LAGRANGIAN THEORY OF HAMILTONIAN REDUCTION*

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Lagrangian formalism corresponding to Hamiltonian reduction procedure is presented. Two versions are considered which lead to unbroken or explicitly broken gauge symmetries, respectively.

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1. Introduction

One of the most powerful methods of constructing integrable models in classical as well as quantum theory is the so called method of Hamiltonian reduction [1–3]. It allows to construct, with the help of symmetry arguments, quite complicated dynamical systems out of a relatively simple ones. The most prominent examples of such systems are Toda [4] and Calogero–Moser–Sutherland ones [5]. The main advantages of the reduction method are: the explicit form of "hidden" symmetry responsible for complete integrability, "algebraization" of the procedure of integration of Hamiltonian equations and the existence of quantum counterpart of the method.

The reduction method can be described as follows. One starts with some symplectic manifold (M,ω) which is a candidate for "large" phase space. A Lie group G acts on (M,ω) in a symplectic way and it is assumed that this action is strongly Hamiltonian. Let g be the Lie algebra of G and let $F_X(p)$ be the Hamiltonian corresponding to the element $X\in g$. The mapping $\mu : M\rightarrow g^*$ defined by $\langle\mu(p),X\rangle\equiv F_X(p)$ is called momentum map. Let us select any G-invariant Hamiltonian H on M. It is easy to see that $\mu(p)$ is a constant of motion for the dynamics generated by H. Let $\alpha\in g^*$ be a fixed vector; under some rather general assumptions

$$P_\alpha=\{p\in M \mid \mu(p)=\alpha\}$$

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is a submanifold of M and is obviously invariant under the flow generated by H. The trouble here is that, typically, P_α is not symplectic, $\omega\mid_{P_\alpha}$ being degenerate. However, this degeneracy can be easily described. Let $G_\alpha \subset G$ be the stability subgroup of α with respect to co-adjoint action. It appears that the null vectors of $\omega\mid_{P_\alpha}$ are exactly the vectors tangent to the orbits of G_α in P_α. Therefore, if $Q_\alpha = P_\alpha/G_\alpha$ is a submanifold (which, again, holds true under quite general assumptions), $(Q_\alpha, \omega\mid_{Q_\alpha})$ is a symplectic manifold. The key point of the method is that the trajectories on P_α when reduced to Q_α remain Hamiltonian with respect to its symplectic structure, the relevant Hamiltonian being simply $H\mid_{Q_\alpha}$. The judicious choice of α allows often to obtain integrable reduced theory.

The functions on Q_α can be viewed as G_α-invariant functions on P_α. This suggests that G_α can be treated as a gauge group inherent in the problem. Such an approach is explicitly described for CMS models in Refs. [6,7]. They differ slightly in philosophy: in Ref. [6] special trick allows to put $G_\alpha = G$ while in Ref. [7] the gauge symmetry group G is explicitly broken to G_α.

In most important examples M is a cotangent bundle, $M = T^*N, \omega$ is a canonical form on M and the action of G on M is obtained by lifting its action on configuration space N. In the present paper we describe, generalizing the results of Refs. [6, 7], the Lagrangian theory of Hamiltonian reduction for such a case.

2. Lagrangian formalism

Let us start with a theory described by generalized coordinates q^i, $i = 1, ..., n$ and the Lagrangian

$$L = L(q, \dot{q})\,. \tag{1}$$

We assume that L is invariant under the action of m-dimensional Lie group G which acts, in general nonlinearly, according to

$$q^i \to q'^i = f^i(q;\varepsilon)\,, \qquad f^i(q, 0) = q^i \tag{2}$$

or, in infinitesimal form

$$q'^i = q^i + f^i_\alpha(q)\varepsilon^\alpha, \qquad f^i_\alpha(q) \equiv \frac{\partial f^i(q;\varepsilon)}{\partial \varepsilon^\alpha}\mid_{\varepsilon=0}\,, \qquad \alpha = 1, ..., m. \tag{3}$$

The composition rule for the transformations (2), (3) reads

$$f^i(f(q;\varepsilon);\eta) = f^i(q;\varphi(\eta,\varepsilon))\,, \tag{4}$$

where $\varphi(\eta, \varepsilon)$ defines the composition rule for group elements.

The velocities transform under (2), (3) as follows

$$\dot{q}'^i = \frac{\partial f^i(q;\varepsilon)}{\partial q^k}\dot{q}^k\,, \tag{5}$$

or infinitesimally

$$\dot{q}'^i = \dot{q}^i + \frac{\partial f^i_\alpha(q)}{\partial q^k}\dot{q}^k\varepsilon^\alpha\,. \tag{6}$$

The condition for L to remain invariant under (2), (3) and (5), (6) reads

$$\frac{\partial L}{\partial q^i}f^i_\alpha(q) + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial f^i_\alpha(q)}{\partial q^k}\dot{q}^k = 0\,, \qquad \alpha = 1, ..., m\,. \tag{7}$$

The conjugated momenta are given by

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i}\,. \tag{8}$$

The Poisson bracket and Hamiltonian are defined in standard way.

The action of G can be readily extended to phase space. The relevant transformation rule for momenta reads

$$p_i = \frac{\partial f^k(q)}{\partial q^i}p'_k\,. \tag{9}$$

It is also straightforward to find the Hamiltonians generating the action of G

$$F_\alpha(q,p) = p_i f^i_\alpha(q)\,. \tag{10}$$

Then

$$\dot{F}_\alpha(q,p) = 0\,, \tag{11}$$

as a consequence of Eq. (7). Also

$$\{F_\alpha(q,p), F_\beta(q,p)\} = c^\gamma_{\alpha\beta}F_\gamma(q,p)\,, \tag{12}$$

which is a consequence of Maurer equations ($c^\gamma_{\alpha\beta}$ are the structure constants of G).

If $\{X^\alpha\}$ denote the basis in g^*, the mapping

$$(q,p) \to F_\alpha(q,p)X^\alpha\,, \tag{13}$$

is the momentum map.

Let us now gauge the G symmetry. We assume that the group parameters are arbitrary differentiable functions of time, $\varepsilon^\alpha = \varepsilon^\alpha(T)$. The transformation rule for $\dot{q}'$s is modified to

$$\dot{q}'^i = \frac{\partial f^i(q;\varepsilon)}{\partial q^k}\dot{q}^k + \frac{\partial f^i(q;\varepsilon)}{\partial \varepsilon^\alpha}\dot{\varepsilon}^\alpha \,. \tag{14}$$

We proceed in a standard way and introduce the gauge variables A^α. In order to find their transformation properties we differentiate both sides of (4) with respect to ε or η and set $\varepsilon = 0$ or $\eta = 0$, respectively. We get

$$\begin{aligned} \frac{\partial f^i(q;\varepsilon)}{\partial q^k} f^k_\alpha(q) &= \frac{\partial f^i(q;\varepsilon)}{\partial \varepsilon^\beta}\, \mu^\beta_\alpha(\varepsilon)\,, \\ \mu^\beta_\alpha &\equiv \frac{\partial \varphi^\beta(\varepsilon,\eta)}{\partial \eta^\alpha}\, |_{\eta=0}\,, \end{aligned} \tag{15}$$

$$\begin{aligned} f^i_\alpha(q') &= \frac{\partial f^i(q;\varepsilon)}{\partial \varepsilon^\beta}\, \tilde{\mu}^\beta_\alpha \varepsilon\,, \\ \tilde{\mu}^\beta_\alpha(\varepsilon) &\equiv \frac{\partial \varphi^\beta(\eta,\varepsilon)}{\partial \eta^\alpha}\, |_{\eta=0}\,, \end{aligned} \tag{16}$$

and $q'^i = f^i(q;\varepsilon)$. Combining both equations we derive

$$f^i_\alpha(q') = \tilde{\mu}^\beta_\alpha(\varepsilon)\lambda^\gamma_\beta(\varepsilon) f^k_\gamma(q)\frac{\partial f^i(q;\varepsilon)}{\partial q^k} \tag{17}$$

with $\lambda^\gamma_\beta(\varepsilon)\mu^\beta_\alpha(\varepsilon) = \delta^\gamma_\alpha$.

Therefore, if A^α transforms as

$$A^\alpha = \lambda^\alpha_\gamma(\varepsilon)\tilde{\mu}^\gamma_\beta(\varepsilon)A'^\beta + \lambda^\alpha_\beta(\varepsilon)\dot{\varepsilon}^\beta\,, \tag{18}$$

the covariant derivative

$$D_t q^i \equiv \dot{q}^i + f^i_\alpha(q)A^\alpha \tag{19}$$

transforms then according to Eq. (5)

$$(D_t q^i)' = \frac{\partial f^i(q;\varepsilon)}{\partial q^k}(D_t q^k)\,. \tag{20}$$

Now, we can construct the Lagrangian invariant under time-dependent G-transformations

$$L = L(q, D_t q)\,. \tag{21}$$

However, we are not yet ready to write out the Lagrangian theory equivalent to the formalism of Hamiltonian reduction. To this end we can proceed in either of two ways presented in Sec. 3 and 4, respectively.

3. Unbroken gauge symmetry

Typically, the momentum map condition breaks G-invariance. It is however, advantageous to keep G-invariance exact, $G_\alpha = G$. This can be achieved by viewing the value of momentum map as dynamical variable.

It is easy to see from Eqs. (10), (17) that F_α transforms according to adjoint representation of G

$$F_\alpha(q', p') = \tilde{\mu}_\alpha^\gamma(\varepsilon)\lambda_\gamma^\beta(\varepsilon)F_\beta(q, p)\,. \tag{22}$$

Let $\{T_\alpha\}$ be any matrix representation of g and let the composition rule refers to exponential parametrization

$$\mathrm{e}^{i\varepsilon^\alpha T_\alpha}\mathrm{e}^{i\eta^\alpha T_\alpha} = \mathrm{e}^{i\varphi^\alpha(\varepsilon,\eta)T_\alpha}\,. \tag{23}$$

It is straightforward to check that the transformation rule (18) is equivalent to

$$A' = UAU^{-1} - i\partial_t UU^{-1}\,, \tag{24}$$

where

$$U \equiv \mathrm{e}^{i\varepsilon^\alpha T_\alpha}, \qquad A \equiv A^\alpha T_\alpha\,. \tag{25}$$

Let us introduce new dynamical variables $v^i, \bar{v}_i$ transforming according to

$$v \to v' = Uv\,, \qquad \bar{v} \to \bar{v}' = \bar{v}U^{-1}\,. \tag{26}$$

We supply (21) with the additional term yielding trivial dynamics for v's

$$\tilde{L} = L(q, D_t q) + \bar{v}(i\partial_t + A^\alpha T_\alpha)v\,. \tag{27}$$

Lagrangian (27) is invariant under the gauge transformations (24) supplemented by

$$\begin{aligned} q^i \to q'^i &= f^i(q; \varepsilon(t))\,, \\ v^a \to v'^a &= U_b^a(t)v^b\,, \\ \bar{v}_a \to \bar{v}'_a &= (U^{-1})_a^b(t)\bar{v}_b\,. \end{aligned} \tag{28}$$

The representation U remains up to now unspecified. In our 1+0-dimensional gauge theory A^α's are not dynamical — they can be eliminated by gauge transformations altogether.

The system defined in Eq. (27) is obviously constrained. Let us perform a standard Dirac analysis. We have

$$\begin{aligned} p_i &\equiv \frac{\partial \tilde{L}}{\partial \dot{q}^i} = \frac{\partial \tilde{L}}{\partial (D_t q^i)}\,, \\ \Pi_\alpha &\equiv \frac{\partial \tilde{L}}{\partial \dot{A}^\alpha} = 0\,, \\ \omega_a &\equiv \frac{\partial \tilde{L}}{\partial \dot{v}^a} = i\bar{v}_a\,, \\ \bar{\omega}^a &\equiv \frac{\partial \tilde{L}}{\partial \dot{\bar{v}}_a} = 0\,. \end{aligned} \tag{29}$$

Therefore, the primary constraints read

$$\begin{aligned} \Pi_\alpha &\approx 0\,, \\ \bar{\omega}^a &\approx 0\,, \\ \omega_a - i\bar{v}_a &\approx 0\,. \end{aligned} \tag{30}$$

Let us impose standard Poisson brackets

$$\{q^i, p_j\} = \delta^i_j,\ \{A^\alpha, \Pi_\beta\} = \delta^\alpha_\beta,\ \{v^a, \omega_b\} = \delta^a_b,\ \{\bar{v}_a, \bar{\omega}^b\} = \delta^b_a \tag{31}$$

and construct the Hamiltonian

$$\begin{aligned} H &= p_i \dot{q}^i - L(q, D_t q) + \omega_a \dot{v}^a - \bar{v}(i\partial_t + A^\alpha T_\alpha)v + u^\alpha \Pi_\alpha \\ &\quad + \bar{u}^a(\omega_a - i\bar{v}_a) + u_a \bar{\omega}^a\,, \end{aligned}$$

or

$$\begin{aligned} H &= p_i \dot{q}^i - L(q, D_t q) - \bar{v} A^\alpha T_\alpha v + u^\alpha \Pi_\alpha + \bar{u}^a(\omega_a - i\bar{v}_a) \\ &\quad + u_a \bar{\omega}^a\,, \end{aligned} \tag{32}$$

where u_a and $\bar{u}^\alpha$ are the appropriate Lagrange multipliers.

Let us look for secondary constraints. Equation

$$\{\Pi_\alpha, H\} = 0 \tag{33}$$

gives secondary constraint

$$F_\alpha(q, p) + \bar{v} T_\alpha v = 0\,. \tag{34}$$

On the other hand

$$\begin{aligned} &\{\bar{\omega}^a, H\} = 0\,, \\ &\{\omega_a - i\bar{v}_a, H\} = 0\,, \end{aligned} \tag{35}$$

give constraints on Lagrange multipliers u_a, $\bar{u}^a$

$$\begin{aligned} \bar{u} &= i(A^\alpha T_\alpha)v\,, \\ u &= -i\bar{v}(A^\alpha T_\alpha)\,. \end{aligned} \tag{36}$$

Therefore,

$$H = p_i\dot{q}^i - L(q, D_t q) + u^\alpha \Pi_\alpha + i\omega(A^\alpha T_\alpha)v - i\bar{v}(A^\alpha T_\alpha)\bar{\omega}\,. \tag{37}$$

Moreover,

$$\{F_\alpha + \bar{v}T_\alpha v, H\} = -c^\gamma_{\alpha\beta}A^\beta(F_\gamma + \bar{v}T_\gamma v)\,, \tag{38}$$

so there are no new constraints.

In order to find the classification into first- and second-class constraints we pass to the equivalent system of constraints. They read

$$\Pi_\alpha \approx 0\,, \tag{39}$$

$$\omega_a - i\bar{v}_a \approx 0\,, \tag{40}$$

$$\bar{\omega}^a \approx 0\,, \tag{41}$$

$$F_\alpha(q,p) - i\omega T_\alpha v + i\bar{v}T_\alpha\bar{\omega} \approx 0\,. \tag{42}$$

It is easy to check that (40) and (41) are second-class constraints while (39) and (42) first-class ones. We define standard Dirac brackets. They read

$$\begin{aligned} \{v^a, \bar{v}_b\}_{\mathrm{D}} &= -i\delta^a_b\,, \\ \{q^i, p_j\}_{\mathrm{D}} &= \delta^i_j\,, \end{aligned} \tag{43}$$

other variables are eliminated by the constraints (40), (41). Therefore we get finally

$$\begin{aligned} H &= p_i\dot{q}^i - L(q, D_t q) + \bar{u}^\alpha \Pi_\alpha - \bar{v}A^\alpha T_\alpha v\,, \\ \Pi_\alpha &\approx 0\,, \\ F_\alpha(q,p) + \bar{v}T_\alpha v &\approx 0\,. \end{aligned} \tag{44}$$

It is also easy to see that

$$p_i\dot{q}^i - L(q, D_t q) = H_0(q,p) - A^\alpha F_\alpha(q,p)\,, \tag{45}$$

where $H_0(q,p)$ is the Hamiltonian corresponding to rigid G-symmetry.

Using this we can write

$$H = H_0(q,p) - A^\alpha(F_\alpha(q,p) + \bar{v}T_\alpha v) + \bar{u}^\alpha \Pi_\alpha\,. \tag{46}$$

By choosing a gauge $A^\alpha = 0$ one finds dynamics determined by $H_0(q,p)$ and restricted to the submanifold P_α given by

$$F_\alpha(q,p) = -\bar{v}T_\alpha v = \text{const}\,. \tag{47}$$

Moreover, the observables $O(q,p)$ should be gauge invariant; for that it is sufficient that they are invariant under rigid G symmetry. So the actual phase space is the submanifold (47) divided by the action of G.

Up to now we have not specified the representation (25). It can be any representation provided one can select vectors $v, \bar{v}$ such that $\bar{v}T_\alpha v$ attains desired form dictated by the (strong) condition that the reduced theory is integrable. We shall not touch upon the question whether this is generally possible. This is at least the case for most interesting examples.

4. Broken gauge symmetry

An alternative approach is based on explicit symmetry breaking. Let us consider the Lagrangian

$$\underline{L} = L(q, D_t q) - \varrho_\alpha A^\alpha\,, \tag{48}$$

where ϱ_α is a fixed element of adjoint representation of G. Let us denote by $G_\varrho \subset G$ the stability subgroup of ϱ_α; for simplicity we assume that the Lie algebra of G_ϱ is spanned by X_α, $\alpha = 1, ..., r \leq m$. Then

$$c^\alpha_{\beta\gamma}\varrho_\alpha = 0\,, \qquad \beta = 1, ..., r\,, \quad \gamma = 1, ..., m\,. \tag{49}$$

The primary constraints are

$$\Pi_\alpha \approx 0\,, \tag{50}$$

while the Hamiltonian reads

$$H = p_i \dot{q}^i - L(q, D_t q) + \varrho_\alpha A^\alpha + u^\alpha \Pi_\alpha\,, \tag{51}$$

or

$$H = H_0(q,p) + A^\alpha(\varrho_\alpha - F_\alpha(q,p)) + u^\alpha \Pi_\alpha\,, \tag{52}$$

here again U^α are Lagrange multipliers related to the constraints (51). Now

$$0 \approx \dot{\Pi}_\alpha = \{\Pi_\alpha, H\} \tag{53}$$

implies

$$F_\alpha(q,p) - \varrho_\alpha \approx 0\,. \tag{54}$$

Differentiating again with respect to time we get

$$0 \approx \dot{F}_\alpha(q,p) = \{F_\alpha(q,p), H\} = -c^\gamma_{\alpha\beta}((F_\gamma - \varrho_\gamma) + \varrho_\gamma)A^\beta \,. \tag{55}$$

Therefore, we get new constraints

$$c^\gamma_{\alpha\beta}\varrho_\gamma A^\beta = 0, \quad \alpha = r+1, ..., m \,. \tag{56}$$

Differentiating once more one arrives at the constraints on Lagrange multipliers

$$c^\gamma_{\alpha\beta}\varrho_\gamma u^\beta = 0, \quad \alpha = r+1, ..., m \,. \tag{57}$$

By assumption G_ϱ is the maximal stability subgroup of ϱ_α. Therefore, $c^\gamma_{\alpha\beta}\varrho_\alpha$, $\alpha, \beta = r+1, ..., m$ is a nonsingular matrix. Eqs. (57) and (58) imply now

$$\begin{aligned} A^\alpha &= 0\,, \\ u^\alpha &= 0\,, \quad \alpha = r+1, ..., m \,. \end{aligned} \tag{58}$$

So, finally, we have the following set of constraints

$$\begin{array}{lll} \Pi_\alpha \approx 0\,, \quad \alpha = r+1, ..., m\,, & & \text{primary constraints} \\ F_\alpha(q,p) - \varrho_\alpha \approx 0\,, \quad \alpha = r+1, ..., m & & \text{secondary constraints} \\ A^\alpha \approx 0\,, \quad \alpha = r+1, ..., m\,, & & \text{secondary constraints} \end{array} \tag{59}$$

while Hamiltonian takes the form

$$H = H_0(q,p) + A^\alpha(\varrho_\alpha - F_\alpha(q,p)) + \sum_{\alpha=1}^{r} u^\alpha \Pi_\alpha \,. \tag{60}$$

The classification into first- and second-class constraints is also easy

$$\begin{array}{ll} \Pi_\alpha \approx 0, \quad \alpha = 1, ..., r\,, & \text{first-class constraints} \\ F_\alpha(q,p) - \varrho_\alpha \approx 0\,, \quad \alpha = 1, ..., r\,, & \text{first-class constraints} \\ \Pi_\alpha \approx 0\,, \quad \alpha = r+1, ..., m\,, & \text{second-class constraints} \\ A^\alpha \approx 0\,, \quad \alpha = r+1, ..., m\,, & \text{second-class constraints} \\ F_\alpha(q,p) - \varrho_\alpha \approx 0\,, \quad \alpha = r+1, ..., m\,, & \text{second-class constraints}\,. \end{array} \tag{61}$$

Dirac brackets allow us to neglect altogether $A^\alpha, \Pi_\alpha, \alpha = r+1, ...m$, while for the remaining variables we obtain

$$\{A, B\}_{\mathrm{D}} = \{A, B\} - \sum_{\alpha,\beta=r+1}^{m} (c^{-1})_{\alpha\beta}\{A, F_\alpha\}\{F_\beta, B\}\,, \tag{62}$$

where $c_{\alpha\beta} \equiv c_{\alpha\beta}^{\gamma}\varrho_{\gamma}$. Hamiltonian reduces now to

$$H = H_0(q,p) + \sum_{\alpha=1}^{r} A^{\alpha}(\varrho_{\alpha} - F_{\alpha}(q,p)) + \sum_{\alpha=1}^{r} u^{\alpha}\Pi_{\alpha} \tag{63}$$

and the first-class constraints are related to G_{ϱ} gauge symmetry. Again it is easy to check that our theory implies Hamiltonian reduction.

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