QUANTIZED MASSIVE FIELDS AND SEMICLASSICAL ELECTRICALLY CHARGED BLACK HOLES

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Constructed within the framework of the Schwinger–DeWitt method, the renormalized stress-energy tensor of the quantized massive scalar, spinor and vector fields in a general spherically-symmetric and static spacetime is employed as a source term of the Einstein field equations. The semiclassical solutions describing the electrically charged black holes are obtained and their properties are studied. Special emphasis is put on the semiclassical extremal configurations: it is shown that the near-horizon geometry, when expanded into a whole manifold, is described by the Bertotti–Robinson line element.

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1. Introduction

One of the most important, but still unresolved issues of modern theoretical physics is the question of a final point of a black hole evaporation. Unfortunately, according to our present understanding, the definite answer to this extremely complicated problem may be obtained only within the full machinery of the (nonexisting as yet) quantum theory of gravity. It is natural, therefore, that as a preliminary step in our way to build up a complete picture of the black hole evolution some simpler models should be considered. It is expected that as long as a black hole mass, M, is greater than the Planck mass, $M_{\rm Pl}$, the semiclassical approach may be safely used and the influence of the quantized fields on the spacetime geometry could be effectively studied. Unfortunately, even this simplified program is hard to execute as the semiclassical approach requires knowledge of the stress-energy tensor of the quantized fields, both massive and massless, for a wide class

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of nonstatic backgrounds, and, moreover, the resulting equations comprise rather complicated system of partial nonlinear differential equations.

For a static black hole and the stress-energy tensor of the quantized fields in the Hartle–Hawking state one has considerable simplifications and the problem becomes tractable. Moreover, as the quantum part of the total stress-energy tensor is expected to be of order $\mathcal{O}(\hbar)$, the back reaction equations could be solved perturbatively with the small parameter taken to be $\epsilon = (M_{\rm Pl}/M)^2$. Since the stress-energy tensor is $\mathcal{O}(\epsilon)$ one expects that the location of the true event horizon of the semicalssical black hole is close to its classical counterpart and so is the horizon defined mass.

It is evident that the success of the method critically depends on the knowledge of the stress-energy tensor itself. And although there exists vast literature devoted solely to constructions of the renormalized stress-energy tensor of quantized fields propagating in the black holes geometries, it is fair to say that at present we are unable to go beyond the linearized version of the semiclassical equations.

For a massless field in the spacetime of the Schwarzschild black hole this program, initiated by York [1], has been carried out in numerous papers investigating various aspects of the back reaction [2–7]. As the stress-energy tensor of the massless field approaches at large distances the form

$$T_a^b = p(h)\,\frac{\pi^2}{90}T^4 \mathrm{diag}\,[-3,\,1,\,1,\,1]\,, \tag{1.1}$$

where T is a black hole temperature and $p(h)$ is the number of helicity states, it is necessary to impose some sort of boundaries.

Typically, in the back reaction calculations one constructs the stress-energy tensor in the classical (unperturbed) spacetime of the black hole characterized by a (bare) mass and seeks for a corrected geometry. The integration constant resulting from integration of the (tt) component of the semiclassical Einstein field equations could be absorbed in the definition of the effective mass M in a process of the finite renormalization. Such a redefinition leads to $\mathcal{O}(\epsilon^2)$ effects in the stress-energy tensor, which are, of course, unimportant in the linearized semiclassical Einstein field equations. This is why it suffices in this approach to construct T_a^b in the classical background.

Recent calculations carried out in the background of the Reissner–Nordström black hole strongly suggest, however, that this approach, *i.e.* back reaction of the quantized fields evaluated in the background of the classical black hole should be abandoned in favor of a self-consistent analysis [8]. Indeed, having at one's disposal the functional dependence of $\langle T_a^b\rangle$ on the metric tensor, one may attempt to solve the system of the semiclassical field equations for a semiclassical line element. This, of course, would allow a more profound analyses of the resulting black hole solutions especially in the extremality limit.

It seems that obvious candidates for such an approach are massive fields that — as is well known — possess some attractive features that make construction of the field fluctuation and stress-energy tensor possible. Indeed, when the Compton length associated with the massive field is much smaller that a characteristic radius of curvature — a case usually referred to as a large mass limit, the particle creation phenomena could be neglected and the effective action, W, may be expanded in inverse powers of m^2. The expansion involves the well-known Hadamard–DeWitt coefficients which are local quantities and are constructed solely from the Riemann tensor, its covariant derivatives to required order and appropriate contractions. This feature opens a possibility to analyze the influence of the quantized fields upon geometry in a self-consistent manner.

This method has been successfully employed in the classical geometries of the Schwarzschild and Reissner–Nordström black holes as well as in the spacetime of the nonlinear black hole [8–14].

Recently the back reaction of the quantized massive scalar field with an arbitrary curvature coupling [15] and conformally invariant massless scalar field [16] on the Reissner–Nordström back hole of a (bare) mass M_0 and an electric charge e has been examined. The stress-energy tensor calculated in a *classical* spacetime of RN black hole has been employed as a source term of the Einstein equations. In this approach one starts with the classical Reissner–Nordströn solution and subsequently introduces quantum corrections.

Here we intend to extend the calculations to the massive spinor and vector fields and investigate the problem from a different perspective. The main objective of this paper is to solve the semiclassical field equations with the total stress-energy tensor describing the classical (electromagnetic) field and the quantized neutral massive field self-consistently and to examine the thus obtained black hole solution with special emphasis put on the extremal configurations. Specifically, we shall expand the discussion of the extremal electrically charged black hole given in Ref. [8].

In what follows we shall confine ourselves to the operators

$$(-\Box + \xi R + m^2)\phi^{(0)} = 0\,, \tag{1.2}$$

$$(\gamma^\mu \nabla_\mu + m)\phi^{(1/2)} = 0\,, \tag{1.3}$$

$$(\delta^\mu_\nu \Box - \nabla_\nu \nabla^\mu - R^\mu_\nu - \delta^\mu_\nu m^2)\phi^{(1)} = 0\,, \tag{1.4}$$

acting on the scalar, spinor, and vector fields, respectively. Here ξ is the curvature coupling constant, and γ^μ are the Dirac matrices obeying standard relations.

2. The renormalized stress-energy tensor

The first nonvanishing term of the renormalized effective action of the quantized massive field in a large mass limit constructed from the (traced) coincidence limit of the Hadamard–Minakshisundaram–DeWitt (HaMiDeW) coefficient $a_3\left(x,x'\right)$ is simply [9–11, 17]:

$$W^{(1)}_{\rm ren}=\frac{1}{32\pi^2m^2}\int g^{1/2}d^4x\left\{\begin{array}{l}[a_3^{(0)}]\\ -tr[a_3^{(1/2)}]\\ tr[a_3^{(1)}]-[a^{(0)}_{3|\xi=0}]\,.\end{array}\right. \tag{2.1}$$

The coefficients a_0, a_1 and a_2 contribute to the divergent part of the action and have to be absorbed into the (quadratic) gravitational action

$$S=\int d^4x\,g^{1/2}\left(\Lambda+R+\alpha R^2+\beta R_{ab}R^{ab}\right) \tag{2.2}$$

by renormalization of the bare coupling constants. The term containing the Kretschmann scalar, $R_{abcd}R^{abcd}$, has been eliminated through the use of the Gauss–Bonnett invariant in four dimensions.

Upon inserting the exact form of $[a_3]$ and performing elementary simplifications one obtains the approximate W_R of the quantized scalar, spinor, and vector field:

$$\begin{aligned}W_R &= \frac{1}{192\pi^2m^2}\int d^4xg^{1/2}\Big(\alpha_1^{(s)}R\Box R+\alpha_2^{(s)}R_{pq}\Box R^{pq}+\alpha_3^{(s)}R^3\\ &\quad-\alpha_4^{(s)}RR_{pq}R^{pq}+\alpha_5^{(s)}RR_{pqab}R^{pqab}-\alpha_6^{(s)}R^p_qR^q_aR^a_p\\ &\quad+\alpha_7^{(s)}R^{pq}R_{ab}R^{a\ b}_{\ p\ q}+\alpha_8^{(s)}\,R_{pq}R^p_{\ cab}R^{qcab}\\ &\quad+\alpha_9^{(s)}R_{ab}^{\ \ pq}R_{pq}^{\ \ cd}R_{cd}^{\ \ ab}\ -\alpha_{10}^{(s)}\,R^{a\ b}_{\ p\ q}R^{p\ q}_{\ c\ d}R^{c\ d}_{\ a\ b}\Big)\\ &= \frac{1}{192\pi^2m^2}\sum_{i=1}^{10}\alpha_i^{(s)}W_i\,,\end{aligned} \tag{2.3}$$

where the spin-dependent numerical coefficients $\alpha_i^{(s)}$ are tabulated in Table I.

Once the approximate effective action is known, the stress-energy tensor could be obtained by functional differentiation of W_R with respect to the metric tensor:

$$\langle T^{ab}\rangle\ =\ \frac{2}{g^{1/2}}\frac{\delta}{\delta g_{ab}}W_R. \tag{2.4}$$

It should be noted, however, that the thus constructed $\langle T^b_a\rangle$ is rather complicated as it consists of over 100 local geometric terms constructed from

TABLE I

The coefficients $\alpha_i^{(s)}$ for the massive scalar, spinor, and vector fields. Note that in order to obtain the result for the massive neutral spinor field one has to multiply the effective action by the factor 1/2.

	$s=0$	$s=1/2$	$s=1$
$\alpha_1^{(s)}$	$\frac{1}{2}\xi^2 - \frac{1}{5}\xi + \frac{1}{56}$	$-\frac{3}{140}$	$-\frac{27}{280}$
$\alpha_2^{(s)}$	$\frac{1}{140}$	$\frac{1}{14}$	$\frac{9}{28}$
$\alpha_3^{(s)}$	$\left(\frac{1}{6}-\xi\right)^3$	$\frac{1}{432}$	$-\frac{5}{72}$
$\alpha_4^{(s)}$	$-\frac{1}{30}\left(\frac{1}{6}-\xi\right)$	$-\frac{1}{90}$	$\frac{31}{60}$
$\alpha_5^{(s)}$	$\frac{1}{30}\left(\frac{1}{6}-\xi\right)$	$-\frac{7}{720}$	$-\frac{1}{10}$
$\alpha_6^{(s)}$	$-\frac{8}{945}$	$-\frac{25}{378}$	$-\frac{52}{63}$
$\alpha_7^{(s)}$	$\frac{2}{315}$	$\frac{47}{630}$	$-\frac{19}{105}$
$\alpha_8^{(s)}$	$\frac{1}{1260}$	$\frac{19}{630}$	$\frac{61}{140}$
$\alpha_9^{(s)}$	$\frac{17}{7560}$	$\frac{29}{3780}$	$-\frac{67}{2520}$
$\alpha_{10}^{(s)}$	$-\frac{1}{270}$	$-\frac{1}{54}$	$\frac{1}{18}$

the curvature tensor, its covariant derivatives and appropriate contractions. This result, for obvious reasons, will not be presented here, and for its exact form as well as the technical details the reader is referred to [12, 13]. Fortunately, despite its complexity there is a wide class of geometries of physical interest in which the result could be successfully applied.

Inspection of Eq. (2.3) reveals some general features of the thus obtained stress-energy tensor. First, it should be noted that $\langle T_a^b\rangle$ naturally divides into 10 purely geometric terms constructed from W_i

$$T^{(i)ab} = \frac{\delta}{\delta g_{ab}} W_i \tag{2.5}$$

that are identical for scalar, spinor and vector fields. The spin of the field is encoded in the numerical coefficients $\alpha_i^{(s)}$. Moreover, one expects that each $T^{(i)ab}$ is covariantly conserved and is regular for regular geometries.

3. Semiclassical Einstein field equations

In this section we shall apply the general formalism to a particular physical situation of the semiclassical electrically charged and spherically symmetric static black hole. We shall assume that the source term of the Einstein field equations consists of both the classical and the quantum part, *i.e.* T_a^b is the sum of the stress-energy tensor of the (classical) electromagnetic field, $T_a^{(\mathrm{em})b}$, and the stress-energy tensor of the massive quantized scalar, spinor or vector field, $\langle T_a^b \rangle$.

Without loss of generality the static and spherically symmetric line element may be written in the form that is useful in the calculations of this type:

$$ds^2 = -\mathrm{e}^{2\psi(r)} f\left(r\right) dt^2 + \frac{dr^2}{f\left(r\right)} + r^2 d\Omega^2 \,, \tag{3.1}$$

where

$$f\left(r\right) = 1 - \frac{2M\left(r\right)}{r} \,. \tag{3.2}$$

We intend to solve the semiclassical (quadratic) Einstein–Maxwell equations

$$R_a^b[g] - \frac{1}{2}R[g]\delta_a^b + \alpha I_a^b[g] + \beta H_a^b[g] + \Lambda\delta_a^b = 8\pi\left(T_a^{(\mathrm{em})b}\left[g\right] + \langle T_a^b\left[g\right]\rangle\right) \,, \tag{3.3}$$

where

$$\begin{aligned} I^{ab} &= \frac{1}{g^{1/2}}\frac{\delta}{\delta g_{ab}}\int d^4x\, g^{1/2}R^2 \\ &= 2R^{;\,ab} - 2RR^{ab} + \tfrac{1}{2}g^{ab}\left(R^2 - 4\Box R\right) \,, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} H^{ab} &= \frac{1}{g^{1/2}}\frac{\delta}{\delta g_{ab}}\int d^4x\, g^{1/2}R_{ab}R^{ab} \\ &= R^{;\,ab} - \Box R^{ab} - 2R_{cd}R^{cbda} + \tfrac{1}{2}g^{ab}\left(R_{cd}R^{cd} - \Box R\right) \,. \end{aligned} \tag{3.5}$$

The electromagnetic part of the total stress-energy tensor for the line element (3.1) is simply

$$T_t^{(\mathrm{em})t} = T_r^{(\mathrm{em})r} = -T_\theta^{(\mathrm{em})\theta} = -T_\phi^{(\mathrm{em})\phi} = -\frac{C_1^2}{8\pi r^4}, \tag{3.6}$$

that is independent of the functions $M\left(r\right)$ and $\psi\left(r\right)$. The integration constant C_1 is interpreted as an electric charge e. On the other hand, the exact form of the quantum part is generally unknown and, therefore, one is forced to refer to some approximations. In this paper we shall employ the Schwinger–DeWitt method, which can be used as long as the Compton length $\lambda_c = m^{-1}$ of the massive field is much less than the characteristic scale of a curvature $\mathbb{L}$. The parameters α, β, and Λ should be determined experimentally and their present values are unknown. It is expected, however, that they are small, otherwise they would lead to various observational effects. In the latter we shall assume that renormalized α, β and Λ vanish.

Employing the stress-energy tensor of the quantized field that functionally depends on the metric tensor we can slightly modify the back reaction program. Indeed, instead of starting from the classical geometry and subsequently constructing the quantum corrections to the metric we can try to solve the semiclassical equations self-consistently. This approach should lead to a physical interpretation of the integration constants.

Because of the Bianchi identities, the semiclassical field equations for the line element (3.1)

$$R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi T_{ab}\,, \tag{3.7}$$

where the total stress-energy tensor is the sum of

$$T_a^b = T_a^{(\mathrm{em})b} + \varepsilon \langle T_a^b \rangle\,, \tag{3.8}$$

reduce to the system

$$G_t^t = -\frac{2}{r^2}\frac{dM\,(r)}{dr} = 8\pi T_t^t \tag{3.9}$$

and

$$G_r^r = G_t^t + \frac{2}{r}\left(1 - \frac{2M\,(r)}{r}\right)\frac{d\psi\,(r)}{dr} \;=\; 8\pi T_r^r\,. \tag{3.10}$$

Here we have introduced the auxiliary parameter ε (not to be confused with the parameter ϵ) that is to be set to 1 at the final stage of calculations.

4. Semiclassical geometry

Let us observe that because of the special form of the stress-energy tensor of the electromagnetic field the difference between time and radial components is zero, and, consequently,

$$T_t^t - T_r^r \sim \mathcal{O}(\varepsilon)\,. \tag{4.1}$$

Now the equation (3.9) could be solved iteratively with the initial condition

$$M(r_+) \;=\; \frac{r_+}{2}\,. \tag{4.2}$$

Indeed, assuming the following expansions

$$M(r) = M_{(0)} + \varepsilon M_{(1)} + \mathcal{O}(\varepsilon^2) \tag{4.3}$$

and

$$\psi(r) = \psi_{(0)} + \varepsilon\psi_{(1)} + \mathcal{O}(\varepsilon^2)\,, \tag{4.4}$$

and collecting the terms with the like powers of ε one obtains

$$\frac{dM_{(0)}}{dr} = \frac{e^2}{2r^2} \tag{4.5}$$

and

$$-\frac{1}{r^2}\frac{dM_{(1)}}{dr} = 4\pi\langle T_t^t(M_{(0)},\psi_{(0)})\rangle = 4\pi\langle T_t^{(0)t}\rangle\,. \tag{4.6}$$

Here the stress-energy tensor $\langle T_t^{(0)t}\rangle$ is constructed for $M(r) = M_{(0)}(r)$ and $\psi(r) = \psi_{(0)} = 0$. It should be noted that in this approach we do not ascribe any particular physical meaning to the function $M_{(0)}$.

Now, solving the Eq. (4.5) with the condition (4.2) and subsequently solving the Eq. (4.6) one has

$$M(r) = \frac{r_+}{2} + \frac{e^2}{2r_+} - \frac{e^2}{2r} - 4\pi\varepsilon\int\limits_{r_+}^{r} dr'(r')^2\langle T_t^{(0)t}\rangle + \mathcal{O}(\varepsilon^2)\,, \tag{4.7}$$

and, consequently, the function $f\,(r)$ assumes the following form

$$f\,(r) = 1 - \frac{r_+}{r} + \frac{e^2}{r^2} - \frac{e^2}{rr_+} + \frac{8\pi\varepsilon}{r}\int\limits_{r_+}^{r} dr'(r')^2\langle T_t^{(0)t}\rangle\,. \tag{4.8}$$

Moreover, from the foregoing analyses it is evident that r_+ may be interpreted as the location of the exact event horizon.

The function $M(r)$ could be written in an alternative form

$$M\,(r) = \frac{r_+}{2} + \tilde{m}\,(r) = m\,(r) - \frac{e^2}{2r}\,. \tag{4.9}$$

It should be noted that

$$\tilde{m}\,(r_+) = 0\,, \tag{4.10}$$

which means that at the event horizon there is no room for quantum effects, and

$$m\,(r) = \frac{r_+}{2} + \frac{e^2}{2r_+} - 4\pi\varepsilon\int\limits_{r_+}^{r} dr'(r')^2\langle T_t^{(0)t}\rangle\,. \tag{4.11}$$

Let us return to the second equation of the system (Eq. 3.10), that, after employing the particular form of the electromagnetic stress-energy tensor and simple rearrangements could be written as follows

$$\psi\left(r\right)=\varepsilon\psi_{1}=4\pi\varepsilon\int\limits_{\infty}^{r}\frac{r'\left(\langle T_{r}^{r}\rangle-\langle T_{t}^{t}\rangle\right)}{1-\frac{2M(r')}{r'}}\,dr'\,, \tag{4.12}$$

where the components of the renormalized stress-energy tensor functionally depend on the metric potential of the general spherically symmetric line element (3.1).

Now we are in a position to determine the semiclassical line element. Indeed, inserting the metric tensor (3.1) into the general expressions that describe the renormalized stress-energy tensor of the massive field in a large mass limit, and, subsequently, employing expansions (4.3) and (4.4) in the thus obtained formulae, collecting the terms with the like powers of ε, and, finally, retaining only the terms that are linear in the auxiliary parameter, after some algebra and massive simplifications, one obtains

$$f\left(r\right)=1-\frac{r_{+}}{r}-\frac{e^{2}}{rr_{+}}+\frac{e^{2}}{r^{2}}+\frac{\varepsilon}{\pi m^{2}}\left(A^{(s)}\left(r\right)+\xi B^{(s)}\left(r\right)\right)\,, \tag{4.13}$$

where

$$\begin{aligned}A^{(0)}(r) \;=\;& \frac{1153}{1960}\frac{e^{4}}{r^{8}}+\frac{5}{112}\frac{{r_{+}}^{2}}{r^{6}}+\frac{13}{280}\frac{e^{2}}{r^{6}}-\frac{1237}{30240}\frac{{r_{+}}^{3}}{r^{7}}-\frac{113}{30240}\frac{1}{r{r_{+}}^{3}}\\ &+\frac{2327}{11340}\frac{e^{6}}{r^{10}}-\frac{613}{1680}\frac{e^{4}r_{+}}{r^{9}}-\frac{613}{1680}\frac{e^{6}}{r^{9}r_{+}}-\frac{1237}{30240}\frac{e^{6}}{r^{7}{r_{+}}^{3}}\\ &+\frac{877}{70560}\frac{e^{2}}{r{r_{+}}^{5}}-\frac{1069}{70560}\frac{e^{4}}{r{r_{+}}^{7}}+\frac{4169}{635040}\frac{e^{6}}{r{r_{+}}^{9}}-\frac{2549}{10080}\frac{e^{2}r_{+}}{r^{7}}\\ &+\frac{5}{112}\frac{e^{4}}{{r_{+}}^{2}r^{6}}-\frac{2549}{10080}\frac{e^{4}}{r_{+}r^{7}}+\frac{1369}{7056}\frac{e^{2}{r_{+}}^{2}}{r^{8}}+\frac{1369}{7056}\frac{e^{6}}{r^{8}{r_{+}}^{2}}\,,\end{aligned} \tag{4.14}$$

$$\begin{aligned}A^{(1/2)}\left(r\right) \;=\;& \frac{3}{280}\frac{{r_{+}}^{2}}{r^{6}}-\frac{27}{140}\frac{e^{2}}{r^{6}}+\frac{3}{280}\frac{e^{4}}{{r_{+}}^{2}r^{6}}-\frac{149}{15120}\frac{{r_{+}}^{3}}{r^{7}}+\frac{1723}{5040}\frac{e^{2}r_{+}}{r^{7}}\\ &+\frac{1723}{5040}\frac{e^{4}}{r_{+}r^{7}}-\frac{149}{15120}\frac{e^{6}}{r^{7}{r_{+}}^{3}}-\frac{2729}{17640}\frac{e^{2}{r_{+}}^{2}}{r^{8}}-\frac{1073}{1764}\frac{e^{4}}{r^{8}}\\ &-\frac{2729}{17640}\frac{e^{6}}{r^{8}{r_{+}}^{2}}+\frac{2687}{10080}\frac{e^{4}r_{+}}{r^{9}}+\frac{2687}{10080}\frac{e^{6}}{r^{9}r_{+}}-\frac{1639}{15120}\frac{e^{6}}{r^{10}}\\ &+\frac{67}{11760}\frac{e^{2}}{r{r_{+}}^{5}}-\frac{13}{15120}\frac{1}{r{r_{+}}^{3}}-\frac{767}{70560}\frac{e^{4}}{r{r_{+}}^{7}}+\frac{451}{70560}\frac{e^{6}}{r{r_{+}}^{9}}\,,\end{aligned} \tag{4.15}$$

$$A^{(1)}(r) = \frac{47849}{10080}\frac{e^2 r_+}{r^7} + \frac{47849}{10080}\frac{e^4}{r_+ r^7} - \frac{577}{280}\frac{e^2}{r^6} - \frac{37}{560}\frac{{r_+}^2}{r^6} - \frac{37}{560}\frac{e^4}{{r_+}^2 r^6}$$
$$+\frac{611}{10080}\frac{{r_+}^3}{r^7} + \frac{611}{10080}\frac{e^6}{r^7 {r_+}^3} - \frac{10393}{3920}\frac{e^2 {r_+}^2}{r^8} - \frac{35449}{3528}\frac{e^4}{r^8}$$
$$-\frac{10393}{3920}\frac{e^6}{r^8 {r_+}^2} + \frac{26879}{5040}\frac{e^4 r_+}{r^9} + \frac{26879}{5040}\frac{e^6}{r^9 r_+} - \frac{31057}{11340}\frac{e^6}{r^{10}}$$
$$-\frac{493}{14112}\frac{e^2}{r {r_+}^5} + \frac{11}{2016}\frac{1}{r {r_+}^3} + \frac{2393}{70560}\frac{e^4}{r {r_+}^7} - \frac{2389}{635040}\frac{e^6}{r {r_+}^9}, \quad (4.16)$$

$$B^{(0)}(r) = \frac{11}{60}\frac{{r_+}^3}{r^7} - \frac{1}{5}\frac{{r_+}^2}{r^6} - \frac{2e^2}{5r^6} - \frac{91}{90}\frac{e^6}{r^{10}} + \frac{1}{60}\frac{1}{r {r_+}^3} - \frac{29}{9}\frac{e^4}{r^8} + \frac{89}{60}\frac{e^4}{r_+ r^7}$$
$$+\frac{89}{60}\frac{e^2 r_+}{r^7} + \frac{e^4}{18 r {r_+}^7} - \frac{31}{30}\frac{e^6}{r^8 {r_+}^2} - \frac{31}{30}\frac{e^2 {r_+}^2}{r^8} + \frac{11}{60}\frac{e^6}{r^7 {r_+}^3}$$
$$+\frac{113}{60}\frac{e^6}{r^9 r_+} - \frac{e^2}{20 r {r_+}^5} - \frac{e^6}{45 r {r_+}^9} + \frac{113}{60}\frac{e^4 r_+}{r^9} - \frac{e^4}{5 {r_+}^2 r^6}, \quad (4.17)$$

and

$$B^{(1/2)}(r) = B^{(1)}(r) = 0. \quad (4.18)$$

Inspection of Eq. (3.10) indicates that knowledge of the function $f(r)$ is sufficient to construct the function $\psi(r)$ to the required order. However, before attempting to solve this equation let us observe that the difference between the (rr) and (tt) components of the stress-energy tensor factorizes as

$$\langle T_r^{(0)r}\rangle - \langle T_t^{(0)t}\rangle = \left(1 - \frac{r_+}{r} + \frac{e^2}{r r_+} - \frac{e^2}{r^2}\right) F(r), \quad (4.19)$$

where $F(r)$ is a regular function, and, consequently, the integral is expected to be finite. Indeed, retaining the $\mathcal{O}(\varepsilon)$ terms in (4.12) one has

$$\psi(r) = \varepsilon\psi_{(1)} = 4\pi\varepsilon \int_\infty^r \frac{r'\left(\langle T_r^{(0)r}\rangle - \langle T_t^{(0)t}\rangle\right)}{1 - \frac{r_+}{r'} + \frac{e^2}{r' r_+} - \frac{e^2}{r'^2}}\, dr'$$
$$= 4\pi\varepsilon \int_\infty^r F(r') r' dr'. \quad (4.20)$$

After rather lengthy calculations one obtains remarkably simple results:

$$\begin{aligned}\psi^{(0)} &= \frac{\varepsilon}{\pi m^2}\left(-\frac{29}{1120}\frac{{r_+}^2}{r^6}-\frac{3}{80}\frac{e^2}{r^6}-\frac{29}{1120}\frac{e^4}{{r_+}^2r^6}\right.\\&\left.+\frac{46}{441}\frac{e^2r_+}{r^7}+\frac{46}{441}\frac{e^4}{r_+r^7}-\frac{229}{1680}\frac{e^4}{r^8}\right)\\&+\frac{\varepsilon\xi}{\pi m^2}\left(\frac{7}{60}\frac{{r_+}^2}{r^6}-\frac{8}{15}\frac{e^2r_+}{r^7}+\frac{7}{30}\frac{e^2}{r^6}\right.\\&\left.+-\frac{8}{15}\frac{e^4}{r_+r^7}\frac{13}{20}\frac{e^4}{r^8}+\frac{7}{60}\frac{e^4}{{r_+}^2r^6}\right),\end{aligned} \tag{4.21}$$

$$\begin{aligned}\psi^{(1/2)} &= \frac{\varepsilon}{\pi m^2}\left(-\frac{11}{1680}\frac{e^4}{{r_+}^2r^6}-\frac{13}{245}\frac{e^4}{r_+r^7}+\frac{37}{1120}\frac{e^4}{r^8}\right.\\&\left.+\frac{7}{120}\frac{e^2}{r^6}-\frac{13}{245}\frac{e^2r_+}{r^7}-\frac{11}{1680}\frac{{r_+}^2}{r^6}\right)\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\psi^{(1)} &= \frac{\varepsilon}{\pi m^2}\left(\frac{131}{3360}\frac{e^4}{{r_+}^2r^6}-\frac{2446}{2205}\frac{e^4}{r_+r^7}+\frac{2141}{1680}\frac{e^4}{r^8}\right.\\&\left.+\frac{173}{240}\frac{e^2}{r^6}-\frac{2446}{2205}\frac{e^2r_+}{r^7}+\frac{131}{3360}\frac{{r_+}^2}{r^6}\right).\end{aligned} \tag{4.23}$$

The form of the line element (3.1) with (4.13)–(4.18) and (4.21)–(4.23) for scalar, spinor and vector fields are the principal results of this paper. The scalar case has been discussed within the framework of the perturbative approach in Ref. [15].

5. Properties of the semiclassical black hole solutions

5.1. Hawking temperature

The Euclidean form of the line element (3.1) obtained from the Wick rotation ($t \to -it$) has no conical singularity as $r \to r_+$, provided the 'time' coordinate is periodic with a period β given by

$$\beta = 4\pi \lim_{r\to r_+} (g_{tt}g_{rr})^{1/2}\left(\frac{d}{dr}g_{tt}\right)^{-1}. \tag{5.1}$$

The surface gravity, κ, is then related to β by means of the standard formula

$$\beta = \frac{2\pi}{\kappa}, \tag{5.2}$$

whereas the Hawking temperature of the black hole, $T_{\rm H}$, is simply given by

$$T_{\rm H} = \frac{\kappa}{2\pi}\,. \tag{5.3}$$

Restricting the general formula (5.1) to the line element (3.1) one easily obtains

$$\kappa = \frac{1}{2}\,{\rm e}^{\psi(r_+)}\frac{df}{dr}_{|r=r_+}\,, \tag{5.4}$$

or, equivalently, making use of the Einstein equations

$$\kappa \cong \kappa_0\left[1+\psi\left(r_+\right)+\frac{4\pi\varepsilon r_+}{\kappa_0}\langle T_t^{(0)t}\rangle_{|r=r_+}\right]\,, \tag{5.5}$$

where

$$\kappa_0 = \frac{1}{2r_+}\left(1-\frac{e^2}{r_+^2}\right)\,. \tag{5.6}$$

The Hawking temperature expressed in terms of the integration constants for neutral massive scalar, spinor and vector field is

$$T_{\rm H}^{(0)} = T_0 \quad +\frac{\varepsilon}{4\pi m^2 r_+^5}\left[\xi\left(\frac{1}{60}-\frac{1}{12}\frac{e^2}{r_+^2}+\frac{11}{90}\frac{e^4}{r_+^4}+\frac{1}{15}\frac{e^6}{r_+^6}\right)\right. \\ \left. -\frac{37}{10080}+\frac{463}{23520}\frac{e^2}{r_+^2}-\frac{33}{1120}\frac{e^4}{r_+^4}+\frac{1229}{90560}\frac{e^6}{r_+^6}\right]\,, \tag{5.7}$$

$$T_{\rm H}^{(1/2)} = T_0 - \frac{\varepsilon}{4\pi m^2 r_+^5}\left(\frac{1}{1008}-\frac{277}{35280}\frac{e^2}{r_+^2}+\frac{113}{10080}\frac{e^4}{r_+^4}-\frac{55}{4704}\frac{e^6}{r_+^6}\right) \tag{5.8}$$

and

$$T_{\rm H}^{(1)} = T_0 + \frac{\varepsilon}{4\pi m^2 r_+^5}\left(\frac{19}{3360}-\frac{3281}{70560}\frac{e^2}{r_+^2}+\frac{25}{224}\frac{e^4}{r_+^4}-\frac{1801}{70560}\frac{e^6}{r_+^6}\right)\,, \tag{5.9}$$

where

$$T_0 = \frac{1}{4\pi r_+}\left(1-\frac{e^2}{r_+^2}\right)\,. \tag{5.10}$$

5.2. Extremal black hole

In order to discuss the semiclassical extremal black holes let us return to the equation (5.4). First, we shall explore the consequences of vanishing of the surface gravity (temperature). Since the surface gravity is defined at the event horizon we have a system of two equations, the first of which is satisfied automatically as $f(r_+) = 0$, whereas the second one

$$\frac{1}{r_+} - \frac{e^2}{r_+^3} + 8\pi\varepsilon r_+ \langle T_t^{(0)t} \rangle_{|r=r_+} = 0 \tag{5.11}$$

is to be used to relate the integration constants r_+ and e. Assuming that the location of event horizon could be expanded as

$$r_+ = r_0 + \varepsilon r_1 + \mathcal{O}(\varepsilon^2)\,, \tag{5.12}$$

where, as before, we do not ascribe any particular physical meaning to r_0 and r_1, one has

$$r_0 = |e| \tag{5.13}$$

and

$$r_1 = -\frac{\Delta r_0^4}{3e^2 - r_0^2}\,, \tag{5.14}$$

where

$$\Delta = 8\pi r_0 \langle T_t^{(0)t} \rangle\,. \tag{5.15}$$

The location of the event horizon expressed in terms of the electric charge is given by

$$r_+ = |e| - \frac{\varepsilon \mu^{(s)}}{720\pi m^2 |e|^3}\,, \tag{5.16}$$

where

$$\mu^{(s)} = \begin{cases} \frac{16}{21} - 4(\xi - \frac{1}{6}) \\ \frac{37}{14} \\ \frac{114}{7} \end{cases} \tag{5.17}$$

for scalar, vector and spinor fields, respectively. Note that the above result could be easily obtained setting $T_{\rm H}^{(s)} = 0$ in (5.7)–(5.9), making use of the expansion (5.12) and retaining $\mathcal{O}(\varepsilon)$ terms.

The problem of the existence of the quantum corrected extremal black holes has recently been a subject of some controversy. On the basis of the perturbative approach carried out in Ref. [18] it was stated that macroscopic zero temperature black holes do not exist, whereas Lowe [19] discussing the similar model has explicitly demonstrated that such configurations could exist. This issue has been further investigated from a more general point of view with the aid of the stress-energy tensor in the large mass limit in [8]. The principal objection to Lowe's demonstration raised in [20] consists in the observation that the corrected event horizon always lies inside the event horizon of the unperturbed (classical) Reissner–Nordström solution. However, in order to determine the quantum corrections to the geometry one has to know the stress-energy tensor inside the event horizon (that in the extremal case also becomes the Cauchy horizon). As there is no possible justification for extending validity of the formulas describing the stress-energy constructed in the exterior region to radii $|e| < r_+$, this, in turn, strongly suggests that the perturbative approach should be abandoned in favor of the self-consistent treatment [8].

In our discussion we employed the approximate stress-energy tensor constructed for a general line element (3.1) rather than the classical Reissner–Nordström solution, and, therefore the objections of Ref. [20] do not apply. One concludes, therefore, that the semiclassical zero temperature black holes do exist, or, to be more exact, that the semiclassical Einstein field equations with the source term given by the stress-energy tensor of the massive fields in the large mass limit allow solutions with vanishing surface gravity (temperature), for which the standard relation holds, although the degenerate event horizon is now located in the classically forbidden region.

5.3. The black hole mass

Till now we have expressed the result in terms of the integration constants e and r_+, which are interpreted as the electric charge and the location of the 'exact' event horizon, respectively. It is possible, however, to express the result in a more familiar form introducing the horizon defined mass M. Indeed, denoting $m\,(r_+) = M$ one has

$$M = \frac{r_+}{2} + \frac{e^2}{2r_+} \tag{5.18}$$

and

$$m\,(r) = M + \delta M\,(r) = M - 4\pi\varepsilon \int\limits_{r_+}^{r} dr'(r')^2 \langle T_t^{(0)t} \rangle\,. \tag{5.19}$$

Now, the natural question arises: what is the relation between the horizon defined mass and the electric charge for the extremal configuration. The answer could be easily obtained by inserting (5.16) in the equation (5.18). After simple calculation one has

$$M = |e| + \mathcal{O}(\varepsilon^2)\,, \tag{5.20}$$

that explicitly demonstrates that for the extremal black holes to order ε the classical relation holds. Thus the ratio of the modulus of the electric charge to the horizon defined mass is 1 but the event horizon is now located at

$$r_+ = M - \frac{\varepsilon\mu^{(s)}}{720\pi m^2 M^3}, \tag{5.21}$$

where $\mu^{(s)}$ is given by (5.17). Note that for the classical Reissner–Nordström black hole with mass M the analogous relation reads

$$r_+ = M = |e|. \tag{5.22}$$

It should be emphasized that M is not the mass that is measured by a distant observer. Indeed, the latter mass is defined as

$$\begin{aligned} M_\infty &= \lim_{r\to\infty} M(r) \\ &= M + \Delta M, \end{aligned} \tag{5.23}$$

and ΔM for massive scalar, spinor and vector fields is given by

$$\begin{aligned} \Delta M^{(0)} &= \frac{1}{\pi m^2}\left[-\frac{4169M^3}{158760r_+^6} + \frac{461M^2}{6615r_+^5} - \frac{6607M}{105840r_+^4} + \frac{3007}{105840r_+^3}\right. \\ &\quad \left. + \xi\left(\frac{4M^3}{45r_+^6} - \frac{11M^2}{45r_+^5} + \frac{41M}{180r_+^4} - \frac{13}{180r_+^3}\right)\right], \end{aligned} \tag{5.24}$$

$$\Delta M^{(1/2)} = \frac{1}{\pi m^2}\left(-\frac{451M^3}{17640r_+^6} + \frac{53M^2}{882r_+^5} - \frac{3289M}{70560r_+^4} + \frac{2521}{211680r_+^3}\right) \tag{5.25}$$

and

$$\Delta M^{(1)} = \frac{1}{\pi m^2}\left(\frac{2389M^3}{158760r_+^6} - \frac{598M^2}{6615r_+^5} + \frac{12071M}{105840r_+^4} - \frac{6197}{158760r_+^3}\right). \tag{5.26}$$

For the extremal configuration these formulas become

$$\Delta M^{(0)} = -\frac{107}{317520\pi m^2 M^3}, \tag{5.27}$$

$$\Delta M^{(1/2)} = -\frac{19}{317520\pi m^2 M^3} \tag{5.28}$$

and

$$\Delta M^{(1)} = -\frac{17}{317520\pi m^2 M^3}. \tag{5.29}$$

The semiclassical Einstein field equations could be solved with a different set of boundary conditions. Indeed, taking

$$M_{\infty} = \lim_{r \to \infty} M(r)\,, \tag{5.30}$$

where M_{∞} is interpreted as the total mass determined by a distant observer one has

$$f(r) = 1 - \frac{2M_{\infty}}{r} + \frac{e^2}{r^2} + \frac{8\pi\varepsilon}{r} \int_{\infty}^{r} dr' (r')^2 \langle T_t^{(0)t} \rangle\,. \tag{5.31}$$

The location of the event horizon, $r_{\rm EH}$, as seen by a distant observer, could be obtained from $f(r_{\rm EH}) = 0$ and it could be easily shown that to $\mathcal{O}(\varepsilon)$

$$r_+ = r_{\rm EH}\,. \tag{5.32}$$

5.4. The near-horizon geometry

Finally, we shall investigate the near-horizon geometry of the extremal semiclassical black hole. In the vicinity of r_+ the line element may be written as

$$ds^2 = -e^{2\psi(r_+)} P(r - r_+)^2 dt^2 + \frac{1}{P(r - r_+)^2} dr^2 + r_+^2 d\Omega^2\,, \tag{5.33}$$

where

$$P = \frac{1}{2} \frac{d^2 f}{dr^2}_{|r=r_+}\,. \tag{5.34}$$

To determine P we shall revert the relation (5.21) to obtain

$$M = r_+ + \frac{\varepsilon \mu^{(s)}}{720\pi m^2 r_+^3} \tag{5.35}$$

and express the result solely in terms of the exact location of the event horizon. Differentiating the function f twice with respect to the radial coordinate, taking the limit $r = r_+$ and making use of (5.35) one arrives at a simple result

$$P = \frac{1}{r_+^2} + \mathcal{O}(\varepsilon^2)\,. \tag{5.36}$$

Introducing a new coordinate

$$r = r_+ \left(1 + \frac{r_+}{e^{\psi(r_+)} y}\right) \tag{5.37}$$

one obtains

$$ds^2 = \frac{r_+^2}{y^2}\left(-dt^2 + dy^2 + y^2 d\Omega^2\right) . \tag{5.38}$$

One concludes, therefore, that expanding the near-horizon geometry of the extremal semiclassical black hole into the whole manifold results in the Bertotti–Robinson spacetime [21, 22]. This situation resembles the classical Reissner–Nordström solution.

5.5. Null geodesics

A further insight into the nature of the semiclassical geometry may be gained from the analysis of test particles. Here we shall limit ourselves to the null geodesics in the extremal configuration satisfying

$$E^2 = \mathrm{e}^{2\psi(r)}\dot{r}^2 + \frac{L^2}{r^2}\mathrm{e}^{2\psi(r)}f(r) , \tag{5.39}$$

where the overdot denotes differentiation with respect to the affine parameter, and E and L are the constants of motion interpreted as the particle's total energy and orbital angular momentum, respectively. Restricting oneself to the circular orbits, *i.e.* $\dot{r} = 0$ one concludes that the second term in the right hand side of (5.39) plays the role of the effective potential:

$$V(r) = \frac{L^2}{r^2}e^{2\psi(r)}f(r) . \tag{5.40}$$

It could be easily shown that to $\mathcal{O}(\varepsilon)$ the equation

$$\frac{d}{dr}V = 0 \tag{5.41}$$

has a minimum at $r = r_+$. Again this behavior resembles the classical Reissner–Nordström spacetime, where the minimum occurs at the degenerate event horizon. The second solution, r_c, gives the location of the maximum of the effective potential (and the radius of an unstable circular orbit) that could be found substituting expansion

$$r_c = r_1 + \varepsilon r_2 \tag{5.42}$$

into (5.41) and collecting the terms with the like powers of ε. Solving the thus obtained system of equations one has

$$r_c^{(s)} = 2r_+ + \frac{\tilde{a}^{(s)}}{720\pi m^2 r_+^3} , \tag{5.43}$$

where

$$\tilde{a}^{(s)} = \begin{cases} \frac{28211}{18816} - \frac{243}{32}(\xi - \frac{1}{6}) \\ \frac{698643}{12544} \\ \frac{691631}{18816} \end{cases} \tag{5.44}$$

for scalar, vector and spinor fields, respectively.

Results propounded in this section strongly indicate that in spite of evident differences between the classical Reissner–Nordström and the quantum-corrected spherically-symmetric and electrically charged black hole solutions there are interesting qualitative similarities, especially in the extremality limit.

6. Concluding remarks

In this paper we have constructed solutions to the semiclassical Einstein field equations describing the spherically-symmetric and electrically charged static black holes with a source term consisting of both classical and quantum parts. The classical contribution to the total stress-energy tensor describes the electromagnetic field whereas the quantum part is constructed for the massive scalar (with arbitrary curvature coupling), spinor and vector field. Obtained solutions are parametrized by two integration constants: the electric charge and the exact (to $\mathcal{O}(\varepsilon)$) location of the event horizon. Although the scalar case have been considered earlier in Ref. [15], it should be noted that there are important differences between the approach adopted in the present paper and that of Taylor, Hiscock and Anderson. Indeed, instead of looking for quantum corrections of the classical geometry caused by the stress-energy tensor evaluated in the background of the Reissner–Nordström geometry, and subsequently defining renormalized mass and corrected location of the event horizon, we employ $\langle T_a^b \rangle$ constructed for a general spherically-symmetric spacetime. Such a choice allows to solve the semiclassical Einstein field equations in a self-consistent way, and makes a more profound treatment of the problem possible. The general forms of the thus obtained line elements have been utilized in the calculations of various characteristics of the semiclassical black holes, such as the temperature and its total mass as seen by a distant observer.

The zero temperature limit of our general solutions leads to the extremal configurations, discussion of which expands and systematizes that of Ref. [8]. Specifically, it is shown that the near-horizon geometry when expanded into a whole manifold is described by the Bertotti–Robinson line element. Moreover, to gain a better understanding of the nature of the semiclassical extremal black holes we study the null geodesics in this geometry.

Finally, we remark that it is possible to expand analyses presented in this paper to the regular solutions of coupled equations of a nonlinear electrodynamics and gravitation. These issues are under active investigations and will be presented elsewhere.

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