]=X_i,\ i=3,5$		
	$[X_1,X_i]=X_{i+2},\ i=2,3$			
$\mathfrak{r}_6^2$	$[X_7,X_i]=iX_i,\quad 1\leq i\leq 4$		8	2
	$[X_7,X_5]=3X_5,$	$[X_7,X_6]=5X_6$		
	$[X_8,X_i]=X_i,\ i=2,3,4,6$	$[X_1,X_i]=X_{i+1}$ i=2,3		
	$[X_2,X_5]=X_6$			
$\mathfrak{r}_6^4$	$[X_7,X_i]=X_i,\quad i=1,3$		8	0
	$[X_7,X_i]=2X_i$,i=4,5	$[X_7,X_6]=3X_6$		
	$[X_8,X_i]=X_i,\ i=2,3,5,6$	$[X_7,X_4]=2X_4$		
	$[X_1,X_i]=X_{i+1}, i=2,5$	$[X_1,X_3]=X_5$		
	$[X_i,X_4]=X_{i+3}, i=2,3$			
$\mathfrak{r}_6^8$	$[X_7,X_i]=X_i,\quad i=1,3$		8	0
	$[X_7,X_i]=3X_i$,i=5,6	$[X_7,X_4]=2X_4$		
	$[X_8,X_i]=X_i,\ 2\leq i\leq 5$	$[X_7,X_6]=2X_6$		
	$[X_1,X_i]=X_{i+1}, i=2,3$	$[X_2,X_5]=X_6$		
	$[X_3,X_4]=-X_6.$			
$\mathfrak{r}_6^9$	$[X_7,X_i]=iX_i,\ 1\leq i\leq 6,$		8	2
	$[X_8,X_i]=X_i,\quad 2\leq i\leq 6,$	$[X_1,X_i]=X_{i+1},\quad 2\leq i\leq 5.$		
$\mathfrak{r}_6^{10}$	$[X_7,X_i]=X_i,\quad i=1,4$		8	0
	$[X_7,X_3]=\frac{1}{2}X_3$	$[X_7,X_5]=\frac{3}{2}X_5$		
	$[X_7,X_6]=2X_6$	$[X_8,X_i]=X_i, i=2,4,6$		
	$[X_8,X_i]=\frac{1}{2}X_i, i=3,5$	$[X_3,X_5]=X_6,$		
	$[X_1,X_i]=X_{i+2}, i=2,3,4$			
$\mathfrak{r}_6^{14}$	$[X_7,X_i]=X_i,\quad i=1,3,5$		8	2
	$[X_7,X_i]=2X_4, i=2,4$	$[X_6,X_i]=X_i,\ i=2,3,4$		
	$[X_1,X_i]=X_{i+1},\ i=2,3$			
$\mathfrak{r}_6^{15}$	$[X_7,X_i]=X_i,\quad i=1,3,6$		8	0
	$[X_7,X_i]=2X_i, i=4,5$	$[X_8,X_i]=X_i, i=2,3,5$		
	$[X_8,X_6]=2X_6$	$[X_1,X_2]=X_3,$		
	$[X_1,X_i]=X_{i+2}, i=3,4$	$[X_2,X_3]=X_6,$		
$\mathfrak{r}_6^{20}$	$[X_7,X_i]=X_i,\quad i=1,3$		8	0
	$[X_7,X_i]=2X_i, i=4,5$	$[X_7,X_6]=3X_6$		
	$[X_8,X_i]=X_i, i=2,3,5$	$[X_1,X_2]=X_3$		
	$[X_1,X_i]=X_{i+2}, i=3,4$	$[X_2,X_4]=X_5$		
$\mathfrak{r}_6^{23}$	$[X_7,X_i]=X_i,\quad i=1,3,6$		8	0
	$[X_7,X_i]=(i-2)X_i,$	$[X_8,X_6]=2X_6$		
	$[X_8,X_i]=X_i,\ 2\leq i\leq 5$	$[X_1,X_i]=X_{i+1},\ i=2,3,4$		

5. Linear contact forms and frobeniusian Lie algebras

The contraction formula enables us to obtain some criteria on the number of invariants of a Lie algebra. This will be of special interest for those classes of Lie algebras which, having an additional structure, can be classified up to contraction. The examples of additional structure we present here are based on properties appearing in differential geometry, such as linear contact forms or a frobeniusian structure. These two properties are related to Heisenberg Lie algebras $\mathfrak{h}_n$, which are well known to be subalgebras of the extended Galilei algebra [24, 25].

5.1. Generalization of the Heisenberg Lie algebra

Given a Lie algebra $\mathfrak{g} = \left(\mathbb{K}^{2n+1}, \mu\right)$ and a linear form ω over $\mathbb{K}^{2n+1}$, the exterior differential of ω relative to μ is given by

$$d\omega_{\mu}\left(X, Y\right) = -\omega\left(\mu\left(X, Y\right)\right), \ \forall X, Y \in \mathbb{K}^{2n+1} . \tag{8}$$

We say that ω is a linear contact form relative to μ if

$$\omega \wedge \left(d\omega_{\mu}\right)^{n} \neq 0 , \tag{9}$$

where $\left(d\omega_{\mu}\right)^{n} = \bigwedge^{n} d\omega_{\mu}$. We observe that the left invariant Pfaff form induced by ω over the Lie groups having $\mathfrak{g}$ as Lie algebra is a contact form in the usual sense. We will simply say that $\mathfrak{g}$ is equipped with a linear contact form.

The motivation to study linear contact forms comes from the analysis of the Lie algebra $\mathfrak{so}(3)$ of the rotation group:

Proposition 4 *Every nonzero linear form ω on $\mathfrak{so}(3)$ is a linear contact form.*

This is easily seen by considering the basis $\{X_1, X_2, X_3\}$ such that

$$[X_j, X_k] = \varepsilon_{ijk} X_i,$$

since for the linear forms $\omega_i, \quad 1 \leq i \leq 3$ we have

$$\omega_i \wedge d\omega_i \neq 0 .$$

This gives a characterization of $\mathfrak{so}(3)$ in terms of contact forms [15]:

Proposition 5 *A Lie algebra $\mathfrak{g}$ all whose linear forms $\omega \in \mathfrak{g}^*$ are linear contact forms is isomorphic to $\mathfrak{so}(3)$.*

This shows that for dimensions $n \geq 3$ we have to relax the condition in order to obtain useful results. We will now see that Lie algebras having a linear contact form can be classified up to contraction, *i.e.*, we can find a family $\mathfrak{F}$ of Lie algebras (the family can eventually reduce to an unique algebra) such that any Lie algebra $\mathfrak{g}$ with a linear contact form contracts to some element of $\mathfrak{F}$ [15].

Theorem 3 *[15] Let $\mathfrak{g} = \left(\mathbb{K}^{2n+1}, \mu\right)$ be a Lie algebra equipped with a linear contact form ω. Then the Heisenberg Lie algebra $\mathfrak{h}_n$ is a contraction of $\mathfrak{g}$.*

We can enumerate two consequences of this Theorem:

1. If $\mathfrak{g}$ has a linear contact form, then the dimension of its centre is at most one.
2. If $\mathfrak{g}$ is semisimple, then its rank is one.

By Theorems 1 and 3, to obtain an upper bound for $\mathcal{N}(\mathfrak{g})$ it suffices to determine $\mathcal{N}(\mathfrak{h}_n)$, which is easily checked to be 1. Observe that it coincides with the dimension of the centre of $\mathfrak{h}_n$. As a consequence we have that any algebra $\mathfrak{g}$ with a linear contact form satisfies

$$\left.\begin{array}{c} \dim \mathfrak{g} = 2n+1,\ n \geq 1 \\ \mathcal{N}(\mathfrak{g}) \leq 1 \end{array}\right\} . \tag{10}$$

In particular, the latter equation provides us a proof of consequence 2. above. Since the number of invariants of a semisimple Lie algebra is the dimension h of a Cartan subalgebra [1], by Theorem 1 we obtain that $h \leq 1$.

5.2. Frobeniusian Lie algebras

Let $\mathfrak{g} = \left(\mathbb{K}^{2n}, \mu\right)$ be a Lie algebra. We say that $\mathfrak{g}$ is frobeniusian if there exists a linear form $\omega \in \mathfrak{g}^*$ such that

$$(d\omega_\mu)^n \neq 0 . \tag{11}$$

Frobeniusian Lie algebras have also been classified up to contraction [16]. In contrast to the previous case, frobeniusian Lie algebras need not contract to the same algebra, but to a parametrized family:

Theorem 4 *[16] Let $\mathfrak{g} = \left(\mathbb{R}^{2n}, \mu\right)$ be a frobeniusian Lie algebra. Then $\mathfrak{g}$ contracts to some element of the following family $\mathfrak{g}\left(\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{n-1-s}\right)$:*

$$\left.\begin{array}{l} [X_1, X_2] = X_1 \\ [X_{2r+1}, X_{2r+2}] = X_1,\ 1 \leq r \leq n-1 \\ [X_2, X_{4k-1}] = \alpha_k X_{4k-1} + \beta_k X_{4k+1},\ k \leq s \\ [X_2, X_{4k}] = \left(-1-\alpha_k\right) X_{4k} - \beta_k X_{4k+2},\ k \leq s \\ [X_2, X_{4k+1}] = -\beta_k X_{4k-1} + \alpha_k X_{4k+1},\ k \leq s \\ [X_2, X_{4k+2}] = \beta_k X_{4k} + \left(-1-\alpha_k\right) X_{4k+2},\ k \leq s \\ [X_2, X_{4s+2k-1}] = -\frac{1}{2} X_{4s+2k-1} + \beta_{k+s-1} X_{4s+2k},\ 2 \leq k \leq n-2s \\ [X_2, X_{4s+2k}] = -\beta_{k+s-1} X_{4k+2s-1} - \frac{1}{2} X_{4s+2k},\ 2 \leq k \leq n-2s \end{array}\right\}, \tag{12}$$

where $0 \leq s \leq \left[\frac{n-1}{2}\right]$ and $\left(\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{n-1-s}\right) \in \mathbb{R}^{n-1}$. The algebras $\mathfrak{g}\left(\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{n-1-s}\right)$ are called frobeniusian model Lie algebras.

We observe that the complex models are obtained by complexification of the preceding algebras. Specifically, for $\mathbb{K} = \mathbb{C}$ we obtain that any complex frobeniusian Lie algebra contracts to some algebra of the following family

$$\left.\begin{array}{l} [X_1, X_2] = X_1 \\ [X_{2r+1}, X_{2r+2}] = X_1,\ 1 \leq r \leq n-1 \\ [X_2, X_{2k+1}] = \lambda_k X_{2k+1},\ 0 \leq\ k \leq n-1 \\ [X_2, X_{2k+2}] = \left(-1-\lambda_k\right) X_{2k+2},\ 0 \leq k \leq n-1 \\ \left(\lambda_1, \dots, \lambda_{n-1}\right) \in \mathbb{C}^{n-1} \end{array}\right\}. \tag{13}$$

Proposition 6 *Let $\mathfrak{g}$ be a frobeniusian Lie algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then $\mathfrak{g}$ has no non-trivial generalized Casimir invariants.*

Proof We prove it for $\mathbb{K} = \mathbb{R}$, the complex case being similar. Observe from (12) that frobeniusian model Lie algebras are solvable with nilradical isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_{n-1}$. Realizing the coadjoint representation of $\mathfrak{g}\left(\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{n-1-s}\right)$ in a functional space of $(2n+1)$ variables denoted by $\{x_1, \dots, x_{2n+1}\}$, we have

$$\widehat{X}_1 = -x_1 \partial_{x_2}, \tag{14}$$

which implies that an invariant F does not depend on x_2. Now, since for any $j \geq 3$ we have

$$\widehat{X}_j = f_j\left(x_3, \dots, x_{2n}\right) \partial_{x_2} + (-1)^{j-1} x_1 \partial_{x_{j+(-1)^{j-1}}}, \tag{15}$$

the function $f_j\left(x_3, \dots, x_{2n}\right)$ expressing the Lie brackets $[X_2, X_j]$ of (12), we obtain that

$$\partial_{x_j} F = 0, \quad j \geq 3 \tag{16}$$

for any invariant F. Finally, considering the representation of X_2:

$$\widehat{X}_2 = -\sum_{j=3}^{2n} g_j(x_3,\dots,x_{2n})\partial_{x_j} + x_1\partial_{x_1}\,, \tag{17}$$

the functions $g_j(x_3,\dots,x_{2n})$ again expressing the brackets of (12), we deduce $\partial_{x_1}F = 0$, which shows that F is a trivial invariant. ■

Observe that frobeniusian model Lie algebras are subalgebras of the semidirect product $\mathfrak{sp}\,(2n,\mathbb{K}) \oplus \mathfrak{h}_n$, $\mathfrak{sp}\,(2n,\mathbb{K})$ being the simple symplectic Lie algebra. This makes frobeniusian algebras interesting for the study of nuclear collective motions [24, 25].

From this result we obtain a sufficiency criterion for a Lie algebra to have no non-trivial invariants:

Proposition 7 *If $\mathfrak{g}$ is a frobeniusian Lie algebra, then $\mathcal{N}\,(\mathfrak{g}) = 0$.*

Proof Since there exists a n-tuple $(\alpha_1,...,\alpha_s,\beta_1,...,\beta_{n-1-s})$ such that $\mathfrak{g}\,(\alpha_1,...,\alpha_s,\beta_1,...,\beta_{n-1-s}) \in \overline{\mathcal{O}\,(\mathfrak{g})}$, the result follows from Theorem 1. ■

Thus the existence of a linear form whose differential is symplectic forces the triviality of invariants of the coadjoint representation of $\mathfrak{g}$. This proposition gives a quite interesting class of algebras to be analyzed with respect to the problem of invariants and deformations. The cohomology of frobeniusian model algebras is known [26], and it has been proven that any nonsolvable frobeniusian Lie algebra contracts on some model $\mathfrak{g}\,(\alpha_1,...,\alpha_s,\beta_1,...,\beta_{n-1-s})$ whose parameters $(\alpha_1,...,\alpha_s,\beta_1,...,\beta_{n-1-s})$ move on a finite union of hyperplanes [26]. Therefore a detailed analysis of these nonsolvable Lie algebras may be approached by cohomological means. We illustrate this by an example. The classification of non-semisimple Lie algebras with nonzero Levi subalgebra $\mathfrak{s}$ in dimensions $n \leq 9$ [27] was motivated by the study of higher dimensional versions of the Bianchi type-IX universe [28]. Obtaining of ten dimensional models in full generality is however not possible due to the great number of parameters involved and the nonexistence of a classification of solvable Lie algebras in dimensions $n \geq 7$. This forces to consider some additional assumption in order to reduce the number of parameters. In the context of frobeniusian Lie algebras, one can ask whether there exist frobeniusian Lie algebras with Levi subalgebra $\mathfrak{s}$ isomorphic to the rotation algebra $\mathfrak{so}(3)$. This reduces to analyze the cohomology of the model algebras in order to obtain the Levi part $\mathfrak{so}(3)$ (this implies severe restrictions on the representations R describing the semidirect product $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$, where $\mathfrak{r}$ is the radical). With some effort it can be shown that solutions to this problem exist, such as for example the parametrized family of Lie algebras $L(\gamma_1,\dots,\gamma_7)$ defined, over the basis $\{X_1,\dots,X_{10}\}$, by the brackets:

$$\left.\begin{array}{lll}
[X_1,X_2]=X_3, & [X_1,X_3]=-X_2, & [X_2,X_3]=X_1,\\
[X_1,X_4]=\frac{1}{2}X_7, & [X_1,X_5]=\frac{1}{2}X_6, & [X_1,X_6]=-\frac{1}{2}X_5,\\
[X_1,X_7]=-\frac{1}{2}X_4, & [X_2,X_4]=\frac{1}{2}X_5, & [X_2,X_5]=-\frac{1}{2}X_4,\\
[X_2,X_6]=\frac{1}{2}X_7, & [X_2,X_7]=-\frac{1}{2}X_6, & [X_3,X_4]=\frac{1}{2}X_6,\\
[X_3,X_5]=-\frac{1}{2}X_7, & [X_3,X_6]=-\frac{1}{2}X_4, & [X_3,X_7]=\frac{1}{2}X_5,\\
[X_4,X_8]=\gamma_5X_4, & [X_5,X_8]=\gamma_5X_5, & [X_6,X_8]=\gamma_5X_6,\\
[X_7,X_8]=\gamma_5X_7, & [X_8,X_{10}]=\gamma_6X_{10}, & [X_9,X_{10}]=\gamma_7X_{10},\\
\multicolumn{3}{l}{[X_4,X_9]=\gamma_1X_4+\gamma_2X_5+\gamma_3X_6+\gamma_4X_7,}\\
\multicolumn{3}{l}{[X_5,X_9]=-\gamma_2X_4+\gamma_1X_5-\gamma_4X_6+\gamma_2X_7,}\\
\multicolumn{3}{l}{[X_6,X_9]=-\gamma_3X_4+\gamma_4X_5+\gamma_1X_6-\gamma_2X_7,}\\
\multicolumn{3}{l}{[X_7,X_9]=-\gamma_4X_4-\gamma_3X_5+\gamma_2X_6+\gamma_1X_7.}
\end{array}\right\}, \quad (18)$$

where $(\gamma_1,\dots,\gamma_7)\in\mathbb{R}^7$. It follows at once that the Levi decomposition of these algebras is $\mathfrak{so}(3)\overrightarrow{\oplus}_{R_4\oplus 3D_0}\mathfrak{r}\,(\gamma_1,\dots,\gamma_7)$, where R_4 is the irreducible representation of $\mathfrak{so}(3)$ of degree four, D_0 is the trivial representation and $\mathfrak{r}\,(\gamma_1,\dots,\gamma_7)$ denotes the radical.

Proposition 8 *The Lie algebras $L(\gamma_1,...,\gamma_7)$ are deformations of the frobeniusian model Lie algebras $\mathfrak{g}(\alpha_1,...,\alpha_s,\beta_1,...,\beta_{n-s-1})$ if and only if*

$$\gamma_1\gamma_6-\gamma_5\gamma_7\neq 0 \tag{19}$$

holds.

Proof The proof follows at once observing that a closed form $\omega\in(L(\gamma_1,...,\gamma_7))^*$ of maximal rank can be reduced to a “canonical form”

$$\omega=(\gamma_6\omega_8\wedge\omega_{10}+\gamma_7\omega_9\wedge\omega_{10})+\sum_{i=1}^{4}\omega_i\,, \tag{20}$$

where

$$
\begin{aligned}
\omega_1 &= -\frac{1}{2}\omega_1 \wedge \omega_7 - \frac{1}{2}\omega_2 \wedge \omega_5 - \frac{1}{2}\omega_3 \wedge \omega_6 + \gamma_5\omega_4 \wedge \omega_8 \\
&\quad + \gamma_1\omega_4 \wedge \omega_9 - \gamma_2\omega_5 \wedge \omega_9 - \gamma_3\omega_6 \wedge \omega_9 - \gamma_4\omega_7 \wedge \omega_9 \,, \\
\omega_2 &= -\frac{1}{2}\omega_1 \wedge \omega_6 + \frac{1}{2}\omega_2 \wedge \omega_4 + \frac{1}{2}\omega_3 \wedge \omega_7 + \gamma_5\omega_5 \wedge \omega_8 \\
&\quad + \gamma_2\omega_4 \wedge \omega_9 + \gamma_1\omega_5 \wedge \omega_9 + \gamma_4\omega_6 \wedge \omega_9 - \gamma_3\omega_7 \wedge \omega_9 \,, \\
\omega_3 &= \frac{1}{2}\omega_1 \wedge \omega_5 - \frac{1}{2}\omega_2 \wedge \omega_7 + \frac{1}{2}\omega_3 \wedge \omega_4 + \gamma_5\omega_6 \wedge \omega_8 \\
&\quad + \gamma_3\omega_4 \wedge \omega_9 - \gamma_4\omega_5 \wedge \omega_9 + \gamma_1\omega_6 \wedge \omega_9 + \gamma_2\omega_7 \wedge \omega_9 \,, \\
\omega_4 &= \frac{1}{2}\omega_1 \wedge \omega_4 + \frac{1}{2}\omega_2 \wedge \omega_6 - \frac{1}{2}\omega_3 \wedge \omega_5 + +\gamma_5\omega_7 \wedge \omega_8 \\
&\quad + \gamma_4\omega_4 \wedge \omega_9 + \gamma_3\omega_5 \wedge \omega_9 - \gamma_2\omega_6 \wedge \omega_9 + \gamma_1\omega_7 \wedge \omega_9 \,.
\end{aligned}
$$

It follows at once that the form being of maximal rank, *i.e.*, satisfying

$$\bigwedge^5 \omega \neq 0\,, \tag{21}$$

is equivalent to impose

$$\gamma_5\gamma_7 - \gamma_6\gamma_1 \neq 0\,. \tag{22}$$

■

Therefore the Lie algebras $L(\gamma_1, \ldots, \gamma_7)$ constitute a parametrized family of non-semisimple Lie algebras with Levi subalgebra isomorphic to $\mathfrak{so}(3)$ and having no non-trivial invariants for the coadjoint representation. We may remark that this is the first example of non-solvable algebras having this property that has appeared in the literature.

Concluding remarks

Theorem 1 provides a necessary condition for a Lie algebra $\mathfrak{g}$ to be a contraction of a Lie algebra $\mathfrak{g}_0$ in terms of the number of invariants of the coadjoint orbit. This result was anticipated for the contractions of the simple

Lie algebra $\mathfrak{sp}\,(3, \mathbb{C})$ in [9], and has been proved formally here for arbitrary Lie algebras.

The practical utility of Theorem 1 is its application to the study of (solvable) Lie algebras having only trivial invariants, by the study of its contractions. Since contractions are also transitive [7, 18], the result can be used to establish lower and upper bounds for the number of invariants of Lie algebras. This could be of interest for those classes of algebras for which a direct determination of the number of invariants is a very difficult problem, due to computational limitations [21]. For certain classes of solvable Lie algebras, whose nilradical generalizes in some sense the nilradicals of Borel subalgebras of simple Lie algebras [29] or metasolvable Lie algebras [30] , the method should also provide useful conclusions. In particular, the use of the formula has provided us with a ten dimensional family of Lie algebras with nontrivial Levi decomposition which are frobeniusian, showing that these algebras constitute an interesting class of algebras for constructing Lie algebras with only trivial invariants for ad^*.

We have also seen that the contraction formula can be used to prove some questions about Lie algebras having less derivations than its dimension, and to prove that the existence of a fundamental set of invariants formed by rational functions is a necessary condition for a Lie algebra to be complete. In particular, the latter algebras seem to form an adequate class to obtain Lie algebras without any non-trivial invariants, as follows from Table II. Their contractions are also of interest, since contractions of solvable Lie algebras are also solvable.

For particular groups, such as the conformal group of space time, where (real) Lie algebras in high dimension appear as subalgebras, the contraction method could be of interest to determine the maximal possible number of invariants.

On the other hand, for certain additional properties, mainly arising from differential geometry, we can always classify the Lie algebras satisfying the property up to contraction, usually obtaining a parametrized family $\mathfrak{F}$. In order to obtain an upper bound for the number of functionally independent invariants for the algebras satisfying the property, it suffices to determine $\mathcal{N}(\mathfrak{F})$ for the elements of the family. Concerning the properties analyzed here, linear contact form and frobeniusian Lie algebras, both model families are of importance for physical applications, since they are deeply related with the Heisenberg Lie algebra. Specifically, one could ask whether any frobeniusian Lie algebra is a subalgebra of $\mathfrak{sp}(2n,\mathbb{K})\oplus\mathfrak{h}_n$, and whether other properties, such as the existence of a symplectic form on the nilradical, can also be solved by this means.

REFERENCES

[1] L. Abellanas, L.M. Alonso, *J. Math. Phys.* **16**, 1580 (1975).

[2] R. Campoamor-Stursberg, *J. Phys. A: Math. Gen.* **35**, 6293 (2002).

[3] R. Campoamor-Stursberg, *J. Math. Phys.* **44**, 771 (2003).

[4] P. Olver, *Applications of Lie Groups to Differential Equations*, New York, Springer 1986.

[5] R. Campoamor-Stursberg, F. Glez Gascon, D. Peralta, *J. Math. Phys.* **42**, 5741 (2001).

[6] I.E. Segal, *Duke Math. J.* **18**, 221 (1951).

[7] E. Inönü, E.P. Wigner, *Proc. Natl. Acad. Sci. U. S.* **39**, 510 (1953).

[8] E.J. Saletan, *J. Math. Phys.* **2**, 1 (1961).

[9] Ait Abdelmalek, X. Leng, J. Patera, P. Winternitz, *J. Phys. A: Math. Gen.* **29**, 7519 (1996).

[10] J. Ancochea Bermudez, M. Goze, *Proc. Amer. Math. Soc.* **127**, 2611 (1999).

[11] J.P. Ndogmo, *J. Phys. A: Math. Gen.* **33**, 2273 (2000).

[12] E.G. Beltrametti, A. Blasi, *Phys. Lett.* **20**, 62 (1966).

[13] D.J. Meng, L. Zhu, *Commun. Algebra* **24**, 4181 (1996).

[14] A. Lichnerowicz, *J. Math. Pures. Appl.* **62**, 269 (1983).

[15] M. Goze, *C. R. A. S. Paris Sér. I* **293**, 425 (1981).

[16] M. Goze, *C. R. A. S. Paris Sér. I* **292**, 813 (1981).

[17] A. Fialowski, J.A. O'Halloran, *Commun. Algebra* **18**, 4121 (1990).

[18] F. Grunewald, O.A. Halloran, *J. Algebra* **116**, 163 (1988).

[19] A.T. Fomenko, V.V. Trofimov, *Integrable Systems on Lie Algebras and Symmetric Spaces*, Gordon and Breach, New York 1988.

[20] V.V. Trofimov, *Izv. Akad. Nauk. SSSR* **43**, 714 (1979).

[21] S. Tremblay, P. Winternitz, *J. Phys. A: Math. Gen.* **31**, 789 (1998).

[22] T. Sato, *Tohoku Math. J.* **23**, 21 (1971).

[23] R. Carles, *Ann. Inst. Fourier* **34**, 65 (1984).

[24] J. Voisin, *J. Math. Phys.* **6**, 1822 (1965).

[25] G. Rosensteel, D.J. Rowe, *Ann. Phys.* **126**, 343 (1980).

[26] J. Ancochea Bermudez, R. Campoamor-Stursberg, *Forum Math.*, to appear.

[27] P. Turkowski, *Linear Alg. Appl.* **171**, 197 (1992).

[28] P. Turkowski, *Phys. Lett.* **B207**, 36 (1988).

[29] J.M. Ancochea Bermudez, R. Campoamor-Stursberg, *J. Pure Appl. Algebra* **170**, 1 (2002).

[30] J.M. Ancochea Bermudez, R. Campoamor-Stursberg, *Transf. Groups* **7**, 307 (2002).