ON FRIEDMANN EQUATION AND THE RADION STABILIZATION IN TWO-BRANE MODELS WITH DYNAMICAL MATTER

Kamila Kowalska and Krzysztof Turzyński

Institute of Theoretical Physics, Warsaw University
Hoża 69, 00-681 Warszawa, Poland
e-mail: Kamila.Kowalska@fuw.edu.pl,
Krzysztof-Jan.Turzynski@fuw.edu.pl

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The consequences of a gravitational stabilization of a two-brane system are studied in the presence of matter and tensions of both branes as well as the cosmological constant in the bulk. An explicit calculation shows that the usual form of the Friedmann equation can be retained in this situation, even though the model is not a realistic one.

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1. Introduction

The string-motivated idea that our Universe is confined to a submanifold of a higher-dimensional space (*a brane*) constitutes a potential solution to the hierarchy problem [1–3]. In such models the enormous hiatus between the electroweak scale and the Planck scale, which results in large radiative corrections to the Higgs scalar mass, can be less troublesome, since new physics is expected at the TeV scale.

On the other hand, addition of the spatial dimensions usually strongly affects cosmology (see *e.g.* [4–7]). In particular, in the case of models with branes, Friedmann equation, describing the evolution of the size of the Universe filled with matter and vacuum energy density, exhibits unusual, quadratic dependence on the energy density, when derived from a higher-dimension formalism.

In the context of the brane scenarios another question arises, whether the separation of the branes is made constant by some underlying, yet unknown physics (*stabilization mechanism*) or slowly varying. It is convenient to investigate this issue in terms of an additional field, a so-called radion, which is just the (55) component of the five-dimensional metric.

There has been a debate in the literature, whether the unusual form of Friedmann equation in five-dimensional cosmologies results from the fact that the extra dimension is implicitly stabilized [9] (which is supported by our calculations) or that the five-dimensional space-time is assumed to be conformally flat [10].

In this work we examine along the lines of Refs. [5–7] the possibility of purely gravitational stabilization of a two-brane system with time-dependent energy density of a one-component perfect fluid on each brane. This is a modest generalization of the models presented in the abovementioned references (in which the case without the fluid was thoroughly discussed), which goes beyond the approximate results of Refs. [9,10]. Our plan is the following. In Sec. 2 we recall Einstein equations in five dimensions and their solutions. Sec. 3 is devoted to studying possible matter content of the branes which allows an equilibrium position for the branes. Sec. 4 contains discussion of the results and the conclusions are presented in Sec. 5.

2. The setup

In this section we recall Einstein equations in five dimensions, as well as their solutions. These results have been more thoroughly discussed in [7] and [4] and we present them for clarity of the following considerations.

We shall consider five-dimensional, $\boldsymbol{Z}_2$-symmetric models with a cosmological constant Λ in the bulk. We shall only consider the anti-de-Sitter bulk ($\Lambda > 0$ in our notation), since in this case the hierarchy problem can potentially be solved. There are also two parallel 3-branes: the *reference* brane, with tension Λ_1 and the *moving* brane, with tension λ_2. These names result from the fact that it is convenient to choose the reference frame in which the former brane is at rest and the latter brane moves in the direction perpendicular to the branes' space directions. In addition, both branes contain a perfect fluid of energy density ρ_1 and ρ_2, respectively. We assume that the branes are maximally symmetric four-dimensional manifolds. This is equivalent to the following ansatz on the five-dimensional metric

$$ds^2 = -n^2(t,y)\,dt^2 + a^2(t,y)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) + dy^2\,, \qquad (2.1)$$

with the reference brane and the moving brane located at $y=0$ and $y=\mathcal{R}(t)$, respectively. This is a straightforward generalization of Friedmann–Robertson–Walker (FRW) metric.

2.1. Einstein equations

Henceforth, we shall use the convention $\dot{x} = \frac{\partial x}{\partial t}$ and $x' = \frac{\partial x}{\partial y}$. Having written the Einstein equations in the bulk

$$R_{AB} = T_{AB} - \frac{1}{3} T g_{AB} + \Lambda g_{AB} \,, \tag{2.2}$$

we obtain the only nontrivial and independent equations for $(\mu, \nu) = (0,0)$, (5,5), (0,5), (r,r)

$$\frac{2}{3} n^2 \Lambda = -3\frac{a''}{a} + nn'' + 3\frac{\dot{n}\dot{a}}{na} + 3nn'\frac{a'}{a} \,, \tag{2.3}$$

$$-\frac{2}{3}\Lambda = -\frac{n''}{n} - 3\frac{a''}{a} \,, \tag{2.4}$$

$$0 = \frac{\dot{a}'}{a} - \frac{n'}{n}\frac{\dot{a}}{a} \,, \tag{2.5}$$

$$-\frac{2}{3} a^2 \Lambda = 2\frac{\dot{a}^2}{n^2} + \frac{a'' a}{n^2} - \frac{a\dot{a}\dot{n}}{n^3} - 2a'^2 - aa'' - \frac{aa'n'}{n} - 2k \,. \tag{2.6}$$

Conservation of energy in the bulk (Bianchi identities) is implied by equations (2.3)–(2.6). There exists a first integral of the Einstein equations

$$\left(\frac{a'}{a}\right)^2 - \left(\frac{\dot{a}}{an}\right)^2 - \frac{\Lambda}{6} - \frac{k}{a^2} + \frac{C}{a^4} = 0 \,, \tag{2.7}$$

where C is a constant of integration. Using (2.5), we obtain

$$a'' = \frac{\Lambda}{6} a + \frac{C}{a^3} \,. \tag{2.8}$$

The solutions of (2.8) are

- $C = 0$:

$$a(t,y) = A(t) \operatorname{ch} \mu y + B(t) \operatorname{sh} \mu y \,, \tag{2.9}$$

where $\mu = \sqrt{\frac{\Lambda}{6}}$,

- $C \neq 0$:

$$a(t,y) = \sqrt{A(t) \operatorname{ch} \nu y + B(t) \operatorname{sh} \nu y + D(t)} \,, \tag{2.10}$$

where $\frac{\nu}{2} = \sqrt{\frac{\Lambda}{6}}$, $C = \frac{\Lambda}{6}\left(A^2(t) - B^2(t) - D^2(t)\right)$.

2.2. Israel conditions — reference brane

Since $\boldsymbol{Z}_2$ symmetry is assumed, the solutions of the Einstein equation in the bulk must also obey the following matching conditions.

$$\lim_{y\to 0^+} \frac{a'}{a} = -\frac{1}{6}\left(\lambda_1+\rho_1\right) , \tag{2.11}$$

$$\lim_{y\to 0^+} \frac{n'}{n} = \frac{1}{6}\left(-\lambda_1+2\rho_1+3p_1\right) , \tag{2.12}$$

where p_i denotes the pressure of the perfect fluid on i-th brane. When the conservation of energy is assumed on the reference brane, Eqs. (2.11) and (2.12) are equivalent.

Israel conditions on the reference brane specify half of the initial conditions for the equation (2.8). Substituting (2.11) into (2.9), we obtain

$$a(t,y) = a_0(t)\left(\operatorname{ch}\mu y - \frac{\lambda_1+\rho_1}{6\mu}\operatorname{sh}\mu y\right) , \tag{2.13}$$

where $a_0(t)$ is the scale factor of the FRW metric on the reference brane. Similarly, substituting (2.11) into (2.10), we obtain the following conditions

$$\sqrt{A+D} = a_0 , \qquad \frac{\nu}{2}\frac{B}{A+D} = -\frac{1}{6}\left(\lambda_1+\rho_1\right) . \tag{2.14}$$

2.3. Friedmann equation

Substituting (2.11) into (2.7) at $y=0$ and choosing the cosmic time to be the proper time on the reference brane (*i.e.* $n(t,y=0)=1$), we obtain

$$H_0^2 = \left(\frac{\dot a_0}{a_0}\right)^2 = -\frac{\Lambda}{6} + \frac{(\lambda_1+\rho_1)^2}{36} - \frac{k}{a_0^2} + \frac{C}{a_0^4} , \tag{2.15}$$

where the last term is sometimes referred to as *dark radiation* [5]. Note the intriguing quadratic dependence of the square of the Hubble constant on the energy density on the reference brane. It is also striking that Friedmann equation (2.15) does not depend explicitly on the matter content nor velocity of the moving brane. There has been a dispute in the literature whether these features are artifacts of the lack of an explicit mechanism for the stabilization of the extra dimension (see *e.g.* [8–10]).

2.4. Israel conditions — moving brane

We shall denote $\xi = \dot{\mathcal{R}}/n$, *i.e.* ξ is the velocity of the moving brane with respect to the reference brane, but this velocity is expressed in terms of the proper time on the moving brane. The matching conditions read

$$\frac{1}{n}\dot{\xi} + \frac{n'}{n}\left(1-2\xi^2\right) = \frac{1}{6}\left(-\lambda_2 + 2\rho_2 + 3p_2\right)\left(1-\xi^2\right)^{3/2}, \qquad (2.16)$$

$$\frac{a'}{a} + \frac{\dot{a}}{an}\xi = \frac{\lambda_2+\rho_2}{6}\sqrt{1-\xi^2}\,. \qquad (2.17)$$

Similarly as before, these equations are related to the conservation of energy on the moving brane.

3. Equilibrium of the radion

In this section, we shall investigate the consequences of the two branes being at rest with respect to each other. We denote $\tau = \mathrm{th}\,\frac{\nu\bar{\mathcal{R}}}{2}$.

Substituting the matching conditions (2.11) into the solution (2.10) of Einstein equations, we obtain

$$a(t,y) = a_0(t)\sqrt{\frac{\frac{C}{a_0^4(t)}\,\mathrm{th}^2\frac{\nu y}{2} + \left(-\frac{\nu}{2} + \left(\frac{\lambda_1}{6} + \frac{\rho_1(t)}{6}\right)\mathrm{th}\frac{\nu y}{2}\right)^2}{\frac{\nu^2}{4}\left(1-\mathrm{th}^2\frac{\nu y}{2}\right)}}\,. \qquad (3.1)$$

The evolution of $a_0(t)$ is given by the Friedmann equation (2.15). Energy conservation and the equation of state $p_1 = w_1\rho_1$ for the matter on the reference brane yield

$$\begin{aligned}\dot{\rho}_1 &= -3H_0(1+w_1)\rho_1\,,\\ \dot{a}_0 &= H_0 a_0\,.\end{aligned} \qquad (3.2)$$

The Eqs. (3.2) result in a power dependence

$$a_0^{-3(1+w_1)} \propto \rho_1\,. \qquad (3.3)$$

When the "moving" brane does not move, $\dot{\mathcal{R}} = 0$, the Eq. (2.17) becomes

$$\frac{\nu}{2}\,\frac{\frac{C}{a_0^4}\tau + \frac{\nu^2}{4}\tau - \left(1+\tau^2\right)\frac{\nu}{2}\left(\frac{\lambda_1}{6}+\frac{\rho_1}{6}\right) + \tau\frac{(\lambda_1+\rho_1)^2}{36}}{\frac{C}{a_0^4}\tau^2 + \left(-\frac{\nu}{2} + \left(\frac{\lambda_1}{6}+\frac{\rho_1}{6}\right)\tau\right)^2} = \frac{\lambda_2}{6} + \frac{\rho_2}{6}\,. \qquad (3.4)$$

It is convenient to keep only $\frac{\nu}{2}$ as a dimensionful parameter and rewrite (3.4) in terms of dimensionless quantities $\frac{\nu^2}{4}\frac{\tilde{C}}{a_0^4}=\frac{C}{a_0^4}$, $\frac{\nu}{2}\sigma_i=\frac{\lambda_i}{6}$, $\frac{\nu}{2}\eta_i=\frac{\rho_i}{6}$ for $i=1,2$

$$\frac{\frac{\tilde{C}}{a_0^4}\tau+\tau-(1+\tau^2)\left(\sigma_1+\eta_1\right)+\tau\left(\sigma_1+\eta_1\right)^2}{\frac{\tilde{C}}{a_0^4}\tau^2+\left(-1+\left(\sigma_1+\eta_1\right)\tau\right)^2}=\sigma_2+\eta_2\,. \tag{3.5}$$

The Eq. (3.5) imposes an algebraic relation between two dynamical quantities a_0 and η_2 whose time evolutions are a priori independent[1]. Even if the condition (3.5) is imposed at certain time t_0, it does not have to hold at later times. In order to check that (3.5) is preserved in the time evolution, one must also check the equality of the first time derivatives of both sides of this equation.

Energy conservation on the moving brane reads

$$\dot{\rho}_2+3\frac{\dot{a}_0(t,\bar{\mathcal{R}})}{a_0(t,\bar{\mathcal{R}})}(\rho_2+p_2)=\dot{\rho}_2+3\mathcal{H}_2(1+w_2)\rho_2=0\,, \tag{3.6}$$

where the Hubble parameter $\mathcal{H}_2$ on the moving brane as it is seen by the observer sitting on the reference brane reads

$$\mathcal{H}_2=H_0\frac{-\frac{\tilde{C}}{a_0^4}\tau^2+\left(1-\left(\sigma_1+\eta_1\right)\tau\right)\left(1+(2+3w_1)\eta_1\tau-\sigma_1\tau\right)}{\frac{\tilde{C}}{a_0^4}\tau^2+\left(-1+\left(\sigma_1+\eta_1\right)\tau\right)^2}\,. \tag{3.7}$$

The time derivative of the left hand side of (3.5) can be calculated using (3.2). The time derivative of the right hand side of (3.5) can be calculated using (3.6), (3.7) and (3.5). The resulting equation can be written in the following form

$$\kappa_2+\kappa_1 a_0^{-4}+\kappa_0 a_0^{-8}=0\,, \tag{3.8}$$

where

$$\begin{aligned}
\kappa_0 &= -\tilde{C}^2\tau^3(1+w_2)(-1+\sigma_2\tau)\,, &(3.9)\\
\kappa_1 &= \tilde{C}\tau\,\big(-\tau(1+w_2)(-1+\sigma_1\tau+\eta_1\tau)(-\sigma_2+\tau+(\sigma_1+\eta_1)(-1+\sigma_2\tau))\\
&\quad+\frac{1}{3}(-1+\sigma_2\tau)(1+w_2)(-1+\sigma_1\tau+\eta_1\tau)(-1+\sigma_1\tau-\tau(2+3w_1)\eta_1)\\
&\quad+\ (-1+\tau^2)(4(-1+\sigma_1\tau)+(1-3w_1)\eta_1\tau)\big)\,, &(3.10)\\
\kappa_2 &= (-1+\sigma_1\tau+\eta_1\tau)^2\Big((-1+\tau^2)(1+w_1)\eta_1+(1+w_2)\\
&\quad\times(-\sigma_2+\tau+(\sigma_1+\eta_1)(-1+\sigma_2\tau))(-1+\sigma_1\tau-\tau(2+3w_1)\eta_1\Big)\,. &(3.11)
\end{aligned}$$

[1] The relation between η_1 and a_0 is established by Eq. (3.2).

If we assume that $\eta_1 = 0$, *i.e.* there is no matter on the reference brane, then κ_i are constant coefficients and, according to Eq. (3.8), must all vanish. For $C \neq 0$ this can only be true for $\sigma_1 = \sigma_2 = \frac{1}{\tau}$. Then $\eta_2 = 0$. For $C = 0$ there is another solution with $\sigma_2 = \frac{\sigma_1 - \tau}{-1+\sigma_1\tau}$. Then $\eta_2 = 0$, again.

When $\eta_1 \neq 0$, the coefficients κ_1 and κ_2 are in general functions of η_1. However, there exists a solution, for which all the coefficients κ_i vanish

$$\sigma_1 = \sigma_2 = \frac{1}{\tau}\,, \qquad w_1 = w_2 = -\frac{1}{3}\,. \tag{3.12}$$

Let us first check if these parameters correspond to a non-singular metric. The scale factor (3.1) on the second brane reads then

$$a(t, \bar{\mathcal{R}}) = a_0(t)\sqrt{\frac{\tau^2}{1-\tau^2}\left(\frac{\tilde{C}}{a_0^4} + \eta_1^2\right)}\,. \tag{3.13}$$

On the other hand, the parameter $n(t, \bar{\mathcal{R}})$ which establishes the relation between the proper times on both branes reads

$$n(t, \bar{\mathcal{R}}) = -\sqrt{\frac{\tau^2}{1-\tau^2}\left(\frac{\tilde{C}}{a_0^4} + \eta_1^2\right)} = -\frac{a(t, \bar{\mathcal{R}})}{a_0(t)} < 0\,. \tag{3.14}$$

Since $n(t, 0) = 1$, this solution is singular.

When the coefficients κ_i do not vanish simultaneously (in particular, $\kappa_0 \neq 0$, since the case $C = 0$ yields $w_1 = w_2 = -1$, contrarily to the assumption), we require the solution of the quadratic equation (3.8)

$$a_0^{-4} = \frac{-\kappa \pm \sqrt{\kappa_1^2 - 4\kappa_0\kappa_2}}{2\kappa_0} \tag{3.15}$$

be consistent with the Eq. (3.3), *i.e.*

$$a_0^{-4} = \alpha\eta_1^2\,. \tag{3.16}$$

This yields the following condition

$$\kappa_1^2 - 4\kappa_0\kappa_2 = \left(2\kappa_0\alpha\rho_1^2 + \kappa_1\right)^2\,. \tag{3.17}$$

The coefficient α can be calculated by comparing the coefficients of the powers of ρ_1 in (3.17). This gives $\alpha = -\frac{1}{36\tilde{C}}, -\frac{2+3w_1}{36\tilde{C}}$. Relation (3.17) between two fourth-rank polynomials in ρ_1 can be satisfied for ρ_1 changing in time if and only if the coefficients of these polynomials are equal. This is equivalent to

$$\sigma_2 = \frac{\sigma_1 - \tau}{-1 + \sigma_1 \tau}, \tag{3.18}$$

$$w_1 = -\frac{1}{3}. \tag{3.19}$$

Since the choice of the reference brane is arbitrary, there must also be $w_2 = -\frac{1}{3}$. Note that the Eq. (3.18) is equivalent to the condition obtained if there are only cosmological constants present [7]. Equation (3.16) becomes then

$$\frac{\tilde{C}}{a_0^4} = -\eta_1^2. \tag{3.20}$$

4. Discussion

Some physical situations, in which our solution might look viable, have been discussed in the literature (see *e.g.* [11]). Although a perfect fluid with $w = -\frac{1}{3}$ cannot be matter nor radiation, it might correspond *e.g.* to a gas of strings.

Substituting (3.20) to (3.1), we obtain

$$a(t,y) = a_0(t)\sqrt{\frac{(-1+\sigma_1\tau_y)(-1+\sigma_1\tau_y+2\eta_1\tau_y)}{1-\tau_y^2}}, \tag{4.1}$$

where $\tau(y) = \mathrm{th}\frac{\nu y}{2} \in [0, \tau]$. The expression under the square root is non-zero for all y, if and only if

$$1 > \sigma_1 \tau(y) \qquad \text{and} \qquad 1 > \sigma_1\tau(y) + 2\eta_1\tau(y). \tag{4.2}$$

Since $\tau(y) < \tau$, it suffices that $\sigma_1 < 1/\tau$ to satisfy the first condition. In order to satisfy the second condition, the initial value of matter density must be appropriately chosen, namely $2\eta_1(t_0) < 1/\tau - \sigma_1$. As the Universe expands, the matter density decreases and (4.2) is then automatically satisfied for $t > t_0$. Moreover,

$$n(t,y) = \frac{(1-\sigma_1\tau(y))^2}{1-\tau(y)^2}\frac{a_0(t)}{a(t,y(\tau))} > 0, \tag{4.3}$$

i.e. this solution is non-singular. Friedmann equation on the first brane reads

$$H_0^2 = \frac{\nu^2}{4}\left(-1+\sigma_1^2\right) + \nu\sigma_1\rho_1 - \frac{k}{a_0^2}, \tag{4.4}$$

i.e. the C-term and the term quadratic in energy density cancel out and the standard form of the Friedmann equation with effective cosmological constant

$$\lambda_1^{\mathrm{eff}} = \frac{\nu^2}{4}\left(-1+\sigma_1^2\right) \tag{4.5}$$

is recovered. In the limit $\sigma_1 \to -1$ our exact result (4.4) agrees with that obtained in [9] for the Randall–Sundrum scenario with small addition of matter.

Our result might seem contradictory to the approximate result obtained for $\nu, \sigma_1, \sigma_2 \to 0$ and $k = 0$ in Ref. [10]

$$H_0^2 = -H_0 \frac{\dot{\mathcal{R}}}{\mathcal{R}} + \frac{\rho_1+\rho_2}{6\mathcal{R}} + \mathcal{O}(\rho_{1,2}^2)\,, \tag{4.6}$$

whereas our method gives for $\dot{\mathcal{R}} = 0$

$$H_0^2 = 0 + \mathcal{O}(\rho_{1,2}^2)\,. \tag{4.7}$$

However, when the two branes are at rest, ρ_1 and ρ_2 are not independent quantities and the relation (2.17) implies that the second term in (4.6) vanishes up to corrections $\mathcal{O}(\rho_{1,2}^2)$. This confirms the hypothesis expressed in [9] that it is the stabilization condition, and not vanishing of the bulk Weyl tensor, as stated in [10], what results in the non-standard form of Friedmann equation in this case. Note that, according to (3.20), the C-term vanishes up to terms quadratic in energy density without assuming the bulk to be a conformally flat space.

A disadvantage of our solution is its instability against small perturbations around the equilibrium position of the second brane. This can be seen explicitly for $t \to \infty$, when $\rho_1, \rho_2 \to 0$ and there are effectively only the cosmological constants. As discussed in [7] such an equilibrium is unstable for $\Lambda > 0$. A more detailed calculation proves the lack of stability for $t < \infty$, too.

5. Conclusions

In this work, we attempted to check if a purely gravitational stabilization of a two-brane system is possible with dynamical matter on the branes. We have shown that, if the equilibrium condition is imposed on the exact solutions of five-dimensional Einstein equations, the square of the energy density is absent from the effective Friedmann equation. Our result support the hypothesis that the quadratic dependence is an artifact of a lack of a stabilization mechanism. Unfortunately, our solution is not stable so it cannot be considered as a realistic one.

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