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VACUUM ENGINEERING AT A PHOTON COLLIDER?

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The aim of this paper is twofold: to provide a rather detailed and self-contained introduction into the physics of the Disoriented Chiral Condensate (DCC) for the photon (and linear) collider community, and to indicate that such physics can be searched and studied at photon colliders. Some side tracks are also occasionally followed during the exposition, if they lead to interesting vistas. For gourmets, the Baked Alaska recipe is given in the appendix.

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1. Introduction

The twentieth century witnessed tremendous progress in our understanding of the fundamental building blocks of matter and their interactions. Not the least role in this success was played by continuous advance in accelerator technologies. At the beginning of the new century, accelerator-based experiments are expected to preserve their leading role in the field of high-energy physics [1].

Over the seven decades since Lowrence's first cyclotron one has observed a nearly exponential growth in effective energies of the accelerators by the increment factor of about 25 per decade (the Levingston law [2]. By the effective energy one means the laboratory energy of particles colliding with a proton at rest to reach the same center of mass energy). At that the cost per unit effective energy has decreased by about four orders of magnitudes. This is indeed a remarkable trend and it was fed by a succession of new ideas and technologies [2]: the principle of phase stability, strong focusing, high impedance microwave devices, superconducting technologies, storage rings and beam cooling.

However the accelerators were becoming ever bigger and more expensive on the whole. We have already entered "the dinosaur era" with monstrous machines and the Levingston tendency is slowing its pace. The problem with the circular e^+e^- colliders is that the synchrotron radiation severely limits maximal attainable energy. It is believed that this technology has reached its limits at LEP and no other bigger project of this type will be ever realized. Instead the linear e^+e^- colliders are considered as a viable alternative. Extension of the existing linear accelerator technology towards higher accelerating gradients and smaller emittance beams is expected to make real a design of the TeV scale linear colliders. Further progress with the conventional techniques is problematic unless some radically new idea appears. In fact the high gradient efficient acceleration is a tough thing. In a free electromagnetic wave the E field is at right angle to the particle momentum and no efficient acceleration can be achieved. For efficient acceleration one has to have matter very near or within the beams. Then energy considerations combined with the survivability of the accelerating structure limits the attainable acceleration gradient [1, 2].

The proton circular colliders still have some reserve left because, owing to the heaviness of the proton, the synchrotron radiation constraint is expected only at very high energies. The Large Hadron Collider (LHC) with 7 TeV proton beams is under construction now. LHC is a very important high-energy physics project and we believe that its results will determine the future shape of the field. An analogous collider with the center of mass energy about 100 TeV seems also feasible and maybe the Very Large Hadron Collider (VLHC) will be the last monstrous dinosaur of this type.

Other possibilities include the muon colliders first suggested by Budker many years ago [3]. Muons, being about 207 times heavier than electrons, experience much less radiative energy losses, which are inversely proportional to the forth power of the particle mass. It seems that the efficient multi-TeV muon colliders can be constructed despite the fact that the muon is an unstable particle [3].

But why all the fuss? Are these future very complex and costly accelerators really necessary? The past research led to the triumph of the Standard Model. At that the revolutionary 70's were followed by decades of the more or less routine verification of the Standard Model wisdom — the situation eloquently expressed by Bjorken some time ago [1]: "a theorist working within the Standard Model feels like an engineer, and one working beyond it feels like a crackpot". Since then "crackpots" have developed a string theory as the main challenge to the standard paradigm [4]. This "Theory of Everything" is full of deep and beautiful mathematical constructs and is generally considered as the most promising road towards understanding fundamental physics. The only trouble with it is that it will be extremely difficult to check experimentally the predictions of this theory, because the most direct predictions refer to the nature of space–time at the Plank scale, $\sim 10^{19}$ GeV, and no experimental method seems to be ever able to access such energies in a foreseeable future. So the string theorists are doomed to face the fatal question "can there be physics without experiments?" [5] for a long time. Therefore, on the one hand, we have a clear experimental and theoretical success up to the electroweak scale, ~ 100 GeV, where the Standard Model reigns, and, on the other hand one has a very ambitious theory without any clues how to check it experimentally. But what lies in between, worth of billions of dollars to spend in future accelerators and detectors, to investigate?

Despite its splendid success, nobody doubts that the Standard Model will break down sooner or later. There are several reasons why the Standard Model cannot be the final theory and why some new physics beyond the Standard Model is expected [6]:

- $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry group defines separate gauge theories with three different coupling constants. The conceptual similarity of these theories is begging for unification.

- The family problem — why are there three quark–lepton families?

- The origin of the quark and lepton masses and mixing angles, as well as of the CP violation.

- Solid experimental evidence of the neutrino oscillations require non-vanishing neutrino masses and therefore some extension of the Standard Model. However, very minimal extension might be sufficient to accommodate neutrino masses.

- The strong CP problem — why is the allowed CP violating θ-term in the QCD Lagrangian very small or absent?

- The hierarchy problem — why is the electroweak scale so different from the Plank scale?

- The cosmological constant problem — why gravity almost does not feel the presence of various symmetry-breaking condensates?

But how far is this expected new physics? The logical structure of the Standard Model itself hints that quite interesting and crucial things can happen in the realm of the next generation of the future colliders. One of the main guiding principles of the Standard Model, which plays a key role in the theory, is gauge symmetry. The historical roots of the gauge invariance are reviewed by Jackson and Okun [7] and the review embraces about two centuries. In fact one can go even further through history, another twenty centuries or so up to the times of the ancient Greece, and find the roots in the most widely known theorem from Euclid's "Elements of Geometry": The sum of the interior angles of a triangle equals 180 degree. Euclid deduces this theorem from the so-called parallel axiom. All efforts to avoid this sophisticated axiom failed and finally led to the discovery of non-Euclidean geometry. But we do not follow this track. Instead, we start to generalize Euclid's 180 degree theorem step by step [8]. The first step involves the concept of exterior angle: the interior angle α and the corresponding exterior angle β are related by $\alpha+\beta=\pi$. Then the theorem immediately generalizes from triangles to arbitrary polygons: The sum of the exterior angles of a polygon equals 2π.

Let us now consider a triangle whose edges are not straight lines but some plane smooth curves. When the unit tangent vector is transported by a length Δl along the smooth curve, it turns through an angle $\Delta\phi$. The limit of the ratio $\Delta\phi/\Delta l$, when $\Delta l\to 0$, defines the geodesic curvature of the curve. Therefore, for such a curved triangle the 180 degree theorem takes the form

$$\sum \text{ext. angles} + \int \text{geod. curv.} = 2\pi\,,$$

where the integral is along the triangle edges. This follows from the fact that any curved triangle can be approximated by a polygon and then the total turning of the tangent along the edges (the integral geodesic curvature of the edges) is given by the sum of the corresponding exterior angles.

One can define the geodesic curvature by using normals instead of tangents, because the normal rotates exactly as the tangent does when a point moves along the curve. The advantage of using normals is that one can generalize the concept of curvature to surfaces which have no unique tangent direction but the direction of the normal is still well defined. The corresponding generalization is called the Gaussian curvature [8] and the 180 degree theorem for a general triangle on a curved surface looks like

$$\sum \text{ext. angles} + \int \text{geod. curv.} + \iint \text{Gaussian curv.} = 2\pi\,. \tag{1}$$

Finally, let D be a domain on the surface whose boundary ∂D is formed by one or more sectionally-smooth curves. We can triangulate D with triangles which have geodesic inside (not belonging to ∂D) edges. For each triangle we will have (1). If we add these equations up and rearrange the angles cleverly we get the Gauss–Bonnet formula [8]

$$\sum \text{ext. angles} + \int\limits_{\partial D} \text{geod. curv.} + \iint\limits_{D} \text{Gaussian curv.} = 2\pi\,\chi(D)\,, \tag{2}$$

where $\chi(D) = v - e + f$, v being the number of vertices, e — the number of edges, and f — the number of triangles in the triangulation; $\chi(D)$ is the topological invariant of D called its Euler characteristic.

The Gauss–Bonnet formula (2) is indeed a long way from the 180 degree theorem, but the potential for generalization is still not exhausted. The ideas of Gauss about the curvature and the geometry on the surface was further generalized by B. Riemann. It was soon realized that most properties of the Riemannian geometry follows from its Levi–Civita parallelism, an infinitesimal parallel transport of the tangent vectors. The important concept of the Levi–Civita connection emerged. All these is the mathematical basis of Einstein's general relativity. Further generalization of the concepts of Levi–Civita connection and curvature to more general, than Riemannian, manifolds lead to the notion of fiber bundles — the mathematical basis of the gauge field theories [9]. Even magnetic monopoles are related to the generalized Gauss–Bonnet theorem [10].

Therefore, both general relativity and gauge theory can be considered as stunning generalizations of the 180 degree theorem of the Euclidean geometry! However, returning to the Standard Model, this is not the whole story. Gauge symmetry is important, very important, in the Standard Model. But the real shape of the world is determined by its spontaneous violation. Then a big question is why and how the $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge symmetry of the Standard Model is broken. So far the phenomenologically adequate answer to this question is given by the introduction of the SU(2)-doublet of scalar fields, the Higgs doublet, whose couplings and vacuum expectation value determine fermion masses and mixings. However there are too many free parameters, not fixed by the theory, indicating that in fact we do not understand what is going on. That is why the discovery of the Higgs boson and investigation of its properties are considered as having the paramount importance.

At this point photon colliders enter the game, because in the $\gamma\gamma$ collisions the Higgs boson will be produced as a single resonance. The idea of photon colliders was proposed many years ago in Novosibirsk [11, 12]. You have to have a linear e^+e^- collider and a powerful laser (several Joules per flash) to realize this idea. High-energy photons are produced by Compton backscattering of the laser light on the high-energy electrons near the interaction point. After the scattering, the photons will have almost the same energy as the initial electrons and small additional angular spread of the order of inverse γ-factor of the initial electron. This additional angular spread does not effect much the resulting $\gamma\gamma$ (or γe) luminosity if the conversion point is close enough to the interaction point. The $\gamma\gamma$ luminosity can be made even larger than the e^+e^- luminosity at the same collider by using the initial electron beams with smaller emittances than allowed in the e^+e^--mode by beam collision effects.

The detailed development of the photon collider idea [13–15] showed that their construction is a quite realistic task and requires a small additional ($\sim 10\%$) investment compared to the linear collider price. The solid state laser technologies with required pulse power and duration already exist. A free electron laser with variable wave length is also an attractive alternative [16].

The expected physics at high-energy photon colliders is really exciting and very rich. It includes [15, 17, 18]:

- Higgs boson physics, both Standard Model and supersymmetric. Especially one should mention the unique opportunity to measure its two photon width, as well as the possibility to explore CP properties of the neutral Higgs boson by controlling the polarizations of the back-scattered photons.
- Search for supersymmetry. In particular, charged sfermions, charginos and charged Higgs bosons will be produced at larger rates in $\gamma\gamma$ collisions than in e^+e^- collisions. The γe option will enable potential discovery of selectrons and neutralinos. The photon collider will also be an ideal place to discover and study stoponium bound states.
- Exploration of the gauge bosons nonlinear interactions.
- Top quark physics.
- QCD-probes in a new unexplored regime.
- Investigation of the photon structure — its hadronic quantum fluctuations cannot be completely determined from the first principles because the large distance effects contribute significantly. Therefore various phenomenological models need experimental input for refinements.
- Search for the low-scale quantum gravity, space–time noncommutativity [19] and extra dimensions.

The last item is exotic enough but one should not forget that [20] "Every time we introduce a new tool, it always leads to new and unexpected discoveries, because Nature's imagination is richer than ours".

In this paper we would like to indicate that the physical program of the photon collider can further be enriched if it is considered as a tool to perturb the QCD vacuum. An interesting phenomenon of the Disoriented Chiral Condensate formation was discussed earlier in the context of hadron–hadron and heavy ion collisions. We believe that photon colliders are also eligible devices to perform such kind of research.

The paper is organized as follows. We begin with the discussion of the linear sigma model, which is used as a QCD substitute in the majority of DCC studies. The idea of the Disoriented Chiral Condensate is explained and investigated in the third section. The fourth section considers the possibility of the DCC production at photon colliders. The Baked Alaska scenario is examined in some details. Quantum state of DCC is explored in the next section. It is mentioned that at photon colliders a direct production of this state might be possible. In the last section we provide some concluding remarks. The references on the subject are very numerous and we list only a few of them. We hope that an interested reader can find independently other important contributions which missed our attention.

2. Linear sigma model

The Lagrangian of quantum chromodynamics (QCD) looks "deceptively simple" [21]. Indeed, it encodes the description of a surprisingly wide range of natural phenomena, from nuclear physics to cosmology, and nevertheless is given by the very compact expression

$$\mathcal{L}_{\rm QCD} = \bar{q}(i\hat{D} - m)q - \frac{1}{2}{\rm Sp}\, G_{\mu\nu}G^{\mu\nu}, \tag{3}$$

where

$$\hat{D} = \hat{\partial} + ig\hat{A}, \;\; G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu], \;\; A_\mu = A_\mu^a \frac{\lambda_a}{2}$$

and $\lambda_a,\ a = 1, \ldots, 8$ are SU(3) Gell-Mann matrices. The theory (QCD) which is defined by this Lagrangian "embodies deep and beautiful principles" and is one of "our most perfect physical theories" [22]. However, if you are interested in applying this "most perfect physical theory" to understand the low-energy experimental data, you will not be particularly happy by discovering at least three reasons [21] for your grievance:

- The Lagrangian (3) describes quark and gluon degrees of freedom, while "correct" degrees of freedom for low energy phenomena are their bound states — various colorless hadrons.
- Unlike quantum electrodynamics, gluons have self-interactions which render QCD in a nonlinear theory with the corresponding increase in the computational complexity.
- At low energies the effective coupling constant is large and usual perturbative methods are not applicable.

However, things are not so bad as they look. It turns out that many important features of the low-energy dynamics are governed by symmetries of the QCD Lagrangian and their breaking patterns. For light quark flavors the QCD Lagrangian possesses (approximate) $\mathrm{U_R}(3) \times \mathrm{U_L}(3)$ chiral symmetry. The corresponding transformations are

$$\begin{aligned} q_\mathrm{R} &\to \mathrm{e}^{i\frac{\lambda_0}{2}\theta^0_\mathrm{R}} q_\mathrm{R}, \quad q_\mathrm{L} \to \mathrm{e}^{i\frac{\lambda_0}{2}\theta^0_\mathrm{L}} q_\mathrm{L}, \\ q_\mathrm{R} &\to \mathrm{e}^{i\frac{\lambda_a}{2}\theta^a_\mathrm{R}} q_\mathrm{R}, \quad q_\mathrm{L} \to \mathrm{e}^{i\frac{\lambda_a}{2}\theta^a_\mathrm{L}} q_\mathrm{L}, \end{aligned} \tag{4}$$

where $\lambda_0 = \sqrt{\frac{2}{3}}$. Fates of these symmetries are different. The first line corresponds to the $\mathrm{U}_V(1) \times \mathrm{U}_A(1)$ transformations with $\theta_{V,A} = \frac{1}{2}\left(\theta^0_\mathrm{L} \pm \theta^0_\mathrm{R}\right)$. The singlet vector current, generated by $\mathrm{U}_V(1)$ transformations, remains conserved in the low-energy limit and the corresponding conserved charge is identified with the baryon number. On the contrary, $\mathrm{U}_A(1)$ symmetry is broken due to quantum anomaly. As a result, η' meson becomes much heavier compared to other pseudoscalars. Non-Abelian symmetries $\mathrm{SU_R}(3) \times \mathrm{SU_L}(3)$, as well as $\mathrm{U}_A(1)$, are further broken spontaneously due to a nonvanishing expectation value of the quark–antiquark condensate: $\langle \bar{q}_\mathrm{R} q_\mathrm{L} \rangle \neq 0$. Eight pseudoscalar mesons (π, K, η) are Goldstone bosons associated with this symmetry breaking pattern $\mathrm{SU_R}(3) \times \mathrm{SU_L}(3) \to \mathrm{SU}_V(3)$. In fact these Goldstone bosons acquire small masses because quark mass terms in the QCD Lagrangian break explicitly the $\mathrm{U_R}(3) \times \mathrm{U_L}(3)$ chiral symmetry.

Having in mind this picture of QCD symmetries and their breaking, one can try to model it by some effective low-energy theory for mesons, which are excitations on the quark–antiquark condensate ground state [23, 24]. One has two kinds of excitations, scalar and pseudoscalar mesons, because

$$\bar{q}_\mathrm{R} q_\mathrm{L} \sim \bar{q}q + \bar{q}\gamma_5 q \,.$$

Therefore, for three light quark flavors, one needs a complex 3×3 matrix field $\Phi_{ab} \sim \bar{q}_{\mathrm{R}b} q_{\mathrm{L}a}$ to parametrize the scalar (S) and pseudoscalar (P) meson nonets:

$$\Phi = S + iP \equiv \frac{\lambda_a}{2}(\sigma_a + i\pi_a) + \frac{\lambda_0}{2}(\sigma_0 + i\pi_0)\,. \tag{5}$$

The imaginary unit is introduced to make the pseudoscalar matrix P Hermitian: iP corresponds to $\bar{q}\gamma_5 q$, but $(\bar{q}\gamma_5 q)^+ = -\bar{q}\gamma_5 q$.

The effective Lagrangian for the field Φ should have the form [25]

$$\mathcal{L} = \mathrm{Sp}\left(\partial_\mu \Phi^+ \partial^\mu \Phi\right) - V(\Phi, \Phi^+) + \mathcal{L}_\mathrm{SB}, \tag{6}$$

where $\mathcal{L}_\mathrm{SB}$ describes symmetry breaking effects and $V(\Phi, \Phi^+)$ stands for self-interactions of the meson field. If we want the theory to be renormalizable (although for effective theories this requirement is not obvious), quartic couplings are at most allowed in $V(\Phi, \Phi^+)$. The chiral transformations (4) read in terms of the Φ field

$$\begin{aligned} \mathrm{U}_V(1) &: \quad \Phi \to \mathrm{e}^{i\frac{\lambda_0}{2}\theta_V^0}\,\Phi \mathrm{e}^{-i\frac{\lambda_0}{2}\theta_V^0} = \Phi, \\ \mathrm{U}_A(1) &: \quad \Phi \to \mathrm{e}^{i\frac{\lambda_0}{2}\theta_A^0}\,\Phi \mathrm{e}^{i\frac{\lambda_0}{2}\theta_A^0} = \mathrm{e}^{i\lambda_0\,\theta_A^0}\,\Phi, \\ \mathrm{SU}_V(3) &: \quad \Phi \to \mathrm{e}^{i\frac{\lambda_a}{2}\theta_V^a}\,\Phi \mathrm{e}^{-i\frac{\lambda_a}{2}\theta_V^a}, \\ \mathrm{SU}_A(3) &: \quad \Phi \to \mathrm{e}^{i\frac{\lambda_a}{2}\theta_A^a}\,\Phi \mathrm{e}^{i\frac{\lambda_a}{2}\theta_A^a}. \end{aligned} \tag{7}$$

Therefore, $\mathrm{Sp}\,(\Phi^+\Phi)$ and $\mathrm{Sp}\,(\Phi^+\Phi)^2$ are invariant under these transformations and the most general form of $V(\Phi, \Phi^+)$ is

$$V(\Phi, \Phi^+) = m^2 \mathrm{Sp}\,(\Phi^+\Phi) + \lambda\,\mathrm{Sp}\,(\Phi^+\Phi)^2 + \lambda' \left[\mathrm{Sp}\,(\Phi^+\Phi)\right]^2 . \tag{8}$$

The symmetry breaking part of the effective Lagrangian has the form

$$\mathcal{L}_{\mathrm{SB}} = \mathrm{Sp}\,H(\Phi + \Phi^+) + c\left[\mathrm{Det}(\Phi) + \mathrm{Det}(\Phi^+)\right] . \tag{9}$$

Here the first term describes explicit symmetry breaking due to nonzero quark masses. The matrix H represents the constant nine-component external field: $H = \frac{\lambda_a}{2}h_a + \frac{\lambda_0}{2}h_0$. In practice isospin symmetry and PCAC are good approximations because u and d quark masses are very small. To preserve these symmetries, the most general possibility is to have only two nonzero constants h_0 and h_8 [26]. h_0 gives a common shift to pseudoscalar (and scalar) masses, while h_8 breaks the $\mathrm{SU}_V(3)$ unitary symmetry down to isospin $\mathrm{SU}_V(2)$ and generates the mass differences between π, K and η, as well as between their parity partners (the phenomenological situation in the scalar nonet is not completely clear yet [25]). The determinant term is invariant under $\mathrm{SU}_V(3)\times\mathrm{SU}_A(3)$ transformations from (7), because $\mathrm{Det}(AB) = \mathrm{Det}(A)\mathrm{Det}(B)$ and $\mathrm{Det}(\mathrm{e}^{i\frac{\lambda_a}{2}\theta^a}) = 1$. However it violates $\mathrm{U}_A(1)$ symmetry down to $Z_A(3)$, because $\mathrm{Det}(\mathrm{e}^{i\lambda_0\theta_A^0}) = 1$ only then $\lambda_0\theta_A^0 = \frac{2\pi}{3}n$, n being an integer. This explicit breaking of $\mathrm{U}_A(1)$ removes the mass degeneracy between η' and π [27,28] and, therefore, is very important for describing the pseudoscalar nonet. Another interesting property of the determinant term is that it gives equal and opposite sign contributions to the masses of the corresponding scalars and pseudoscalars [28]. Therefore, the large splitting between scalars and pseudoscalars is expected solely from the fact that η' is much heavier than π [28]. This is exactly the situation observed in experiment. Physics behind the determinant term is related to the $\mathrm{U}_A(1)$ quantum anomaly, mentioned above, caused by nonperturbative effects in the QCD vacuum due to instantons [29]. Note that the $i\,[\mathrm{Det}(\Phi) - \mathrm{Det}(\Phi^+)]$ term is not allowed as it violates P and CP [30]. Indeed, under charge conjugation $\Phi \to \Phi^T$, which does not change the determinant. While under parity $\Phi \to \Phi^+$ and $\mathrm{Det}(\Phi) - \mathrm{Det}(\Phi^+)$ changes the sign.

The linear sigma model, as defined by (6), (8) and (9), has six free parameters to be fixed from experiment: m^2, λ, λ', c, h_0 and h_8. Five parameters can be fixed by using experimental information from the pseudoscalar sector alone, for example [25, 31], pion and kaon masses, the average squared mass of the η and η' mesons $0.5\,(m_\eta^2 + m_{\eta'}^2)$, and two decay constants f_π and f_K. To fix the λ' coupling constant, which violates the OZI rule [32], some experimental information from the scalar sector is required, for example [25], the sigma meson mass. The other scalar masses, the scalar and pseudoscalar mixing angles, and the difference $m_{\eta'}^2 - m_\eta^2$ are then predicted quite reasonably [25, 31, 32].

To summarize, the linear sigma model is an attractive effective theory candidate for description of the low energy QCD dynamics. Phenomenologically, it is quite successful and explains various puzzles concerning scalar and pseudoscalar mesons [32]:

- why the pion and kaon are light
- why the η' is so heavy
- why the scalar mesons are much heavier than pseudoscalars
- why the sigma meson is so light compared to other scalars
- the pseudoscalar and scalar mixing angles
- the accidental degeneracy of the $a_0(980)$ and $f_0(980)$ mesons
- the strong coupling of $f_0(980)$ to $K\bar{K}$
- two photon widths of $a_0(980)$ and $f_0(980)$ mesons

In the next sections we will be interested in some qualitative features of the dynamics described by the linear sigma model. At that we will make further simplification by neglecting the effects of the strange quark. In the two flavor case, one can assume that the field Φ in the Lagrangian is the 2×2 complex matrix. However, SU(2) has a unique property among SU(N) groups, its fundamental representation being equivalent to its complex conjugate. Owing to this property, two linear combinations $\Phi + \tau_2 \Phi^* \tau_2$ and $\Phi - \tau_2 \Phi^* \tau_2$ both transform irreducibly under the $\mathrm{SU_R}(2) \times \mathrm{SU_L}(2)$ group [24]. Each of them has only two independent complex matrix elements. Therefore, it is possible to construct two flavor linear sigma models by using only four lightest mass eigenstates $\pi^\pm$, π^0 and σ. Hence we take

$$\Phi = \frac{1}{2}\sigma + \frac{i}{2}\,\vec{\pi}\cdot\vec{\tau},$$

τ_i being the Pauli matrices. Then

$$\Phi^+ \Phi = \frac{1}{4}\left(\sigma^2 + \vec{\pi}^{\,2}\right), \quad \mathrm{Sp}\,(\Phi^+ \Phi)^2 = \frac{1}{2}\left[\mathrm{Sp}\,(\Phi^+ \Phi)\right]^2 = \frac{1}{8}\left(\sigma^2 + \vec{\pi}^{\,2}\right)^2$$

and, therefore, the Lagrangian takes the form (up to the irrelevant constant term)

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu \sigma\, \partial^\mu \sigma + \frac{1}{2}\,\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - \frac{\lambda_s}{4}\left(\sigma^2 + \vec{\pi}^{\,2} - v^2\right)^2 + H\sigma, \tag{10}$$

here

$$\lambda_s = \lambda' + \frac{1}{2}\,\lambda, \quad v^2 = -\frac{m^2}{\lambda_s}, \quad H = h_0 \,.$$

This is the classic linear sigma model of Gell-Mann and Levy [33]. Its free parameters λ_s, v and H (the strength of the symmetry preserving term, the location of its minimum and the strength of the symmetry-breaking term) can be fixed by using pion and sigma masses and PCAC as follows [34]. In the chiral limit (then $H = 0$) the linear sigma model potential

$$V_S = \frac{\lambda_s}{4}\left(\sigma^2 + \vec{\pi}^{\,2} - v^2\right)^2$$

has a famous "Mexican hat" shape. Therefore, the chiral symmetry is spontaneously broken because the sigma field develops a nonzero vacuum expectation value $\langle\sigma\rangle = v$ (the pion field, being pseudoscalar, cannot acquire a nonzero vacuum expectation value without violating parity). The symmetry breaking term $V_{\mathrm{SB}} = -H\sigma$ tilts the Mexican hat and now $\langle\sigma\rangle = \sigma_0 \neq v$. Shifting the sigma field by its vacuum expectation value, $\sigma = \sigma_0 + \sigma'$, and isolating quadratic terms $\frac{m_\sigma^2}{2}\sigma'^{\,2}$ and $\frac{m_\pi^2}{2}\,\vec{\pi}^{\,2}$ in the potential $V_S + V_{\mathrm{SB}}$, we get meson masses

$$m_\pi^2 = \lambda_s(\sigma_0^2 - v^2)\,, \quad m_\sigma^2 = \lambda_s(3\sigma_0^2 - v^2)\,. \tag{11}$$

The vacuum expectation value σ_0 is determined from the condition

$$\left.\frac{\partial V\,(\sigma, \vec{\pi})}{\partial \sigma}\right|_{\vec{\pi}=0} = 0,$$

which gives

$$H = \lambda_s \sigma_0 (\sigma_0^2 - v^2) = \sigma_0 m_\pi^2. \tag{12}$$

Besides (11) and (12), we need one more relation to determine four quantities λ_s, v, H and σ_0. This relation is given by PCAC:

$$\partial^\mu \vec{J}_{5\mu} = f_\pi m_\pi^2\,\vec{\pi}. \tag{13}$$

Indeed, the axial-vector current $\vec{J}_{5\mu}$ is nothing but the Noether current associated with the $\mathrm{SU}_A(2)$ transformations

$$\varPhi \to \mathrm{e}^{i\frac{\tau_i}{2}\theta_i}\, \varPhi\, \mathrm{e}^{i\frac{\tau_i}{2}\theta_i}. \tag{14}$$

In terms of the σ and $\vec{\pi}$ fields, the infinitesimal form of Eq. (14) reads

$$\delta\sigma = -\pi_i\theta_i\,, \quad \delta\pi_i = \sigma\theta_i\,. \tag{15}$$

The divergence of $\vec{J}_{5\mu}$ is given by the Gell-Mann–Levy equation [27]

$$\partial^\mu J^i_{5\mu}(x) = -\frac{\partial(\delta\mathcal{L})}{\partial\theta_i(x)}\,,$$

where $\delta\mathcal{L}$ is the variation of the Lagrangian under (15) with space–time dependent parameters $\theta_i(x)$, which equals

$$\delta\mathcal{L} = \sigma\partial_\mu\pi_i\,\partial^\mu\theta_i - \pi_i\partial_\mu\sigma\,\partial^\mu\theta_i - H\pi_i\theta_i\,.$$

Therefore,

$$\partial^\mu J^i_{5\mu}(x) = -\frac{\partial(\delta\mathcal{L})}{\partial\theta_i(x)} = H\pi_i$$

and comparing with PCAC Eq. (13) we get

$$H = f_\pi m_\pi^2\,. \tag{16}$$

Now from (11), (12) and (16) it is easy to get

$$\sigma_0 = f_\pi\,, \quad \lambda_s = \frac{m_\sigma^2 - m_\pi^2}{2f_\pi^2}\,, \quad v^2 = \frac{m_\sigma^2 - 3m_\pi^2}{m_\sigma^2 - m_\pi^2}\, f_\pi^2\,, \quad H = f_\pi m_\pi^2\,. \tag{17}$$

The precise values of these parameters are largely immaterial having in mind idealized nature of the model. In any case, they can be estimated from (17) if needed. For example, for $m_\sigma = 600$ MeV one gets: $\lambda_s \sim 20$, $v \sim 90$ MeV and $H \sim (120\ \mathrm{MeV})^3$.

3. Disoriented chiral condensate

The linear sigma model potential in the limit $H \to 0$ has a degenerate minimum at $\sigma^2 + \vec{\pi}^2 = v^2$ (in this limit $m_\pi = 0$ and $v = f_\pi$). The vacuum state, we believe our world is based on, points in the σ-direction, $\langle\sigma\rangle = f_\pi$, $\langle\vec{\pi}\rangle = 0$, and, therefore, spontaneously violates the chiral symmetry. The natural question is whether one can change the vacuum state by some perturbation. The following analogy is helpful here: $\mathrm{SU}(2)\times\mathrm{SU}(2)$ is locally isomorphic to O(4); therefore, the order parameter $\langle \Phi \rangle$ of the linear sigma model can be considered as some analog of spontaneous magnetization of the O(4) Heisenberg ferromagnetic. Then, changing the vacuum state in the relativistic field theory, which assumes an infinite system, is analogous to rotating all spins in the infinite magnet simultaneously and is clearly impossible. Our universe, although not infinite, is quite large and hence at first sight we have no means to alter its vacuum state: only one QCD vacuum state is realized in our world, all other chirally equivalent vacuum states being unreachable and thus unphysical. However experience with real magnets suggests that this simple argument (as well as virtually all other no–go theorems) may point not so much to the real impossibility but to the need of more elaborate imagination. In the case of ferromagnet it is relatively simple to change the magnetization in some large enough volume. All what is needed is to apply an external magnetic field. Even such a comparatively weak field as Earth's magnetic field can do the job. We are tempting here to indicate one interesting application of this effect [35]. Above the Curie temperature the rotational invariance is restored in the ferromagnet and there is no spontaneous magnetization — all record of the previous magnetization is lost. As lava from a volcano cools below the Curie temperature the Earth's magnetic field aligns the magnetization of the ferromagnetic grains. By studying such solidified lavas (basalt rocks), geophysicists have reconstructed a history of the Earth's magnetic field with a striking result that the Earth's magnetic field has flip-flopped many times, once in every half million years, on the average. But this is not the most interesting part of the story. Investigation of the ocean floor magnetization revealed a surprising strip structure. Successive strips of normally and reversely magnetized rock lied symmetrically on both sides of the volcanic mid-Atlantic ridge. The explanation of this enigma comes from plate tectonics. On each side of the ridge the tectonic plates are pulled away, one of it towards Europa and the other towards America. Lava, emerges from the middle, solidifies, sticks to the plates and is also pulled away with the magnetic field orientation recorded in it. So the oceanic floor seems to be a gigantic tape recorder for reversals of the Earth's magnetic field! This discovery was crucial in recognition of Alfred Wegener's theory of continental drift — the idea which initially was met with enormous resistance from geophysicists.

Long ago Lee and Wick argued [36] that an analogous domain formation phenomenon is also possible in the case of quantum field theory with degenerate vacuum and in principle there should exist a possibility of flipping the ordinary vacuum in a limited domain of space to an abnormal one. "The experimental method to alter the properties of the vacuum may be called vacuum engineering" [37]. It seems that a new generation of the very high energy heavy ion and hadron colliders may provide a practical tool for such vacuum engineering. The scientific significance of this possibility can hardly be overestimated, because "if indeed we are able to alter the vacuum, then we may encounter some new phenomena, totally unexpected" [37].

Disoriented chiral condensate formation is one of the new phenomena which may happen in very high energy collisions [38, 39]. In such a collision there is some probability that a high multiplicity final state will be produced with high entropy. Collision debris form a hot shell expanding in all directions nearly at the velocity of light. This shell effectively shields the inner region up to hadronization time and then it breaks up into individual hadrons. The hadronization time can be quite large [40] and during all this time the inner region has no idea about the chiral orientation of the normal, outside vacuum. Therefore, if the inner vacuum is perturbed enough in first instants of the collision to forget its orientation, then almost certainly it will relax back in the ground state other than the σ-direction. Of course, the explicit symmetry breaking ($\sim H$) term lifts the vacuum degeneracy. However, the corresponding tilting of the "Mexican hat" is small and will not effect the initial stage evolution significantly [41]. Therefore, it is not unlikely that some high energy collisions can lead to the formation of relatively large space domains where the chiral condensate is temporarily disoriented. At later times such Disoriented Chiral Condensate will relax back to the normal vacuum by emitting coherent burst of pion radiation.

But how can the initial vacuum be excited? A short time after the collision of the order of 0.3–0.8 fm$/c$ the energy density in the interior of the collision region drops enough to make meaningful the introduction of σ and π collective modes [42]. After this time the classical dynamics of the system is reasonably well described by the linear sigma model. However, initially the σ and π fields are surrounded by a thermal bath. So we need the sigma model at finite temperature. To reveal a simple physical picture behind the phenomenon, we will use the following simplified approach [43, 44]. Let us decompose fields into the slowly varying classical part (the condensate) and high frequency thermal fluctuations

$$\phi(x) = \phi_{\rm cl}(x) + \delta\phi(x)\,.$$

By definition the thermal average $\langle\phi\rangle_{\rm th} = \phi_{\rm cl}$ and $\langle\delta\phi\rangle_{\rm th} = 0$. Therefore the thermal averaged symmetric potential, which determines evolution of $\phi_{\rm cl}$ at initial times, until the effects of the explicit symmetry breaking term become significant, has the form (we have suppressed isospin indices for a moment)

$$\langle V_S\rangle_{\rm th} = \frac{\lambda_s}{4}\left(\phi_{\rm cl}^2 + \langle(\delta\phi)^2\rangle_{\rm th} - v^2\right)^2 .$$

To calculate $\langle(\delta\phi)^2\rangle_{\rm th}$, let us decompose $\delta\phi(x)$ into the annihilation and creation operators

$$\delta\phi(x) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a(\vec{k}) e^{-ik\cdot x} + a^+(\vec{k}) e^{ik\cdot x} \right), \qquad (18)$$

with $\omega_k = \sqrt{\vec{k}^2 + m^2}$ and (our normalization corresponds to $(2\pi)^{-3}$ particles per unit volume)

$$[a(\vec{k}), a^+(\vec{k'})] = \delta(\vec{k} - \vec{k'}) \, .$$

At the thermal equilibrium the thermal bath is homogeneous over the (large) spatial volume V. Therefore,the thermal fluctuations are the same at every point inside V and $\langle(\delta\phi)^2\rangle_{\rm th}$ can be replaced by its spatial average

$$\langle(\delta\phi)^2\rangle_{\rm th} \to \frac{1}{V}\int d\vec{x} \langle(\delta\phi)^2\rangle_{\rm th} \, .$$

Substituting here (18) we get

$$\langle(\delta\phi)^2\rangle_{\rm th} \to \frac{1}{V}\int \frac{d\vec{k}}{2\omega_k} \langle a(\vec{k})a^+(\vec{k}) + a^+(\vec{k})a(\vec{k})\rangle_{\rm th} . \qquad (19)$$

We assumed that the chemical potential of the field ϕ is small, so that the probability of finding its two quanta simultaneously in a unit volume is negligible, and hence

$$\langle a(\vec{k})a(-\vec{k})\rangle_{\rm th} \approx 0 \, , \quad \langle a^+(\vec{k})a^+(-\vec{k})\rangle_{\rm th} \approx 0 \, .$$

However,

$$\langle aa^+ + a^+a\rangle_{\rm th} = 2\langle a^+a\rangle_{\rm th} + [a, a^+]$$

and the second term gives a temperature independent constant. Actually this contribution in (19) is infinite and should be cured by renormalization (that is subtracted). The nontrivial finite part is

$$\langle(\delta\phi)^2\rangle_{\rm th} = \frac{1}{V}\int \frac{d\vec{k}}{\omega_k} \langle a^+(\vec{k})a(\vec{k})\rangle_{\rm th} .$$

However, $a^+(\vec{k})a(\vec{k})$ is the number density operator (in momentum space). Hence, its thermal average is given by the Bose–Einstein distribution

$$\langle a^+(\vec{k})a(\vec{k})\rangle_{\rm th} = \frac{N}{e^{\omega_k/T} - 1} \, ,$$

where $N = V/(2\pi)^3$ is the total number of ϕ-particles in the volume V. Finally,

$$\langle(\delta\phi)^2\rangle_{\rm th} = \int \frac{d\vec{k}}{(2\pi)^3} \, \frac{1}{\omega_k(e^{\omega_k/T} - 1)} \, . \qquad (20)$$

In a high temperature limit $T \gg m$, (20) is simplified to

$$\langle(\delta\phi)^2\rangle_{\rm th} = \int \frac{d\vec{k}}{(2\pi)^3}\, \frac{1}{|\vec{k}|(\mathrm{e}^{|\vec{k}|/T}-1)} = \frac{T^2}{2\pi^2}\int\limits_0^\infty \frac{x\,dx}{\mathrm{e}^x-1} = \frac{T^2}{12}\,. \tag{21}$$

Let us now restore the isotopic content of our theory. Each isotopic mode gives a contribution (20) to the effective thermal potential. However σ-meson is too heavy. Therefore, we assume $\omega_k/T \gg 1$ for it and neglect its contribution. There remain three pionic modes. Pions, on the contrary, are light and we neglect their masses. Then the thermal effective potential takes the form

$$\langle V_S\rangle_{\rm th} = \frac{\lambda_s}{4}\left(\sigma^2+\vec{\pi}^2+\frac{T^2}{4}-v^2\right)^2 . \tag{22}$$

The minimum energy configuration corresponds to

$$\langle\sigma\rangle = \sqrt{v^2-\frac{T^2}{4}}\,.$$

Therefore, the σ-condensate completely melts down at $T_{\rm c}=2v\approx 180$ MeV. Above this phase transition point the vacuum configuration corresponds to $\langle\sigma\rangle\approx 0$. In fact, the σ-condensate never melts completely down (for temperatures for which the linear sigma model still makes sense), because of the $\sim H$ term. However, near the critical temperature this residual value of the σ-condensate (which minimizes $V\approx\frac{\lambda_s}{4}\sigma^4-H\sigma$) is quite small

$$\langle\sigma\rangle \approx \left(\frac{4H}{\lambda_s}\right)^{1/3} \approx 3\ \mathrm{MeV} \ll f_\pi .$$

Temperatures of the order $T_{\rm c}$ can be reached in very high energy collisions. Then, in some small volume, chiral condensate is melted and all information about the "correct" orientation of the chiral order parameter is lost. What happens when this volume cools down? Again an analogy with magnets is helpful. If a magnet is heated above the Curie temperature and then slowly cooled, it loses its spontaneous magnetization. This happens because many small domains are formed with magnetization direction changing at random from domain to domain, so that there is no net magnetization. Therefore, if we want to have a large DCC domain, slow cooling in thermal equilibrium is not the best choice. Indeed, it was argued [45] that in such circumstances the size of DCC domains remains small. Hopefully, the interior of the collision fireball is cooled very rapidly due to fireball expansion. Rajagopal and Wilczek found [34,45] that in such an out of equilibrium process larger DCC domains can be formed. This is analogous to the quenching technique in magnet production from a melted alloy. The physical mechanism which operates here is the following [41]. After a quench, the temperature suddenly drops to zero and, therefore, the dynamics will be governed by the zero temperature Lagrangian. If the cooling process is very rapid, the field configuration does not have time to follow the sudden change in the environment. Therefore, immediately after the quench fields do not have vacuum expectation values. Hence, the system finds itself in a strongly out of equilibrium situation, namely near the top of the "Mexican hat". The vacuum expectation values will begin to develop while the system is rolling down towards the valley of the symmetric potential, but this will take some time. Meanwhile the Goldstone modes (pions) will be tachyonic:

$$m_\pi^2 = \lambda_s \left(\langle\sigma\rangle^2 - v^2 \right) < 0, \tag{23}$$

if $\langle\sigma\rangle$ is small. Therefore, the oscillation frequencies $\omega_k = \sqrt{\vec{k}^2 + m_\pi^2}$ will be imaginary for long enough wavelengths and they will grow with time exponentially. The zero mode is the one which is amplified most effectively. As a result, a large sized correlated region will be formed with nearly a uniform field. When the fields approach the bottom of the potential and $\langle\sigma\rangle$ gets close to its zero temperature value, this mechanism ceases to operate. Therefore, a natural question is how fast the rolling down takes place and whether the zero mode has enough time to be significantly amplified. To answer this question, one should consider the evolution of the σ and $\vec{\pi}$ fields, according to the linear sigma model. During this evolution we have

$$\partial^\mu \vec{J}_\mu = 0, \quad \partial^\mu \vec{J}_{5\mu} = H\vec{\pi}. \tag{24}$$

Derivation of the second equation (PCAC) was given earlier. At that the axial-vector current is

$$J_{5\mu}^i = \frac{\partial(\delta\mathcal{L})}{\partial(\partial^\mu\theta_i)} = \sigma\partial_\mu\pi_i - \pi_i\partial_\mu\sigma.$$

The conserved vector current $\vec{J}_\mu$ is the Noether current associated with the $\mathrm{SU}_V(2)$ transformations from (7) and a similar procedure will give [27]

$$\vec{J}_\mu = \vec{\pi} \times \partial_\mu\vec{\pi}.$$

To make the problem analytically tractable, we idealize the initial conditions and assume that the whole collision energy is initially localized in the infinitesimally thin pancake to an infinite transverse extent [46]. Then the fields can depend only on the longitudinal coordinate x. Besides, such initial conditions are invariant under the longitudinal boosts. Therefore, in fact the fields can only depend on the proper time $\tau=\sqrt{t^2-x^2}$. Then $\partial_\mu=\frac{x_\mu}{\tau}\frac{d}{d\tau}$ with the (Minkowskian) 2-vector $x^\mu=(t,x)$. Therefore,

$$\vec{J}_\mu=\frac{x_\mu}{\tau}\,\vec{\pi}\times\dot{\vec{\pi}},\quad \vec{J}_{5\mu}=\frac{x_\mu}{\tau}\left(\vec{\pi}\dot{\sigma}-\sigma\dot{\vec{\pi}}\right),$$

where the dot denotes differentiation with respect to τ. Using $\partial^\mu x_\mu=2$, we get

$$\partial^\mu\vec{J}_\mu=\frac{2}{\tau}\,\vec{\pi}\times\dot{\vec{\pi}}+\tau\frac{d}{d\tau}\left(\frac{\vec{\pi}\times\dot{\vec{\pi}}}{\tau}\right)=\frac{1}{\tau}\frac{d}{d\tau}\left(\tau\vec{\pi}\times\dot{\vec{\pi}}\right)$$

and

$$\partial^\mu\vec{J}_{5\mu}=\frac{1}{\tau}\frac{d}{d\tau}\left[\tau\left(\vec{\pi}\dot{\sigma}-\sigma\dot{\vec{\pi}}\right)\right].$$

Therefore, the conservation of the vector current and PCAC (24) imply

$$\vec{\pi}\times\dot{\vec{\pi}}=\frac{\vec{a}}{\tau},\quad \vec{\pi}\dot{\sigma}-\sigma\dot{\vec{\pi}}=\frac{\vec{b}}{\tau}+\frac{H}{\tau}\int\limits_{\tau_0}^{\tau}\tau'\vec{\pi}(\tau')\,d\tau'\,, \tag{25}$$

with $\vec{a}$ and $\vec{b}$ as integration constants. Initially, far from the valley of the symmetric potential, the symmetry breaking term $H\vec{\pi}$ plays an insignificant role and can be neglected. Then (25) shows that $\vec{a}\cdot\vec{b}=0$ and the triad $\vec{a},\ \vec{b},\ \vec{c}=\vec{a}\times\vec{b}$ forms a convenient axis for decomposition of isovectors. The first equation of (25) indicates that $\pi_a=0$ and, hence, (25) is equivalent to the system

$$\pi_b\dot{\pi}_c-\pi_c\dot{\pi}_b=\frac{a}{\tau},\ \ \pi_b\dot{\sigma}-\sigma\dot{\pi}_b=\frac{b}{\tau},\ \ \pi_c\dot{\sigma}-\sigma\dot{\pi}_c=0\,. \tag{26}$$

Because of the last equation, the motion in the (π_b,π_c,σ)-space is planar

$$\frac{\pi_c}{\sigma}=k=\mathrm{const.}$$

Then, the first two equations give

$$k=\frac{a}{b}\,.$$

To simplify the discussion, we assume $b\gg a$. Then $\pi_c\approx 0$ and the motion plane coincides with the (π_b,σ) plane. Let us introduce the polar coordinates in this plane

$$\pi_b=f\sin\theta\,,\quad \sigma=f\cos\theta\,. \tag{27}$$

Then, (26) gives

$$f^2\dot{\theta} = -\frac{b}{\tau}\,. \tag{28}$$

The equation for the radial coordinate f can be derived from the equation of motion

$$\Box\vec{\pi} = -\lambda_s\left(\sigma^2 + \vec{\pi}^2 - v^2\right)\vec{\pi}$$

with $\Box = \partial_\mu\partial^\mu$ as the d'Alembertian. In our case, this equation is equivalent to

$$\frac{1}{\tau}\frac{d}{d\tau}\left(\tau\frac{d\pi_b}{d\tau}\right) = -\lambda_s\left(f^2 - v^2\right)\pi_b\,.$$

Substituting here (27) and using (28), we get the radial equation

$$\ddot{f} + \frac{\dot{f}}{\tau} = \frac{b^2}{f^3\tau^2} - \lambda_s\left(f^2 - v^2\right)f\,. \tag{29}$$

At later times the difference $g = (f-v)/v$ is expected to be small. Therefore,

$$\left(f^2 - v^2\right)f \approx 2v^3 g$$

and (29) reduces to

$$\ddot{g} + \frac{\dot{g}}{\tau} = \frac{b^2}{v^4\tau^2} - 2\lambda_s v^2 g\,.$$

Introducing a new dimensionless variable $s = \sqrt{2\lambda_s}v\tau$ (note that in the $H = 0$ limit $m_\sigma = \sqrt{2\lambda_s}v$), we get the inhomogeneous Bessel equation

$$s^2\frac{d^2g}{ds^2} + s\frac{dg}{ds} + s^2 g = \left(\frac{b}{v^2}\right)^2. \tag{30}$$

Therefore, the solution can be expressed through the Bessel functions $J_0(s)$ and $Y_0(s)$. Hence, for large proper times it will exhibit a damped oscillatory behavior. For example, for large $s \gg 1$,

$$J_0(s) \approx \sqrt{\frac{2}{\pi s}}\cos\left(s - \frac{\pi}{4}\right).$$

Large enough compared to what number? The inhomogeneous term in (30), which is a reminiscence of the influence of the angular motion on the radial motion, is characterized by a dimensionless number b/v^2, which we assume to be much greater than one. Therefore, the asymptotic value of s can be estimated to be $s \sim b/v^2$, which translates into the proper time

$$\tau_{\mathrm{R}} \sim \frac{b}{\sqrt{2\lambda_s v^3}}. \tag{31}$$

This gives us an estimate of the rolling-down time.

Let us now consider the process of formation and growth of correlated domains in a scalar quantum field theory after a quench from an equilibrium disordered initial state at the temperature $T_i = T$ to a final state at $T_f \approx 0$ [47]. For unstable modes the instantaneous quench will be mimicked by a time dependent mass

$$m^2(t) = m_i^2\, \Theta(-t) - m_f^2\, \Theta(t),$$

which is tachyonic at $t > 0$. In the decomposition

$$\phi(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a(\vec{k}) u_k(t) \mathrm{e}^{i\vec{k}\cdot\vec{x}} + a^+(\vec{k}) u_k^*(t) \mathrm{e}^{-i\vec{k}\cdot\vec{x}} \right), \tag{32}$$

the corresponding mode functions $u_k(t)$ obey

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + m^2(t) \right] u_k(t) = 0$$

and initially (for $t < 0$) we have $u_k(t) = \mathrm{e}^{-i\omega_k t}$, $\omega_k = \sqrt{m_i^2 + \vec{k}^2}$. For $t > 0$ the solution is

$$u_k(t) = A_k \mathrm{e}^{\alpha_k t} + B_k \mathrm{e}^{-\alpha_k t}, \tag{33}$$

with $\alpha_k = \sqrt{m_f^2 - \vec{k}^2}$ (we will concentrate on unstable modes so that $\vec{k}^2 < m_f^2$). Matching the $t > 0$ and $t < 0$ solutions and their first derivatives at $t = 0$, we can determine the A_k and B_k coefficients

$$A_k = \frac{1}{2} \left(1 - i\, \frac{\omega_k}{\alpha_k} \right), \quad B_k = \frac{1}{2} \left(1 + i\, \frac{\omega_k}{\alpha_k} \right). \tag{34}$$

The information about the domain size is encoded in the equal time correlation function (spatially averaged over the volume V)

$$G(\vec{x},t) = \frac{1}{V} \int d\vec{y} \langle \phi(\vec{x}+\vec{y},t) \phi(\vec{y},t) \rangle_{\mathrm{th}}. \tag{35}$$

Indeed, if x is not greater than the domain size L_{D}, then $\phi(\vec{x}+\vec{y},t)\phi(\vec{y},t) \approx \phi^2(\vec{y},t)$ and the integral (35) should be near its maximal value $G(\vec{0},t)$. On the contrary, if $x \gg L_{\rm D}$, then the integral (35) averages to zero. Substituting (32) into (35) and remembering that for our approximations

$$\langle aa\rangle_{\rm th} \approx 0, \quad \langle a^+a^+\rangle_{\rm th} \approx 0,$$

we get

$$\begin{aligned} G(\vec{x},t) &= \frac{1}{V}\int \frac{d\vec{k}}{2\omega_k}|u_k(t)|^2 \left[\langle a(\vec{k})a^+(\vec{k})\rangle_{\rm th}{\rm e}^{i\vec{k}\cdot\vec{x}} + \langle a^+(\vec{k})a(\vec{k})\rangle_{\rm th}{\rm e}^{-i\vec{k}\cdot\vec{x}}\right] \\ &= \frac{1}{V}\int \frac{d\vec{k}}{2\omega_k}|u_k(t)|^2 \left[\langle a(\vec{k})a^+(\vec{k})\rangle_{\rm th} + \langle a(-\vec{k})a^+(-\vec{k})\rangle_{\rm th} + \delta(\vec{0})\right]{\rm e}^{i\vec{k}\cdot\vec{x}}\,. \end{aligned} \tag{36}$$

To understand the meaning of $\delta(\vec{0})$, let us return one step backward and write

$$\begin{aligned} \frac{|u_k(t)|^2}{2\omega_k}\,\delta(\vec{0}) &= \int \frac{d\vec{k'}}{(2\pi)^3}\,\delta(\vec{k}-\vec{k'})\,{\rm e}^{-i\vec{y}\cdot(\vec{k}-\vec{k'})}\,\frac{u_k^*(t)}{\sqrt{2\omega_k}}\,\frac{u_{k'}(t)}{\sqrt{2\omega_{k'}}}\,d\vec{y} \\ &= \frac{|u_k(t)|^2}{2\omega_k}\int\frac{d\vec{y}}{(2\pi)^3} = \frac{|u_k(t)|^2}{2\omega_k}\,\frac{V}{(2\pi)^3}\,. \end{aligned}$$

Besides, as was explained above,

$$\langle a^+(\vec{k})a(\vec{k})\rangle_{\rm th} = \frac{N}{{\rm e}^{\omega_k/T}-1}, \quad N = \frac{V}{(2\pi)^3}\,.$$

Therefore, (36) takes the form

$$G(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^3}\,\frac{1}{2\omega_k}\,|u_k(t)|^2\,\coth\left(\frac{\omega_k}{2T}\right){\rm e}^{i\vec{k}\cdot\vec{x}}\,.$$

After the quench $T \approx 0$ and hence $\coth\left(\frac{\omega_k}{2T}\right) \approx 1$. To study the growth of domains, one should subtract the contributions that were already present before the quench [47]

$$\tilde{G}(\vec{x},t) = G(\vec{x},t) - G(\vec{x},0) = \int \frac{d\vec{k}}{(2\pi)^3}\,\frac{1}{2\omega_k}\,\left[|u_k(t)|^2 - 1\right]\,{\rm e}^{i\vec{k}\cdot\vec{x}}\,.$$

But (33) and (34) imply

$$|u_k(t)|^2 = \frac{1}{2}\left(1+\frac{\omega_k^2}{\alpha_k^2}\right)\cosh\left(2\alpha_k t\right) + \frac{1}{2}\left(1-\frac{\omega_k^2}{\alpha_k^2}\right).$$

Therefore, the contribution of unstable modes in the domain growth is controlled by

$$\tilde{G}(\vec{x},t) = \frac{1}{2}\int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{2\omega_k}\left(1+\frac{\omega_k^2}{\alpha_k^2}\right)\left[\cosh\left(2\alpha_k t\right)-1\right]\mathrm{e}^{i\vec{k}\cdot\vec{x}}$$
$$= \frac{1}{4\pi^2 r^2}\int\limits_0^{m_f}(kr)\sin\left(kr\right)\frac{1}{\omega_k}\left(1+\frac{\omega_k^2}{\alpha_k^2}\right)\sinh^2\left(\alpha_k t\right)dk\,, \quad (37)$$

where $r=|\vec{x}|$ and $k=|\vec{k}|$. We are interested in the large t asymptote of this function. Then

$$\sinh^2\left(\alpha_k t\right)\to \mathrm{e}^{2\alpha_k t}$$

and

$$\tilde{G}(\vec{x},t)\to\frac{m_i^2+m_f^2}{16\pi^2 r^2}\int\limits_0^{m_f}dk\,\frac{\sin\left(kr\right)}{\omega_k\alpha_k^2}\,\mathrm{e}^{g(k)}\,,$$

where

$$g(k)=2\alpha_k t+\ln\left(kr\right).$$

Note that $g(k)$ has a maximum at $k_0\approx\sqrt{\frac{m_f}{2t}}$ and

$$g''(k_0)\approx-\frac{4t}{m_f}.$$

Therefore

$$g(k)\approx g(k_0)-\frac{2t}{m_f}\left(k-k_0\right)^2$$

and for large t the function $\mathrm{e}^{g(k)}$ has a very sharp peak at $k=k_0$. Near this point $(\omega_k\alpha_k^2)^{-1}$ is a slowly varying function. Therefore

$$\tilde{G}(\vec{x},t)\to\frac{k_0}{16\pi^2 r}\,\frac{m_i^2+m_f^2}{m_i m_f^2}\,\mathrm{e}^{2m_f t}\,I,\quad I=\int\limits_0^{m_f}dk\,\mathrm{e}^{-\frac{2t}{m_f}\left(k-k_0\right)^2}\sin\left(kr\right),$$

where we have used

$$\frac{1}{\omega_{k_0}\alpha_{k_0}^2}\approx\frac{1}{m_i m_f^2}\,,\quad g(k_0)\approx 2m_f t+\ln\left(k_0 r\right).$$

The remaining integral

$$I\approx\int\limits_{-\infty}^{\infty}dk\,\mathrm{e}^{-\gamma\left(k-k_0\right)^2}\sin\left(kr\right)=\sqrt{\frac{\pi}{\gamma}}\sin\left(k_0 r\right)\mathrm{e}^{-\frac{r^2}{4\gamma}}\,,\;\;\gamma=\frac{2t}{m_f}.$$

As we can see [47, 48]

$$\tilde{G}(\vec{x},t) \approx \tilde{G}(\vec{0},t)\, \frac{\sin(k_0 r)}{k_0 r}\, \mathrm{e}^{-\frac{m_f r^2}{8t}} \,.$$

Therefore, the domain size grows with time, according to the Cahn–Allen scaling relation [48, 49]

$$L_{\mathrm{D}}(t) = \sqrt{\frac{8t}{m_f}} \,. \tag{38}$$

Remembering our estimate (31) for the rolling-down time and taking a mean value of (23) as an estimate for m_f^2: $m_f^2 = \frac{1}{2}\lambda_s v^2$, we get

$$L_{\mathrm{D}} = \frac{1}{v}\sqrt{\frac{8b}{\lambda_s v^2}} \approx 1.4\ \mathrm{fm}\ \sqrt{\frac{b}{v^2}} \,. \tag{39}$$

Assuming Gaussian initial fluctuations of the fields and of their derivatives, it can be shown [46] that the probability of the initial strength b of the axial-vector current to be large is exponentially suppressed. Therefore, our estimate (39) shows that typically DCC domains are quite small, unless b is large enough in some rare occasions. One concludes that "formation of an observable DCC is likely to be a rather natural but rare phenomenon" [50].

To close this section, let us indicate some review articles about the DCC phenomenon [40, 42, 50, 51], where an interested reader can find further discussions and references to the original literature, which is quite numerous.

4. DCC at a photon collider

Usually DCC formation is considered in the context of heavy ion or hadron–hadron collisions. We see no reason why gamma–gamma collisions have to be discriminated in this respect. The basic interaction for the photon is of course an electromagnetic interaction with charged particles, and to lowest order in α the photon appears as a point-like particle in its interactions. However, according to quantum field theory, the photon may fluctuate into a virtual charged fermion–antifermion pair. In this way strong interactions come into play through quark–antiquark fluctuations. While the high-virtuality part of such quark–antiquark fluctuations can be calculated perturbatively, the low-virtuality part cannot. The latter is usually described phenomenologically by a sum over low-mass vector-meson states — the Vector Meson Dominance (VMD) ansatz. Therefore, the effective photon state vector has the form [52, 53]

$$|\gamma\rangle = \sqrt{Z_3}|\gamma_{\mathrm{bare}}\rangle + \sum_V c_V|V\rangle + \sum_q c_q|q\bar{q}\rangle + \sum_l c_l|l^+l^-\rangle \,, \tag{40}$$

where [52]

$$c_V^2 = \frac{4\pi\alpha}{f_V^2}\,, \quad \frac{f_\rho^2}{4\pi} \approx 2.2\,, \quad \frac{f_\omega^2}{4\pi} \approx 23.6\,, \quad \frac{f_\phi^2}{4\pi} \approx 18.4\,, \quad \frac{f_{J/\Psi}}{4\pi} \approx 11.5\,.$$

The coefficients of the perturbative $|q\bar{q}\rangle$ part depend on the scale μ at that the photon is probed. Namely [52]

$$c_q^2 = \frac{\alpha}{\pi} e_q^2 \ln \frac{\mu^2}{k_0^2}\,,$$

where $|e_q| = \frac{1}{3}, \frac{2}{3}$ is the quark charge and k_0 is an unphysical parameter separating the low- and high-virtuality parts of the quark–antiquark fluctuations.

The last term in (40) describes fluctuations into lepton pairs and is uninteresting inasmuch as the hadronic final state is concerned. The coefficient of the first bare-photon term is given by

$$Z_3 = 1 - \sum_V c_V^2 - \sum_q c_q^2 - \sum_l c_l^2$$

and is close to unity.

Therefore, for some fraction of time the photon behaves like a hadron. This fraction is quite small, about 1/400 [54], but for hadronic final states this smallness is overcompensated by the fact that in its hadron facet the photon experiences strong interactions.

According to (40), one has six different types of possible interactions in the high-energy photon–photon collisions [52]:

- Both photons turn into hadrons (vector mesons) and the partons of these hadrons interact with each other.
- One photon turns into a hadron and its partons interact with the quark–antiquark fluctuation of another photon.
- Both photons fluctuate perturbatively into quark–antiquark pairs and subsequently these fluctuations interact with each other.
- A bare photon interacts with the partons of the hadron which another photon was turned into.
- A bare photon interacts with the quark–antiquark fluctuation of another photon.
- Bare photons interact directly in a hard process.

In fact, in the total hadronic cross sections, the first two event classes dominate, the bulk of the contribution coming from the $\rho^0\rho^0$ component of the first class [52]. Therefore, high-energy photon–photon collisions are very much similar to the hadron–hadron collisions, and if DCC can be formed in the latter case, it will be formed also in the former case. Of course, the photon–photon cross section is strongly reduced compared to the hadron–hadron cross sections (about 10^5 times). However, it is not improbable that a great deal of this smallness is overcome by somewhat more favorable conditions for the DCC formation in the gamma–gamma collisions than in the proton–(anti)proton collisions. The argument goes as follows. As has been mentioned above, for boost-invariant initial conditions, when the field depends only on the proper time $\tau = \sqrt{t^2 - x^2}$, the d'Alembertian equals

$$\Box = \frac{1}{\tau}\,\frac{d}{d\tau}\left(\tau\frac{d}{d\tau}\right) = \frac{d^2}{d\tau^2} + \frac{1}{\tau}\,\frac{d}{d\tau}\,.$$

The second term describes the decrease of energy in a covolume due to longitudinal expansion and brings an effective "friction", which is necessary for quenching, into the equation of motion [46, 55]. The transverse ($D = 2$) and spherical ($D = 3$) expansions can be modeled analogously if one assumes that the field depends only on $\tau = \sqrt{t^2 - \sum\limits_{i=1}^{D} x_i^2}$. Then

$$\Box = \frac{d^2}{d\tau^2} + \frac{D}{\tau}\,\frac{d}{d\tau}\,.$$

Therefore, the larger is D the more efficient is the quenching and the spherical expansion seems to be the most favorable for pion zero mode amplification [55, 56]. This simple observation is confirmed by a more detailed study [56]. However, to organize an isotropically expanding fireball is not a trivial task even in head-on hadron–hadron collisions. Constituent quarks inside hadrons become "black" at high energies, and for the projectile remnants not to spoil the isotropic expansion, one may wait for a rare event when these black disks inside the projectiles are aligned [57]. The probability that all six constituents are aligned in colliding protons during a head-on collision is $p_1 \sim \left(\frac{r_q^2}{r_p^2}\right)^5$, while the analogous probability for the four constituents of $\rho^0\rho^0$ collisions is $p_2 \sim \left(\frac{r_q^2}{r_\rho^2}\right)^3$. Taking $r_q \approx \frac{1}{2} r_\rho \approx \frac{1}{3} r_p$, we get $p_2/p_1 \approx 10^3$. Therefore, photon–photon collisions seem to be more favorable in this respect.

Even if the "right" fireball is prepared, the odds of the large DCC domain formation are usually small. In [55] this probability was found to be about 10^{-3}. Remember that our preceding considerations indicate that one needs large initial strength of the axial-vector SU(2)-current. In gamma–gamma collisions this initial strength is expected to be enhanced due to chiral anomaly effects, analogous to what was considered in [58] for heavy-ion collisions. The effects of chiral anomaly can be incorporated in the linear sigma model by adding the following interaction Lagrangian

$$\mathcal{L}_{\rm anom.} = \frac{\alpha}{4\pi f_\pi}\,\pi^0 \epsilon_{\mu\nu\sigma\tau} F^{\mu\nu} F^{\sigma\tau} = \frac{\alpha}{\pi f_\pi}\,\pi^0 \vec{E}\cdot\vec{H}\,.$$

Under $\mathrm{SU}_A(2)$ transformations (15), we have

$$\delta\mathcal{L}_{\rm anom.} = \frac{\alpha}{\pi f_\pi}\,\vec{E}\cdot\vec{H}\,\sigma\,\theta_3\,.$$

Therefore, according to the Gell-Mann–Levi equation (we neglect explicit symmetry breaking):

$$\partial^\mu J^3_{5\mu}(x) = -\frac{\partial(\delta\mathcal{L})}{\partial\theta_3(x)} = -\frac{\alpha}{\pi f_\pi}\,\vec{E}\cdot\vec{H}\,\sigma$$

and the axial-vector current is no longer conserved even in the limit of zero quark masses. As a result, electromagnetic fields can lead to the enhancement of the axial-vector current strength in the π^0-direction. The corresponding induced strength equals

$$b_3 = -\frac{\alpha}{\pi f_\pi}\int E(\vec{\tau})\cdot H(\vec{\tau})\,\sigma(\tau)\,\tau d\tau. \tag{41}$$

As was shown in [58], the expected effects are small in relativistic heavy-ion collisions, but nevertheless this initial small "kick" can have substantial effect on the DCC formation. Unfortunately we can not use (41) to estimate how big is the kick in gamma–gamma collisions — the application of the sigma model makes sense only after some time after the collision, while the effects of the chiral anomaly on the b_3 magnitude are confined to the first instants of the collision.

It was suggested [59, 60] that in hadron–hadron collisions DCC could be formed through the "Baked Alaska" scenario. However, as we have mentioned above, the considerable part of the $\gamma\gamma\to$ hadrons cross section is due to the $\rho^0\rho^0$ mechanism. Therefore, the Baked Alaska model should work for gamma–gamma collisions too. So let us take a closer look at it.

Normally Baked Alaska is a delightful dessert where ice cream is covered by meringue and then baked very quickly in a hot oven without melting the ice cream (you can find the recipe in the appendix. Try it and enjoy).

In physics context, however, this term firstly appeared as denoting a model for nucleation of the B phase of superfluid ^{3}He inside the supercooled A phase [61–63]. The surface tension at the boundary between the A and B phases is anomalously large. Therefore usual small bubbles of the B phase, created inside the A phase by thermal fluctuations, are energetically not profitable. Hence, they shrink and vanish. Only for a very large bubble the volume energy gain overcomes the surface energy and the bubble begins to grow. However, it is virtually impossible to create such a gigantic critical bubble by thermal fluctuations. The experiment, nevertheless, discovered a high enough nucleation rate. To explain the puzzle, Leggett suggested that the nucleation was assisted by cosmic rays [61]. Secondary electrons from the passage of a cosmic-ray muon through the liquid create hot spots in ^{3}He by depositing several hundred eV energy in small volumes. Inside such a "fireball" the Cooper pairs of the ^{3}He atoms are broken and, therefore, the normal, Fermi-liquid phase of the ^{3}He is restored. Yet, the fireball expands quickly and becomes a "Baked Alaska": a cold core surrounded by a hot, thin shell of normal fluid. There is some probability that after the core is cooled below the superfluidity phase transition temperature, it finds itself in the B phase. This B phase core can expand to larger than the critical radius because for some time it is shielded from the A phase bulk by the expanding hot shell, thereby eliminating surface energy price of the A–B boundary layer. When the shielding shell finally disappears, the B phase bubble is larger than the critical one and, therefore, expands further until it fills the whole vessel.

Let us return to Baked Alaskas produced by high-energy gamma–gamma collisions. Suppose DCC is formed inside the Baked Alaska core with the misalignment angle θ. That is inside the DCC region one has

$$\langle\sigma\rangle_{\mathrm{DCC}} = f_\pi \cos\theta, \quad \langle\vec{\pi}\rangle_{\mathrm{DCC}} = f_\pi \sin\theta\, \vec{n},$$

$\vec{n}$ being a unit vector in isospin space. Outside the fireball one has the normal vacuum:

$$\langle\sigma\rangle = f_\pi, \quad \langle\vec{\pi}\rangle = 0.$$

Finally, when the shielding shell of hot hadronic matter disappears, the DCC relaxes to this outside normal vacuum by emitting coherent low energy pions. Hadronization of the shell also produces mainly pions and, therefore, generates a background to the DCC signal. Simple considerations allow one to estimate the numbers of the DCC and background pions [60]. Energy density in the DCC region is higher than in the normal vacuum because of the symmetry breaking term $V_{\mathrm{SB}} = -H\sigma$. The difference is

$$\Delta\epsilon_V = -H\langle\sigma\rangle_{\mathrm{DCC}} + H\langle\sigma\rangle = Hf_\pi(1-\cos\theta) = 2f_\pi^2 m_\pi^2 \sin^2\frac{\theta}{2}.$$

Therefore, the total volume energy available for pion radiation from the DCC decay is

$$E_V = \frac{8\pi}{3} R^3 f_\pi^2 m_\pi^2 \sin^2\frac{\theta}{2}\,,$$

where R is the fireball radius at the moment of hadronization. However, pions radiated from the DCC are nonrelativistic (in the DCC rest frame). Therefore, the expected average number of such pions is (we have assumed that $\langle \sin^2\frac{\theta}{2}\rangle = \frac{1}{2}$)

$$N_V \approx \frac{E_V}{m_\pi} = \frac{4\pi}{3} R^3 f_\pi^2\, m_\pi\,. \tag{42}$$

One can assume [60] that at the moment of hadronization the shell consists of one densely packed layer of pions, each having the radius $r_\pi \approx \frac{1}{2} m_\pi^{-1} \approx 0.7$ fm. Therefore, the number of background pions from the fireball shell is

$$N_b \approx \frac{4R^2}{r_\pi^2}\,. \tag{43}$$

For a large DCC bubble of the radius $R \approx 10 r_\pi \approx 7$ fm, the above given estimates imply

$$N_V \approx \frac{4\pi}{3}\, 125 \left(\frac{f_\pi}{m_\pi}\right)^2 \approx 250\,, \quad N_b \approx 400\,.$$

There will also be coherent pions associated with the surface energy of the interface between the DCC and outside vacuum. The energy density in the interface is dominated by the contribution due to gradients of the fields. If one assumes that the interface thickness is the same as for the hadronized shell, that is $d \approx 2r_\pi \approx m_\pi^{-1}$, then

$$\epsilon_S \approx \frac{1}{2}\left[(\Delta\vec{\pi})^2 + (\Delta\sigma)^2\right] \approx \frac{f_\pi^2}{2d^2}\left[\sin^2\theta + (1-\cos\theta)^2\right] = 2\frac{f_\pi^2}{d^2}\,\sin^2\frac{\theta}{2}\,.$$

The corresponding total average energy is thus

$$E_S \approx 4\pi\, R^2\, d\,\langle \epsilon_S\rangle \approx 4\pi\, R^2\, f_\pi^2\, m_\pi.$$

Pions originated from the surface layer of thickness d will have characteristic momenta $p_\pi \sim 1/d \approx m_\pi$ and energy $E_\pi = \sqrt{p_\pi^2 + m_\pi^2} \approx \sqrt{2} m_\pi$. Therefore the expected average number of surface pions is

$$N_S \approx \frac{E_S}{E_\pi} \approx 2\sqrt{2}\pi R^2\, f_\pi^2 \approx 50\sqrt{2}\pi \left(\frac{f_\pi}{m_\pi}\right)^2 \approx 105\,.$$

As we can see, the DCC signal from such a large single domain is quite prominent. This becomes especially evident if one realizes that there is a large probability, $\sim 10\%$, that almost all of these 250 nonrelativistic signal pions are charged ones, with only a few neutral pion admixture. The probability that such a huge isospin-violating fluctuation happens in the background pions is, of course, completely negligible. This striking feature of the DCC signal follows from the following simple geometrical argument [64] (the inverse-square-root distribution, discussed below, was independently rediscovered many times by different authors. See [40] for relevant references). Let the pion field in a single DCC domain be aligned along a fixed isospin direction $\vec{n} = (\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$. Classically the radiation is proportional to the square of the field strength. Therefore the fraction of neutral pions f, emitted during relaxation of such a DCC domain, equals

$$f = \frac{|\pi_3|^2}{\sum\limits_{a=1}^{3} |\pi_a|^2} = \cos^2\theta\,. \tag{44}$$

The probability for f to be in the interval $(f,\, f+df)$ is given by

$$P(f)df = \left[P(\cos\theta) + P(-\cos\theta)\right]\, d\cos\theta\,. \tag{45}$$

Any orientation of the unit vector $\vec{n}$ is equally valid. Therefore the probability $P(\cos\theta)d\cos\theta$ for finding $\cos\theta$ in the interval $(\cos\theta,\, \cos\theta + d\cos\theta)$ equals

$$\frac{1}{4\pi}\int\limits_{\cos\theta}^{\cos\theta+d\cos\theta} d\cos\theta \int\limits_0^{2\pi} d\phi = \frac{1}{2}\, d\cos\theta\,.$$

This implies $P(\cos\theta) = \frac{1}{2}$. Therefore, from (45) the probability density for the neutral fraction f is

$$P(f) = \frac{d\cos\theta}{df}\,,$$

while from (44) $df = 2\cos\theta\, d\cos\theta = 2\sqrt{f}\, d\cos\theta$, and we finally obtain

$$P(f) = \frac{1}{2\sqrt{f}}\,. \tag{46}$$

This inverse-square-root distribution is drastically different from what is expected for noncoherent pion production: the binomial-distribution which for large pion multiplicities N turns into a narrow Gaussian centered at $f = \frac{1}{3}$:

$$P_{\rm nc}(f) = C_N^{Nf}\left(\frac{1}{3}\right)^{Nf}\left(\frac{2}{3}\right)^{N(1-f)} \longrightarrow \frac{3N}{2\sqrt{\pi}}\, \mathrm{e}^{-\frac{9N\left(f-\frac{1}{3}\right)^2}{4}}\,.$$

For example, the probability that f does not exceeds 0.01 according to (46) equals

$$P(f \leq 0.01) = \int\limits_0^{0.01} \frac{df}{2\sqrt{f}} = \sqrt{0.01} = 10\% \ \ !!\,,$$

while the binomial-distribution predicts

$$P(f \leq 0.01) = \int\limits_0^{0.01} P_{\rm nc}(f)\, df \approx 0.01\, \frac{3N}{2\sqrt{\pi}}\, {\rm e}^{-\frac{N}{4}} \ll 1\,, \quad \text{for } N \gg 1\,.$$

However, as we have seen above, it is not easy to produce large DCC domain. For many small DCC domains, with random vacuum orientations, the effect of the inverse-square-root distribution will be washed out by averaging over the orientations (it was, however, argued in [65] that the later case of small DCC domains may lead to enhanced baryon–antibaryon production within the framework of the Skyrmion picture of the nucleon). The intuitive reason why it is difficult to grow up a large DCC bubble is the following. Up to now the only mechanism discussed by us for the DCC formation, was the spinodal instability which operates when the fields are rolling-down from the top of the Mexican hat potential towards the valley. But this rolling-down time is small unless the initial strength of the axial-vector current is large (see Eq. (31)). Fortunately, there exists a parametric resonance mechanism [66] which can further assist the DCC formation, after the spinodal instability is over. To illustrate the physical idea, let us again return to the Blaizot–Krzywicki model [46] with boost-invariant initial conditions. The equation of motion for the pion field is

$$\Box \vec{\pi} = -\lambda_s \left(\sigma^2 + \vec{\pi}^2 - v^2\right) \vec{\pi} \tag{47}$$

with

$$\Box = \frac{d^2}{d\tau^2} + \frac{1}{\tau}\, \frac{d}{d\tau}\,.$$

At later times the fields are near the true vacuum. So we assume

$$\sigma \approx f_\pi, \ \ \vec{\pi} = \pi(\tau)\vec{n}\,.$$

The above pionic field describes a (small) disoriented chiral condensate aligned along a fixed unit vector $\vec{n}$ in isospace — the result of preceding spinodal instability. If we neglect nonlinear terms, we get in the zeroth approximation (note that $\lambda_s \left(f_\pi^2 - v^2\right) = m_\pi^2$)

$$\ddot{\pi}^{(0)}(\tau) + \frac{\dot{\pi}^{(0)}(\tau)}{\tau} + m_\pi^2\, \pi^{(0)}(\tau) = 0\,. \tag{48}$$

This equation is equivalent to the Bessel equation and its general solution is a linear combination of the Bessel functions $J_0(m_\pi \tau)$ and $Y_0(m_\pi \tau)$. Therefore in the large τ limit one expects damped oscillations

$$\pi^{(0)}(\tau) = \frac{A}{\sqrt{\tau}} \cos\left(m_\pi \tau + \varphi\right),$$

with A and φ as some constants. Now consider a fluctuation (in the $\vec{n}$-direction) $\pi^{(1)}(\tau)$ around the zero-mode $\pi^{(0)}(\tau)$: $\pi(\tau) = \pi^{(0)}(\tau) + \pi^{(1)}(\tau)$. Keeping only the terms linear in $\pi^{(1)}(\tau)$ we get from (47) and (48)

$$\ddot{\pi}^{(1)}(\tau) + \frac{\dot{\pi}^{(1)}(\tau)}{\tau} + m_\pi^2\, \pi^{(1)}(\tau) = -3\lambda_s \pi^{(0)\,2}(\tau)\, \pi^{(1)}(\tau),$$

or (we have neglected unimportant $\frac{3\lambda_s A^2}{2\tau}\, \pi^{(1)}(\tau)$ term)

$$\ddot{\pi}^{(1)}(\tau) + \frac{\dot{\pi}^{(1)}(\tau)}{\tau} + \omega_0^2 \left[1 + \frac{q}{\tau} \cos\left(\omega \tau + 2\varphi\right)\right] \pi^{(1)}(\tau) = 0,$$

where $\omega_0 = m_\pi$, $\omega = 2m_\pi$ and $q = \frac{3\lambda_s A^2}{2m_\pi^2}$. Therefore, one expects a parametric resonance because $\omega = 2\omega_0$.

For more general initial conditions, parametric instabilities are expected for the low momentum pion modes [67]. The energy of the σ-field oscillations around $\sigma = f_\pi$ can also be pumped into pionic modes through the parametric resonance [66, 68]. In this case, naively only modes with $k \sim \sqrt{\frac{m_\sigma^2}{4} - m_\pi^2} \approx$ 270 MeV can be amplified, because oscillation frequency in the σ-direction is $m_\sigma = 600$ MeV. However, the energy can be redistributed in the long wavelength modes due to nonlinearity.

An interesting example of the parametric instability is Faraday waves [69] — surface waves parametrically excited in a vertically vibrating container of fluid when the vibration amplitude exceeds a certain threshold. The resulting standing waves on the fluid surface can form funny intricate patterns [70]. Even a more closer analog is given by the induction phenomenon in quartic Fermi–Pasta–Ulam (FPU) chains [71] : the energy, initially supplied to a single harmonic mode, remains in this mode over a certain period, called the induction time, when it is abruptly transferred to other harmonic modes. The original explanation [71] involves nonlinear parametric instabilities similar to the one described above. It should be mentioned, however, that our above arguments, in favor of the exponential growth of fluctuations due to the parametric resonance, are heuristic. A naive perturbation theory, implicit in these arguments, is not adequate for such nonlinear problems. In the case of the FPU chains, it was argued that a more correct treatment was provided by shifted-frequency perturbation theory [72] or by Krylov–Bogoliubov–Mitropolsky averaging technique [73]. We are not aware of similar studies in the context of DCC dynamics, but the reality of parametric instabilities is indirectly confirmed by numerical studies [34, 74]: the observed amplification of long-wavelength pionic modes last much longer than expected solely from the spinodal instabilities. Besides, the amplification of pionic modes with $k \approx 270$ MeV was clearly demonstrated.

5. Quantum state of the disoriented chiral condensate

The eventual decay of the DCC is a quantum process, because one registers pions and not the classical field. Therefore, the natural question about the DCC quantum state $|\eta\rangle_{\rm DCC}$ arises. The usual way of quantizing some classical field configuration is to use coherent states which are eigenstates of the annihilation operator [75]

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle\,. \tag{49}$$

Decomposing

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle\,, \quad |n\rangle = \frac{(a^+)^n}{\sqrt{n!}}\,|0\rangle$$

and using $a|n\rangle = \sqrt{n}\,|n-1\rangle$, we get from (49) the recurrent relation $\sqrt{n}\,c_n = \alpha\,c_{n-1}$. Therefore

$$c_n = \frac{\alpha^n c_0}{\sqrt{n!}}$$

and

$$|\alpha\rangle = c_0\,\exp\left(\alpha a^+\right)|0\rangle\,.$$

However,

$$\langle\alpha|\alpha\rangle = \sum_n \langle\alpha|n\rangle\langle n|\alpha\rangle = \sum_n |c_n|^2 = |c_0|^2 \exp\left(\alpha^*\alpha\right),$$

and, therefore, the normalization condition $\langle\alpha|\alpha\rangle = 1$ determines c_0 up to a phase. Finally

$$|\alpha\rangle = \exp\left(-\frac{\alpha^*\alpha}{2} + \alpha a^+\right)|0\rangle\,. \tag{50}$$

The generalization of this construction to the DCC classical field configuration $f(\vec{x})$ is [76] (the isospin indices are suppressed)

$$|\eta\rangle_{\rm DCC} = \exp\left(-\frac{1}{2}\int d\vec{k} f^*(\vec{k}) f(\vec{k}) + \int d\vec{k} f(\vec{k}) a^+(\vec{k})\right)|0\rangle\,, \tag{51}$$

where $f(\vec{k})$ is the Fourier transform of the DCC classical field.

However, it was argued [77,78] that the quantum state of the disoriented chiral condensate is expected to be a squeezed state [79], if the parametric amplification mechanism, discussed above, indeed plays a crucial role in the DCC formation. To explain why, let us consider a one-dimensional, unit mass, quantum parametric oscillator with the Hamiltonian

$$\hat{H}(t) = \frac{1}{2}\left[\hat{p}^2 + \omega^2(t)\,\hat{x}^2\right],$$

where the $\hat{p}$ and $\hat{x}$ operators are time-independent in the Schrödinger picture and obey the canonical commutation relation ($\hbar = 1$):

$$[\hat{x}, \hat{p}] = i\,.$$

Quantum state vector $|\psi\rangle$ of this oscillator is determined by the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}(t)\,|\psi\rangle\,. \tag{52}$$

Lewis and Riesenfeld gave [80] a general method of solving the Schrödinger equation by using explicitly time-dependent invariants which are solutions of the quantum Lieuville–Neumann equation

$$\frac{\partial \hat{I}}{\partial t} + i[\hat{H}, \hat{I}] = 0\,. \tag{53}$$

It turns out that the eigenstates of such a Hermitian invariant $\hat{I}(t)$ are just the desired solutions of the Schrödinger equation up to some time-dependent phase factor. Let us demonstrate this remarkable fact [80] . The eigenvalues of the Hermitian operator $\hat{I}(t)$ are real. Therefore, from (53) one easily gets

$$i\langle\lambda'|\,\frac{\partial \hat{I}}{\partial t}\,|\lambda\rangle = (\lambda - \lambda')\langle\lambda'|\hat{H}(t)|\lambda\rangle\,, \tag{54}$$

where $|\lambda\rangle$ is an eigenvector of the operator $\hat{I}(t)$ with the eigenvalue λ:

$$\hat{I}(t)\,|\lambda\rangle = \lambda\,|\lambda\rangle\,. \tag{55}$$

In particular

$$\langle\lambda|\,\frac{\partial \hat{I}}{\partial t}\,|\lambda\rangle = 0\,. \tag{56}$$

By differentiating (55) with respect to time and taking the scalar product with $|\lambda'\rangle$, we get

$$\langle\lambda'|\,\frac{\partial\hat{I}}{\partial t}\,|\lambda\rangle = (\lambda-\lambda')\langle\lambda'|\,\frac{\partial}{\partial t}\,|\lambda\rangle + \;\delta_{\lambda\lambda'}\,\frac{\partial\lambda}{\partial t}\,. \tag{57}$$

For $\lambda=\lambda'$, we get from (57) and (56)

$$\frac{\partial\lambda}{\partial t} = 0\,.$$

That is the eigenvalues of the operator $\hat{I}(t)$ are time-independent (as it should be for the invariant operator). From (57) and (54) one gets

$$(\lambda-\lambda')\langle\lambda'|\,i\frac{\partial}{\partial t}\,|\lambda\rangle = (\lambda-\lambda')\langle\lambda'|\hat{H}(t)|\lambda\rangle$$

and, therefore

$$\langle\lambda'|\,i\frac{\partial}{\partial t}\,|\lambda\rangle = \langle\lambda'|\hat{H}(t)|\lambda\rangle\,, \quad \text{if}\;\; \lambda'\neq\lambda\,. \tag{58}$$

If (58) is also satisfied for the diagonal matrix elements ($\lambda=\lambda'$), then we can immediately deduce that $|\lambda\rangle$ is a solution of the Schrödinger equation. But this may not be the case for our particular choice of eigenvectors. Nevertheless, in this case we can still adjust the phases of the eigenvectors in such a way that the new eigenstates

$$|\lambda\rangle' = \mathrm{e}^{i\theta_\lambda(t)}\,|\lambda\rangle\,, \quad \theta_\lambda(0)=0\,,$$

insure the validity of (58) for all λ, λ'. All what is needed is to choose the time-dependent phases $\theta_\lambda(t)$ in such a way that one has

$$\langle\lambda|\,\mathrm{e}^{-i\theta_\lambda}\left(i\frac{\partial}{\partial t}\right)\mathrm{e}^{i\theta_\lambda}\,|\lambda\rangle = \langle\lambda|\hat{H}(t)|\lambda\rangle\,,$$

or

$$\frac{d\theta_\lambda}{dt} = \langle\lambda|\,i\frac{\partial}{\partial t} - \hat{H}(t)\,|\lambda\rangle\,.$$

Therefore

$$\theta_\lambda(t) = \int\limits_0^t \langle\lambda|\,i\frac{\partial}{\partial\tau} - \hat{H}(\tau)\,|\lambda\rangle\,d\tau\,.$$

To summarize, we can take any set $|\lambda\rangle$ of the eigenstates of the invariant operator $\hat{I}(t)$ and express the general solution of the Schrödinger equation (52) as a linear combination

$$|\psi\rangle = \sum_\lambda C_\lambda \exp\left\{ i\int\limits_0^t \langle\lambda|\, i\frac{\partial}{\partial\tau} - \hat{H}(\tau)\,|\lambda\rangle\, d\tau \right\} |\lambda\rangle\,, \tag{59}$$

with some time-independent coefficients C_λ.

To find the invariant $\hat{I}(t)$, let us firstly construct its classical counterpart $I(t)$ by the simple and transparent method of Eliezer and Gray [81]. The equation of motion

$$\ddot{x} + \omega^2(t)\, x = 0$$

can be viewed as the x-projection of a two-dimensional auxiliary motion governed by the equation

$$\ddot{\vec{r}} + \omega^2(t)\,\vec{r} = 0\,, \tag{60}$$

where $\vec{r} = x\,\vec{i} + y\,\vec{j}$. In the polar coordinates

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi\,,$$

and

$$\ddot{\vec{r}} = \left(\ddot{\rho} - \rho\,\dot{\varphi}^2\right)\vec{e}_\rho + (\rho\,\ddot{\varphi} + 2\dot{\rho}\,\dot{\varphi})\,\vec{e}_\varphi\,,$$

where the unit basic vectors are

$$\vec{e}_\rho = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}, \quad \vec{e}_\varphi = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}\,.$$

Therefore, (60) in the polar coordinates takes the form

$$\ddot{\rho} - \rho\,\dot{\varphi}^2 + \omega^2(t)\rho = 0, \quad \rho\,\ddot{\varphi} + 2\dot{\rho}\,\dot{\varphi} = \frac{1}{\rho}\,\frac{d(\rho^2\dot{\varphi})}{dt} = 0\,.$$

The second equation implies

$$\rho^2\dot{\varphi} = L = \text{const.}$$

This is nothing but the conservation of the angular momentum for the auxiliary planar motion, when the first equation can be rewritten as

$$\ddot{\rho} + \omega^2(t)\rho = \frac{L^2}{\rho^3}\,.$$

Let us now remark that (for unit mass)

$$p = \dot{x} = \dot{\rho}\cos\varphi - \rho\dot{\varphi}\sin\varphi = \frac{\dot{\rho}x - L\sin\varphi}{\rho}\,.$$

Therefore,

$$L\sin\varphi = \dot\rho x - \rho p\,, \quad L\cos\varphi = \frac{L}{\rho}x\,,$$

and

$$\frac{L^2}{\rho^2}x^2 + (\rho p - \dot\rho x)^2 = L^2 = \text{const.}$$

The Ermakov–Lewis invariant [82, 83] corresponds to the particular case when the angular momentum L has a unit value

$$I(t) = \frac{1}{2}\left[\frac{x^2}{\rho^2} + (\rho p - \dot\rho x)^2\right].$$

It is straightforward to check that its quantum counterpart

$$\hat I(t) = \frac{1}{2}\left[\frac{\hat x^2}{\rho^2} + (\rho\hat p - \dot\rho\hat x)^2\right] \tag{61}$$

obeys the quantum Lieuville–Neumann equation (53) if the auxiliary function $\rho(t)$ satisfies the Ermakov–Milne–Pinney equation [84]

$$\ddot\rho + \omega^2(t)\rho = \frac{1}{\rho^3}\,. \tag{62}$$

To find eigenvectors of the operator $\hat I(t)$, let us note that

$$\hat I(t) = b^+(t)b(t) + \frac{1}{2}\,, \tag{63}$$

where we have introduced time-dependent "creation" and "annihilation" operators

$$b(t) = \frac{1}{\sqrt 2}\left[\frac{\hat x}{\rho} + i(\rho\,\hat p - \dot\rho\,\hat x)\right], \quad b^+(t) = \frac{1}{\sqrt 2}\left[\frac{\hat x}{\rho} - i(\rho\,\hat p - \dot\rho\,\hat x)\right]. \tag{64}$$

It can be immediately checked that one indeed has the canonical commutation relation

$$[b(t), b^+(t)] = 1\,. \tag{65}$$

Equations (63) and (65) indicate that the eigenvectors of $\hat I(t)$ are b-number states

$$|n; b\rangle = \frac{(b^+(t))^n}{\sqrt{n!}}\,|0; b\rangle\,, \tag{66}$$

where the b-vacuum state is defined by the condition

$$b(t)\,|0; b\rangle = 0\,. \tag{67}$$

Using these eigenvectors, one can construct the solution of the Schrödinger equation, as described above. In the DCC case, however, it is preferable to express the quantum state vector in terms of the pion creation and annihilation operators a^+, a. So we need a relation between two sets of the creation–annihilation operators a^+, a and b^+, b (the Bogoliubov transformation). The pionic modes correspond to the late time asymptotes $\omega(t) \to \omega(\infty) = \omega_0$. The solution of the Ermakov–Milne–Pinney equation for $\omega(t) = \omega_0 = \mathrm{const}$ is

$$\rho = \frac{1}{\sqrt{\omega_0}}\,.$$

Therefore, the "pionic" creation–annihilation operators have the standard form

$$a = \frac{1}{\sqrt{2\omega_0}}\left[\omega_0\,\hat{x} + i\hat{p}\right], \quad a^+ = \frac{1}{\sqrt{2\omega_0}}\left[\omega_0\,\hat{x} - i\hat{p}\right]. \tag{68}$$

The comparison of (68) and (64) gives the desired Bogoliubov transformation

$$b(t) = \alpha(t)\,a + \beta^*(t)\,a^+, \quad b^+(t) = \beta(t)\,a + \alpha^*(t)\,a^+, \tag{69}$$

where

$$\begin{aligned}\alpha(t) &= \frac{1}{2\sqrt{\omega_0}}\left[\frac{1}{\rho(t)} + \omega_0\,\rho(t) - i\,\dot{\rho}(t)\right],\\ \beta(t) &= \frac{1}{2\sqrt{\omega_0}}\left[\frac{1}{\rho(t)} - \omega_0\,\rho(t) + i\,\dot{\rho}(t)\right].\end{aligned} \tag{70}$$

It can be checked that

$$|\alpha(t)|^2 - |\beta(t)|^2 = 1\,.$$

Therefore, up to an irrelevant common phase, we can take

$$\alpha(t) = \cosh r(t)\,, \quad \beta(t) = \mathrm{e}^{i\delta(t)}\sinh r(t)\,. \tag{71}$$

The Bogoliubov transformation (69) can be viewed as an unitary transformation

$$b = \hat{S}(z)\,a\,\hat{S}^+(z)\,, \quad b^+ = \hat{S}(z)\,a^+\,\hat{S}^+(z)\,, \tag{72}$$

where $\hat{S}(z)$ is the so-called squeezing operator [79]

$$\hat{S}(z) = \exp\left[\frac{1}{2}\left(z\,a^+a^+ - z^*\,aa\right)\right], \quad z = r\,\mathrm{e}^{i(\delta+\pi)}\,. \tag{73}$$

Indeed, using the Campbell–Hausdorf formula

$$\mathrm{e}^{\hat{B}}\hat{A}\mathrm{e}^{-\hat{B}} = \hat{A} + [\hat{B},\hat{A}] + \frac{1}{2!}[\hat{B},[\hat{B},\hat{A}]] + \frac{1}{3!}[\hat{B},[\hat{B},[\hat{B},\hat{A}]]] + \ldots$$

and summing up the resulting infinite series, one can check that (72) and (69) are equivalent, if α and β are given by (71).

Therefore, the DCC quantum state is expected to have the form

$$|\eta\rangle_{\rm DCC} = \hat{S}(z)|\,\Psi_0\rangle\,,$$

where $|\,\Psi_0\rangle$ is the DCC initial state before the onset of the parametric amplification. If $|\,\Psi_0\rangle$ is a coherent state, then the resulting $|\eta\rangle_{\rm DCC}$ state will be the one called the squeezed state [79]. The actual parameters of this state (for example z) are determined by the initial conditions and are hard (if not impossible) to estimate from the theory alone.

Let us now recall the isospin, for a moment. The isospin generators are

$$\vec{I} = \int \vec{\pi}(x) \times \dot{\vec{\pi}}(x)\, d\vec{x}\,.$$

The classical pion field $\vec{\pi}$ of the DCC points in some fixed direction, $\vec{n}$ in the isospace. If its time derivative $\dot{\vec{\pi}}$ also points in the same direction then $\vec{I}=0$ and we will have an isosinglet state. One can expect such a situation in ρ–ρ collisions (which dominates in the $\gamma\gamma \to$ hadrons cross section), because the "vacuum cleaning" effect, which precedes the DCC formation, in this case is mainly due to two colliding isospin blind gluon walls. It is easy to construct an isosinglet squeezed state by just exponentiating the apparently isoscalar operator

$$-\sum_{i=1}^{3} a_i^+ a_i^+ = 2a_+^+ a_-^+ - a_3^+ a_3^+\,,$$

where $a_\pm^+ = \frac{\mp 1}{\sqrt{2}}(a_1^+ \pm i a_2^+)$ are the charged-pion creation operators. The resulting squeezed state is [78]

$$|\,\Psi\rangle = N\,\exp\left\{\frac{\alpha}{2}\left(2a_+^+ a_-^+ - a_3^+ a_3^+\right)\right\}|0\rangle\,.$$

Well, this expression does not seem, at first glance, to correspond to the canonical form (73) of the squeezing operator. In fact, it indeed gives a squeezed state. This is clear from the following normal-ordered form of the squeezing operator [79]

$$S(r{\rm e}^{i\theta}) = N\,\exp\left(\frac{\alpha}{2}\,a^+a^+\right)\sum_{n=0}^{\infty}\frac{({\rm sech}\,r-1)^n}{n!}(a^+)^n(a)^n\,\exp\left(-\frac{\alpha}{2}\,aa\right), \tag{74}$$

where $\alpha = {\rm e}^{i\theta}\tanh r$ and

$$N = \frac{1}{\sqrt{\cosh r}} = \left(1-|\alpha|^2\right)^{1/4}.$$

What is the probability $P(m;n)$ that $|\Psi\rangle$ decays by producing in total $2n$ pions and among them $2m$ neutral pions and equal numbers of positively and negatively charged pions? According to the standard rules of quantum mechanics

$$P(m;n) = |\langle m;n|\,\Psi\rangle|^2,$$

where the normalized state $|m;n\rangle$ is defined through

$$|m;n\rangle = \frac{1}{\sqrt{(2m)!}}\,\frac{1}{(n-m)!}\,(a_+^+ a_-^+)^{(n-m)}\,(a_3^+)^{2m}\,|0\rangle\,.$$

However,

$$\begin{aligned}
|\,\Psi\rangle &= N\sum_{k=0}^{\infty}\frac{(\alpha/2)^k}{k!}\,\left(2a_+^+a_-^+ - a_3^+a_3^+\right)^k\,|0\rangle \\
&= N\sum_{k=0}^{\infty}\frac{(\alpha/2)^k}{k!}\,\sum_{l=0}^{k} C_k^l\,2^{k-l}\,\sqrt{(2l)!}\,(l-k)!\,|l;k\rangle\,.
\end{aligned}$$

Therefore,

$$P(m;n) = \left|N\,\frac{(\alpha/2)^n}{n!}\,C_n^m\,2^{n-m}\,\sqrt{(2m)!}\,(n-m)!\right|^2 = N^2\,|\alpha|^{2n}\,\frac{(2m)!}{(m!\,2^m)^2}\,.$$

One can prove by induction in n that

$$\sum_{m=0}^{n}\frac{(2m)!}{(m!\,2^m)^2} = \frac{(2n+1)!}{(n!\,2^n)^2}.$$

This enables one to express $P(m;n)$ as a product of two probabilities:

$$P(m;n) = P_1(n)P_2(m;n)\,,$$

where $P_1(n)$ is the probability that one will find the total number of $2n$ pions after the state $|\,\Psi\rangle$ decays

$$P_1(n) = N^2\,|\alpha|^{2n}\,\frac{(2n+1)!}{(n!\,2^n)^2}\,.$$

Note that

$$\sum_{n=0}^{\infty}\frac{(2n+1)!}{(n!\,2^n)^2}\,|\alpha|^{2n} = 1+\frac{3}{2}\,|\alpha|^2+\frac{3\cdot 5}{2\cdot 4}\,|\alpha|^4+\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\,|\alpha|^6+\cdots = (1-|\alpha|^2)^{-3/2}\,.$$

Therefore, the normalization coefficient N should be $N=(1-|\alpha|^2)^{3/4}$, and this is exactly what is expected from (74) for the product of three properly normalized single-mode squeezed states of Cartesian pions.

More interesting for us is the second factor, $P_2(m;n)$, the probability that one finds $2m$ neutral pions in such a $2n$-pion final state [60, 85]:

$$P_2(m;n)=\frac{(n!\,2^n)^2}{(2n+1)!}\,\frac{(2m)!}{(m!\,2^m)^2}\,.$$

In fact, $P_2(m;n)$ is a particular case of the Polyá distribution [86]. If m and n are both large, one can use the Stirling formula

$$n!\approx\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$$

to get

$$P(m;n)\approx\frac{1}{2n}\,\frac{1}{\sqrt{\frac{m}{n}}}\,.$$

Therefore the same inverse-square-root distribution (46) is recovered in the continuum limit.

A few more words about coherent and squeezed states, in order to make them more familiar. If $[\hat{A},\hat{B}]$ is a c-number, then [87]

$$\mathrm{e}^{\hat{A}+\hat{B}}=\mathrm{e}^{\hat{A}}\,\mathrm{e}^{\hat{B}}\,\mathrm{e}^{-\frac{1}{2}[\hat{A},\hat{B}]}\,.$$

Using this theorem, we get

$$\exp\left[\alpha\,a^+-\alpha^*\,a\right]=\exp\left[-\frac{1}{2}\alpha^*\,\alpha\right]\,\exp\left(\alpha\,a^+\right)\,\exp\left(-\alpha^*\,a\right).$$

But $\exp\left(-\alpha^*\,a\right)|0\rangle=|0\rangle$. Therefore, the coherent state $|\alpha\rangle$ can be generated by the unitary displacement operator: $|\alpha\rangle=\hat{D}(\alpha)\,|0\rangle$, where

$$\hat{D}(\alpha)=\exp\left[\alpha\,a^+-\alpha^*\,a\right]. \tag{75}$$

Now let us consider the ground state for a harmonic oscillator (we have abandoned the unit mass restriction but will still keep $\hbar=1$):

$$\Psi_0(x)\equiv\langle x|0\rangle=\left[2\pi\sigma_0^2\right]^{-1/4}\,\exp\left[-\left(\frac{x}{2\sigma_0}\right)^2\right],\quad \sigma_0^2=\frac{1}{2m\omega}\,,$$

and calculate the effect of the displacement operator on it. For the harmonic oscillator

$$a=\frac{1}{\sqrt{2m\omega}}\left[m\omega\,\hat{x}+i\,\hat{p}\right],\quad a^+=\frac{1}{\sqrt{2m\omega}}\left[m\omega\,\hat{x}-i\,\hat{p}\right].$$

Hence,

$$\alpha\, a^+ - \alpha^*\, a = ip_0\, \hat{x} - ix_0\, \hat{p}\,,$$

where

$$x_0 = \sqrt{\frac{2}{m\omega}}\,\mathrm{Re}\,\alpha\,, \quad p_0 = \sqrt{2m\omega}\,\mathrm{Im}\,\alpha\,.$$

In the coordinate representation $\hat{p} = -i\frac{\partial}{\partial x}$; therefore ,

$$\begin{aligned}\Psi_{cs}(x) \equiv \langle x|\alpha\rangle &= \exp\left[ip_0 x - x_0\frac{\partial}{\partial x}\right]\,\Psi_0(x)\\ &= \left[2\pi\sigma_0^2\right]^{-1/4}\,\exp\left[-\left(\frac{x-x_0}{2\sigma_0}\right)^2 + ip_0 x - i\,\frac{x_0\,p_0}{2}\right].\end{aligned}$$

As we see, for harmonic oscillator, the coherent state is a Gaussian which is displaced from the origin by x_0. It has the ground state width σ_0 and a phase linearly dependent on the position x. Schrödinger discovered [88] such a state as early as 1926 while seeking "unspreading wave packets".

What about the squeezed vacuum state $\Psi_{s0} = \hat{S}(z)\,\Psi_0$? Note that (for simplicity we will take $z = r$ to be real)

$$a^+a^+ - aa = -i(\hat{x}\,\hat{p} + \hat{p}\,\hat{x}\,).$$

Then

$$\frac{-ir}{2}\,[\hat{x}\,\hat{p} + \hat{p}\,\hat{x}, \hat{x}] = -r\hat{x}, \quad \frac{-ir}{2}\,[\hat{x}\,\hat{p} + \hat{p}\,\hat{x}, \hat{p}] = r\hat{p},$$

and the Campbell–Hausdorf formula will give

$$\hat{S}(r)\,\hat{x}\,\hat{S}^+(r) = \mathrm{e}^{-r}\,\hat{x}, \quad \hat{S}(r)\,\hat{p}\,\hat{S}^+(r) = \mathrm{e}^{r}\,\hat{p}.$$

Therefore

$$\hat{S}(r)\,\hat{H}\,\hat{S}^+(r) = \frac{\mathrm{e}^{2r}\hat{p}^2}{2m} + \frac{m\omega^2\mathrm{e}^{-2r}\hat{x}^2}{2} = -\frac{1}{2m}\,\frac{\partial^2}{\partial(\mathrm{e}^{-r}x)^2} + \frac{m\omega^2}{2}\,(\mathrm{e}^{-r}x)^2,$$

and comparing the equations which Ψ_{s0} and Ψ_0 are satisfying

$$\hat{S}(r)\,\hat{H}\,\hat{S}^+(r)\,\,\Psi_{s0} = \frac{\omega}{2}\,\Psi_{s0}\,, \quad \hat{H}\,\,\Psi_0 = \frac{\omega}{2}\,\Psi_0\,,$$

we conclude that

$$\Psi_{s0}(x) = \Psi_0(\mathrm{e}^{-r}x) = \left[2\pi\sigma_0^2\right]^{-1/4}\,\exp\left[-\left(\frac{x}{2\mathrm{e}^r\sigma_0}\right)^2\right].$$

In a more general case, the squeezed state $\Psi_{ss} = \hat{D}(\alpha)\hat{S}(r)\,\Psi_0$ is a Gaussian

$$\Psi_{ss}(x) = \left[2\pi\sigma_0^2\right]^{-1/4} \exp\left[-\left(\frac{x-x_0}{2\sigma}\right)^2 + ip_0x - i\,\frac{x_0\,p_0}{2}\right],$$

where

$$\sigma = \mathrm{e}^r\sigma_0\,.$$

Therefore, the squeezed state differs from the coherent state only by squeezing in (if $\mathrm{e}^r < 1$), or squeezing out (if $\mathrm{e}^r > 1$) the width of the ground state Gaussian. The momentum-space wave function is squeezed oppositely.

For further discussion see [79, 89]. Let us note that squeezed states were also discovered long ago (in 1927) by Kennard [87]. Both the Schrödinger and Kennard works had little impact, and both coherent and squeezed states, which are now cornerstones of quantum optics, were rediscovered after decades. As Nieto remarks, "To be popular in physics you have to either be good or lucky. Sometimes it is better to be lucky. But if you are going to be good, perhaps you should not be too good".

As the last, but not the least, question of this section, let us ask whether a quantum state of the disordered chiral condensate $|\eta\rangle_{\rm DCC}$ may be produced without any intermediate phase transitions altogether, through the quantum reaction $\gamma\gamma \to |\eta\rangle_{\rm DCC}$. The functional integral methods and stationary phase approximation (semiclassical approximation) are natural tools to study the scattering amplitudes between initial wave packet states and certain final coherent states [90, 91]. In this article we are not really interested in actual calculations of this type. Our aim at the beginning is more humble — to provide some arguments that such a quantum transition is indeed possible and interesting. So we will consider an oversimplified toy model with the (second quantized) Hamiltonian

$$\hat{H} = \omega\, a^+a + 2\omega\, b^+b + g\,(ba^+a^+ + b^+aa),$$

with $2\omega = m_\pi$. Here the b-mode mimics neutral pions, a-mode — photons, and the interaction term with $g = \frac{\alpha\, m_\pi^2}{\pi\, f_\pi}$ imitates the $\pi^0\gamma\gamma$ interaction due to axial anomaly. Such Hamiltonians are used in quantum optics to study the second-harmonic generation [92, 93]. We further assume that all available initial energy, $\sqrt{s}$, is accumulated in the a-mode. That is, the assumed initial a-mode occupancy is

$$N_a = \frac{\sqrt{s}}{\omega} = \frac{2\sqrt{s}}{m_\pi}\,.$$

It can be easily checked that $[\hat{H}_0, \hat{H}_{\rm int}] = 0$, where

$$\hat{H}_0 = \omega\, a^+a + 2\omega\, b^+b, \quad \hat{H}_{\rm int} = g\,(ba^+a^+ + b^+aa)\,.$$

This means that both $\hat{H}_0$ and $\hat{H}_{\rm int}$ are constants of motion. We can simply forget about $\hat{H}_0$, because it just gives an irrelevant common phase factor $\exp{(iN_a\omega t)}$ in the evolution operator. Hence the nontrivial part of the initial-state evolution is given by the relation

$$|\,\Psi(t)\rangle = {\rm e}^{i\hat{H}_{\rm int}t}\,|N_a;0\rangle\,.$$

However, $N_a \gg 1$; therefore, as far as the b-mode initial development is concerned, we can replace the a and a^+ operators in $\hat{H}_{\rm int}$ by the c-numbers α and α^*, at that $|\alpha|^2 = N_a$. In this approximation

$$|\,\Psi(t)\rangle = {\rm e}^{\beta\, b^+ - \beta^*\, b}\,|0\rangle\,, \quad \beta = ig\,\alpha^2\,t\,. \tag{76}$$

As we see, a coherent state of b-quanta is being formed. The mean number of quanta in this state grows with time (until the approximation considered breaks down) as follows:

$$N_b = |\beta|^2 = (gN_a t)^2\,. \tag{77}$$

To estimate the terminal time, let us note that the variance of the number of quanta in the coherent state (76) also equals $|\beta|^2$ (see, for example, [89]). Therefore, the development time for the coherent state (76) can be estimated from the energy–time uncertainty relation

$$\sqrt{N_b}\, m_\pi t \sim 1\,.$$

Using this estimation, we finally get from (77) the relation

$$\sqrt{s} = \frac{\pi\, N_b}{2\,\alpha}\, f_\pi\,.$$

Therefore, for example, $N_b \sim 100$ implies $\sqrt{s} \sim 2$ TeV.

Having in mind a very crude and heuristic nature of our arguments, we admit that we may easily be wrong by an order of magnitude. Nevertheless, the main indications of the above exercise that the axial anomaly can lead to the generation of a pionic coherent state in gamma–gamma collisions, and that the efficient generation requires not fantastically high center-of-mass energies certainly seems interesting and deserves further study.

6. Concluding remarks

"Have no fear of perfection — you'll never reach it" — Salvador Dali once remarked. At the end of our enterprise we reluctantly realized how true the second half of this quotation is. Therefore, we abandon an unrealistic dream to produce the perfect review of the disoriented chiral condensate and try to finish here. It is important to finish at right time, is not it? One American general began his talk with the following sentence: "My duty is to speak, and your duty is to listen. And I hope to end my duty before you end yours". We hope that the reader has not yet end his duty, because there is one topic which should be touched a little before we finish.

The disoriented chiral condensate is a very attractive idea, and it has some solid theoretical support behind it, as we tried to demonstrate above. But are there any experimental indications in favor of its real existence? In fact there are some exotic cosmic ray events, called Centauros, where one may suspect the DCC formation. Centauros were discovered in high-mountain emulsion chamber cosmic ray experiments [94]. Typically, the detectors used in such experiments consist of the upper and lower chambers separated by the carbon target. Each chamber is a sandwich of the lead absorber and the sensitive layers. The normal cosmic ray event is usually generated by the primary interaction at about 500–1000 m altitude above the detector apparatus. About one third of the products of the primary interaction are neutral pions. Each neutral pion decays into two γ quanta. Therefore, roughly one γ quantum is expected per a charged particle in the primary interaction. When the interaction products reach the upper chamber the numbers of electronic and photonic secondaries are much increased through the electromagnetic shower formation. Therefore, the upper chamber usually detects several times more particles than the lower chamber, because the electromagnetic component is strongly suppressed by the carbon layer, leaving mainly the hadronic component to be detected by the lower chamber. A big surprise was the discovery of events with the contrary situation. Such events were named "Centauros" because it was not possible to guess their lower parts from the upper ones.

The first Centauro was observed in 1972 at the Chacaltaya high mountain laboratory [94]. It was initiated by the primary interaction at a relatively low altitude, at only (50 ± 15) m above the detector. Therefore, the event was very clean, that is was almost not distorted by electromagnetic and nuclear cascades in the atmospheric layer above the chambers. After correcting for the hadron detection efficiency and for the influence of the secondary atmospheric interactions, the event can be interpreted as the production of only one electromagnetic (e/γ) particle and 74 hadrons with the total interaction energy ≈ 330 TeV.

Afterwards some more Centauros were found. Namely [94], the Chacaltaya experiment observed 8 unequivocal Centauros, and two experiments at Pamir found 3 and 2 more Centauros. But no clean Centauros were found in Kanbala and Fuji experiments — the puzzle which still remains a mystery [94]. However, if the definition of Centauro is somewhat relaxed and all hadron-rich species are considered, then such Centauro-like anomalies constitute about 20% of events with the total visible energy ≥ 100 TeV [94], hence they are by no means rare phenomena at such high energies.

Of course the DCC formation is a candidate explanation of Centauro events. However such explanation is not without difficulties [94]. For example, the large transverse momenta observed in the Centauro events is difficult to explain in the DCC scenario. It is not also evident that the Centauro hadrons are pions. If they are mostly baryons instead, then an alternative explanation may be provided by strangelets [95].

Let us note, however, that if one takes the DCC explanation of Centauros seriously, some predictions immediately follow. First of all, it may happen that the DCC domain is produced in the cosmic-ray interactions with significant transverse velocity. In this case the coherent pions from the DCC decay will constitute "coreless jet" in the laboratory frame, with pions in the jet having small (< 100 MeV) relative transverse momenta [40]. Interestingly, such hadron-rich events, called Chirons, were really observed in both Chacaltaya and Pamir experiments [94].

If the DCC is aligned along the π^0-direction in isospace, then a particular anti-Centauro event is expected with neutral pion fraction f close to unity. For example, one has not very small probability that the neutral pion fraction is in the interval $0.99 \leq f \leq 1$:

$$P(0.99 \leq f \leq 1) = \int\limits_{0.99}^{1.0} \frac{df}{2\sqrt{f}} \approx 0.5\% \,.$$

No such anti-Centauro events were observed in the Chacaltaya and Pamir experiments. However, some anti-Centauros were reported in the Japanese–American JACEE experiment, in which the emulsion chambers were flown near the top of the atmosphere by balloons [94]. By the way, this experiment has not seen any Centauro events — another mystery puzzle of this cosmic ray Centauro business.

Of course Centauro-like events were searched in accelerator experiments [94]. The first searches have been performed by UA1 and UA5 experiments at CERN even before the DCC idea was suggested. Both experiments found no Centauro candidates in the central rapidity region.

The estimated average energy of Cosmic-ray Centauro events is about 1740 TeV [94]. If Centauros are formed in nucleon–nucleon collisions, this energy threshold translates into $\sqrt{s} \approx 1.8$ TeV in the c.m. frame — roughly the Tevatron energy. This observation maybe explains the failure of UA1 and UA5 experiments, where the maximal available energy was $\sqrt{s} \approx 0.9$ TeV, and makes Fermilab experiments more attractive in this respect. However, one has to bear in mind that there is a crucial kinematic difference between cosmic-ray and collider experiments [94]: the cosmic-ray experiments generally detect particles from the fragmentation rapidity region whereas the collider experiments are mainly focused on the central rapidity region. Therefore, the fact of observation of cosmic-ray Centauros does not automatically guarantee that these beasts can be found in Tevatron experiments.

A small test experiment Mini–Max (T-864) [96] at Tevatron was specially designed for a search of DCC in the forward region. The results of this experiment [97] are consistent with the generic production mechanism and show no evidence of the presence of DCC. Despite this failure to find DCC, the Mini–Max experiment was an important benchmark. It was demonstrated for the first time that it is possible to work in the very forward region with severe background conditions. Much was learned in both detector operation and data analysis which should prove useful in future more elaborate efforts of this kind.

Central rapidity region Centauros were searched in the CDF experiment at Tevatron with negative result [94]. Another major Tevatron detector D0 is also suitable for such searches, as the Monte Carlo study shows [94].

A serious effort to study possible DCC formation in heavy ion collisions was undertaken in the CERN SPS fixed target experiment WA98 [98]. Again no DCC signal was found in the central $158\,A$ GeV Pb + Pb collisions.

A majority of future heavy ion experiments at RHIC and LHC have plans to look for the Centauro phenomena [94]. The kinematic conditions at which cosmic ray Centauros are produced will be accessible at RHIC. Therefore the corresponding experiments (PHOBOS, STAR, PHENIX and BRAHMS) are very interesting in light of Centauro investigation. At LHC the energy accessible in Pb+Pb central collisions will be much higher than the expected threshold energy for the Centauro production. Besides, Pb+Pb collisions at LHC will produce a very high baryon number density in the forward rapidity region. To study the novel phenomena expected in such high baryochemical potential environment, the CASTOR detector, as the part of the ALICE experiment, was designed [99]. Its main goal is the Centauro and strangelet search in the very forward rapidity region in nucleus–nucleus collisions.

We believe that future photon–photon colliders are also good places to look for the DCC production. Some hints were given above that the DCC formation conditions might be even more favorable at photon–photon colliders rather than at proton–proton (or proton–antiproton) colliders. Here we present one more argument of this kind which deals with the very different roles played by gluons in mesons and baryons. Mesons can be considered as a quark–antiquark pair connected by a gluon string (flux tube). Therefore, the gluon field configuration in mesons is, in some sense, topologically trivial. In baryons one has a quite different picture [100–102]. According to the common wisdom, baryons are three-quark bound states. In high energy pp or heavy ion collisions the valence quark distributions in the projectiles will be significantly Lorentz-contracted because a typical fraction of the proton's momentum carried by a valence quark is $\sim 1/3$. Therefore, one expects that the constituent quarks of the colliding protons will not have enough time to interact significantly during the high-energy collisions and hence it is difficult to stop them. If the baryon number of the projectile is associated with their valence quarks, which is the naive expectation, then the ready prediction from the above given collision picture will be that the baryon number flow should be concentrated at large positive and negative rapidities, with a nearly zero net baryon number at central rapidities. Surprisingly, this is not the case supported by experiments. On the contrary, experiments suggest that the valence quarks do not carry the proton's baryon number and the flow of the baryon number can be separated from the flow of the valence quarks [101, 102]! But then what is the mysterious fourth constituent of the proton which traces its baryon number? In QCD the baryon is represented by a gauge invariant, non-local, color singlet operator. In fact, the gauge invariance constraint severely restricts the possible forms of such composite operator, leaving only one possibility (α, β, γ are the color indices, the flavor indices are suppressed for simplicity) :

$$B = \epsilon_{\alpha\beta\gamma}\left[\hat{T}(x_1,x)q(x_1)\right]^{\alpha}\left[\hat{T}(x_2,x)q(x_2)\right]^{\beta}\left[\hat{T}(x_3,x)q(x_3)\right]^{\gamma} .$$

Here $\hat{T}(x_i,x)$ is the open string operator (the Wilson line), or parallel transporter of the quark field $q(x_i)$ from the point x_i to the point x, where the three strings join. This string operator is an analog of the well known Aharonov–Bohm phase in QED and is given by the path-ordered exponent

$$\hat{T}(x_i,x) = P\,\exp\left(ig\int\limits_{x_i}^{x} A_\mu\,dx^\mu\right), \quad A_\mu = A_\mu^a\frac{\lambda_a}{2} .$$

Therefore, the gluon strings (flux tubes) inside a baryon have nontrivial Y-shaped topology and one finds a novel object there — the string junction. This string junction is just the fourth constituent of the baryon which traces its baryon number [101]. The string junction can be more easily stopped in the high-energy collisions, because, being formed from the soft gluons, it is not Lorentz-contracted and always has enough time to interact.

Now we have the following picture of the high-energy pp collisions [101]: the valence quarks are stripped-off and produce jets in the fragmentation regions. In some events, one or both of the string junctions are stopped in the central rapidity region producing a violent gluon sea containing one or two twists. On the contrary, no twists are expected in the gluon sea produced by high-energy photon–photon ($\rho\rho$) collisions. We believe that the latter situation is more favorable for the Baked Alaska scenario and, therefore, for the DCC production through this mechanism.

As a final remark, let us note that the DCC formation is just one interesting collective effect expected in high-energy collisions. Other exotic phenomena are also worth to be searched. Let us mention a few: the possible formation of the pion and eta strings during the chiral phase transition [103], creation of the parity and CP violating metastable vacuum bubbles [104], production of QCD Buckyballs — femtometer scale gluon junction networks (QCD analog of the nanoscale carbonic Fullerenes) [102]. Vacuum engineering at photon colliders promises to be an exciting adventure and we suspect that one may encounter "totally unexpected" new phenomena during such exploration: "There are more things in Heaven and Earth, Horatio, than are dreamt of in your philosophy" [105].

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Appendix: the Baked Alaska recipe

Here we reproduce the recipe from [106].

Ingredients:

- 3 egg whites
- 1/2 cup of sugar
- 1 cup of really hard, frozen ice cream
- 1 big, thick, hard cookie
- Baking sheet
- Aluminum foil
- Hand mixer
- A grown up!!

Directions:

1. Have your grown up heat your oven to 500 degrees Fahrenheit.
2. Cover the baking sheet with aluminum foil.
3. Put the egg whites into a bowl and use the mixer to beat them for about five minutes until they're stiff.
4. Keep beating the egg whites while adding the sugar a little at a time until they're fluffy and shiny. ("The name for this stuff is meringue!")
5. Put your cookie on the baking sheet.
6. Put your scoop of ice cream on top of the cookie. Make sure it doesn't hang over the edge of the cookie.
7. Completely cover the cookie and the ice cream with the meringue.
8. Put it in the oven for three to five minutes until the meringue is a delicate, light brown.
9. Take it out of the oven, put it on a plate, and eat up!

REFERENCES

[1] The future of high-energy experimental physics is discussed in W.K. Panofsky, SLAC-PUB-3357, 1984; J.D. Bjorken, *Int. J. Mod. Phys.* **A16**, 483 (2001); SLAC-PUB-5361, Invited talk given at 25th Int. Conf. on High Energy Physics, Singapore, Aug 2–8, 1990.

[2] W.K. Panofsky, M. Breidenbach, *Rev. Mod. Phys.* **71**, S121 (1999); J.D. Lawson, RL-83-082, Invited talk given at Int. Europhysics Conf. on High Energy Physics, Brighton, England, Jul 20–27, 1983; RAL-87-016, Invited talk given at 1987 Particle Accelerator Conf., Wash., D.C., Mar 16–19, 1987.

[3] For the concept of muon colliders see C.M. Ankenbrandt *et al.*, *Phys. Rev. ST Accel. Beams* **2**, 081001 (1999); A.N. Skrinsky, *Sov. Phys. Usp.* **25**, 639 (1982). And references therein.

[4] For informal discussion of the string theory see, for example, E. Witten, `hep-th/0212247`. See also D.I. Olive, *Phil. Trans. Roy. Soc. Lond.* **A329**, 319 (1989).

[5] G. 't Hooft, *Int. J. Mod. Phys.* **A16**, 2895 (2001).

[6] F. Wilczek, *Int. J. Mod. Phys.* **A13**, 863 (1998); **A16**, 1653 (2001).

[7] J.D. Jackson, L.B. Okun, *Rev. Mod. Phys.* **73**, 663 (2001). See also L. O'Raifeartaigh, N. Straumann, `hep-ph/9810524`.

[8] S.S. Chern, *Am. Math. Mon.* **86**, 339 (1979); **97**, 679 (1990). A nice historical account of the Gauss-Bonnet theorem is given in D.H. Gottlieb, *Am. Math. Mon.* **103**, 457 (1996).

[9] For an introduction in the geometry of gauge theories see, for example, M. Daniel, C.M. Viallet, *Rev. Mod. Phys.* **52**, 175 (1980).

[10] L.H. Ryder, *J. Phys. A* **13**, 437 (1980).

[11] I.F. Ginzburg, G.L. Kotkin, V.G. Serbo, V.I. Telnov, *JETP Lett.* **34**, 491 (1981).

[12] I.F. Ginzburg, G.L. Kotkin, V.G. Serbo, V.I. Telnov, *Nucl. Instrum. Methods Phys. Res.* **205**, 47 (1983); I.F. Ginzburg, G.L. Kotkin, S.L. Panfil, V.G. Serbo, V.I. Telnov, *Nucl. Instrum. Methods Phys. Res.* **A219**, 5 (1984).

[13] V.I. Telnov, *Nucl. Instrum. Methods Phys. Res.* **A294**, 72 (1990); *Nucl. Instrum. Methods Phys. Res.* **A355**, 3 (1995).

[14] V. Telnov, *Nucl. Instrum. Methods Phys. Res.* **A455**, 63 (2000); *Int. J. Mod. Phys.* **A13**, 2399 (1998).

[15] B. Badelek *et al.*, `hep-ex/0108012`.

[16] A.M. Kondratenko, E.V. Pakhtusova, E.L. Saldin, *Sov. Phys. Dokl.* **27**, 476 (1982); E.L. Saldin, E.A. Schneidmiller, M.V. Yurkov, *Nucl. Instrum. Methods Phys. Res.* **A472**, 94 (2001).

[17] M.M. Velasco *et al.*, `hep-ex/0111055`.

[18] E. Boos *et al.*, *Nucl. Instrum. Methods Phys. Res.* **A472**, 100 (2001).

[19] I. Hinchliffe, N. Kersting, Y.L. Ma, `hep-ph/0205040`.

[20] F. Dyson, *Am. Math. Mon.* **103**, 800 (1996).

[21] B.R. Holstein, `hep-ph/9911449`.

[22] F. Wilczek, *Nucl. Phys.* **A663**, 3 (2000).

[23] J. Gasser, H. Leutwyler, *Ann. Phys.* **158**, 142 (1984); *Nucl. Phys.* **B250**, 465 (1985); U.G. Meissner, *Rep. Prog. Phys.* **56**, 903 (1993); G. Ecker, *Prog. Part. Nucl. Phys.* **35**, 1 (1995); B.L. Ioffe, *Phys. Usp.* **44**, 1211 (2001).

[24] J. Schechter, `hep-ph/0112205`.

[25] J.T. Lenaghan, D.H. Rischke, J. Schaffner-Bielich, *Phys. Rev.* **D62**, 085008 (2000).

[26] M. Napsuciale, S. Rodriguez, *Int. J. Mod. Phys.* **A16**, 3011 (2001).

[27] S. Gasiorowicz, D.A. Geffen, *Rev. Mod. Phys.* **41**, 531 (1969).

[28] P. Carruthers, R.W. Haymaker, *Phys. Rev.* **D4**, 1808; 1815 (1971); J. Schechter, Y. Ueda, *Phys. Rev.* **D3**, 2874 (1971).

[29] G. 't Hooft, *Phys. Rep.* **142**, 357 (1986).

[30] R.D. Pisarski, F. Wilczek, *Phys. Rev.* **D29**, 338 (1984).

[31] N.A. Tornqvist, *Eur. Phys. J.* **C11**, 359 (1999).

[32] M. Napsuciale, `hep-ph/0204170`.

[33] M. Gell-Mann, M. Levi, *Nuovo Cim.* **16**, 705 (1960).

[34] K. Rajagopal, F. Wilczek, *Nucl. Phys.* **B404**, 577 (1993).

[35] W.J. Kious, R.I. Tilling
http://pubs.usgs.gov/publications/text/dynamic.html

[36] T.D. Lee, G.C. Wick, *Phys. Rev.* **D9**, 2291 (1974).

[37] T.D. Lee, *Particle Physics And Introduction To Field Theory*, New York 1981.

[38] A.A. Anselm, *Phys. Lett.* **B217**, 169 (1989); A.A. Anselm, M.G. Ryskin, *Phys. Lett.* **B266**, 482 (1991).

[39] J.D. Bjorken, *Int. J. Mod. Phys.* **A7**, 4189 (1992); J.D. Bjorken, K.L. Kowalski, C.C. Taylor, `hep-ph/9309235`.

[40] J.D. Bjorken, SLAC-PUB-6488, Lectures given at Workshop on Continuous Advances in QCD, Minneapolis, MN, 18-20 Feb 1994.

[41] Z. Huang, *Phys. Rev.* **D49**, 16 (1994).

[42] J.D. Bjorken, *Acta Phys. Pol. B* **28**, 2773 (1997).

[43] D.A. Kirzhnits, A.D. Linde, *Sov. Phys. JETP* **40**, 628 (1975), [*Zh. Eksp. Teor. Fiz.* **67**, 1263 (1974)]; *Ann. Phys.* **101**, 195 (1976); D.G. Caldi, S. Nussinov, *Phys. Rev.* **D29**, 739 (1984).

[44] For more rigorous treatment of symmetry restoration in the thermal field theory see L. Dolan, R. Jackiw, *Phys. Rev.* **D9**, 3320 (1974). A basic introduction to the thermal field theory is given, for example, in T. Altherr, *Int. J. Mod. Phys.* **A8**, 5605 (1993).

[45] K. Rajagopal, F. Wilczek, *Nucl. Phys.* **B399**, 395 (1993).

[46] J.P. Blaizot, A. Krzywicki, *Phys. Rev.* **D50**, 442 (1994); *Phys. Rev.* **D46**, 246 (1992).

[47] M.J. Bowick, A. Momen, *Phys. Rev.* **D58**, 085014 (1998); D. Boyanovsky, D. s. Lee, A. Singh, *Phys. Rev.* **D48**, 800 (1993); D. Boyanovsky, *Phys. Rev.* **E48**, 767 (1993).

[48] S.P. Kim, C.H. Lee, *Phys. Rev.* **D62**, 125020 (2000).

[49] Theory of phase ordering kinetics is reviewed in A.J. Bray, *Adv. Phys.* **43**, 357 (1994).

[50] A. Krzywicki, `hep-ph/9405244`.

[51] J.P. Blaizot, A. Krzywicki, *Acta Phys. Pol. B* **27**, 1687 (1996); Z. Huang, `hep-ph/9501366`; K. Rajagopal, `hep-ph/9703258`; S. Gavin, *Nucl. Phys.* **A590**, 163C (1995); I.M. Dremin, A.V. Leonidov, *Usp. Fiz. Nauk* **165**, 759 (1995).

[52] G.A. Schuler, T. Sjostrand, *Z. Phys.* **C73**, 677 (1997).

[53] R. Engel, J. Ranft, *Phys. Rev.* **D54**, 4244 (1996).

[54] P. Chen, T.L. Barklow, M.E. Peskin, *Phys. Rev.* **D49**, 3209 (1994).

[55] A. Krzywicki, J. Serreau, *Phys. Lett.* **B448**, 257 (1999).

[56] J. Randrup, *Phys. Rev. Lett.* **77**, 1226 (1996).

[57] J.D. Bjorken, *Acta Phys. Pol. B* **23**, 637 (1992).

[58] H. Minakata, B. Muller, *Phys. Lett.* **B377**, 135 (1996); M. Asakawa, H. Minakata, B. Muller, *Phys. Rev.* **D58**, 094011 (1998).

[59] J.D. Bjorken, K.L. Kowalski, C.C. Taylor, SLAC-PUB-6109 Presented at 7th Les Rencontres de Physique de la Vallee d'Aoste: Results and Perspectives in Particle Physics, La Thuile, Italy, 7–13 Mar 1993; G. Amelino-Camelia, J.D. Bjorken, S.E. Larsson, *Phys. Rev.* **D56**, 6942 (1997).

[60] K.L. Kowalski, C.C. Taylor, `hep-ph/9211282.`

[61] A.J. Leggett, *Phys. Rev. Lett.* **53**, 1096 (1984).

[62] A.J. Leggett, *J. Low Temp. Phys.* **87**, 571 (1992); P. Schiffer, D.D. Osheroff, A.J. Leggett, *Prog. Low Temp. Phys.* vol. XIV, ed. W.P. Halperin, L.P. Pitaevskii, Elsevier, Amsterdam 1995, pp. 159-211.

[63] *Phys. Today* **45N6**, 20 (1992).

[64] See, for example, S. Gavin in Ref. [51].

[65] J.I. Kapusta, A.M. Srivastava, *Phys. Rev.* **D52**, 2977 (1995); J.I. Kapusta, S.M. Wong, *J. Phys. G* **28**, 1929 (2002).

[66] S. Mrówczyński, B. Müller, *Phys. Lett.* **B363**, 1 (1995).

[67] S. Maedan, *Phys. Lett.* **B512**, 73 (2001).

[68] H. Hiro-Oka, H. Minakata, *Phys. Rev.* **C61**, 044903 (2000); D.I. Kaiser, *Phys. Rev.* **D59**, 117901 (1999).

[69] See, for example, J. Miles, D. Henderson, *Annu. Rev. Fluid Mech. 1990* **22**, 143 (1990); J. Bechhoefer, B. Johnson, *Am. J. Phys.* **64**, 1482 (1996).

[70] Some examples of such patterns can be found here:
http://www.cmp.caltech.edu/ lifshitz/patterns.html
For comprehensive review of pattern formation see M. C. Cross, P. C. Hohenberg, *Rev. Mod. Phys.* **65**, 851 (1993).

[71] N. Saitô, N. Hirotomi, A. Ichimura, *J. Phys. Soc. Jap.* **39**, 1431 (1975).

[72] G. Christie, B.I. Henry, *Phys. Rev.* **E58**, 3045 (1998).

[73] K. Yoshimura, *Phys. Rev.* **E62**, 6447 (2000).

[74] T.C. Petersen, J. Randrup, *Phys. Rev.* **C61**, 024906 (2000); J. Randrup, *Heavy Ion Phys.* **9**, 289 (1999).

[75] For review see, for example, W.M. Zhang, D.H. Feng, R. Gilmore, *Rev. Mod. Phys.* **62**, 867 (1990).

[76] R.D. Amado, F. Cannata, J.P. Dedonder, M.P. Locher, B. Shao, *Phys. Rev.* **C50**, 640 (1994).

[77] I.I. Kogan, *JETP Lett.* **59**, 307 (1994).

[78] R.D. Amado, I.I. Kogan, *Phys. Rev.* **D51**, 190 (1995).

[79] Extensive bibliography about squeezed states can be found in V.V. Dodonov, J. Opt. B: Quantum Semiclass. Opt. **4**, R1 (2002), (http://www.df.ufscar.br/ quantum/publications/job4-s98.pdf).
Excellent informal introduction is given by M.M. Nieto, LA-UR-84-2773.

[80] H.R. Lewis, W.B. Riesenfeld, *J. Math. Phys.* **10**, 1458 (1969).

[81] C.J. Eliezer, A. Gray, SIAM, *J. Appl. Math.* **30**, 463 (1976).

[82] H.R. Lewis, *J. Math. Phys.* **9**, 1976 (1968).

[83] Ermakov's method, which dates back to 1880, is reviewed in P.B. Espinoza, `math-ph/0002005`. See also R.S. Kaushal, *Int. J. Theor. Phys.* **37**, 1793 (1998). The Courant-Snyder invariant, used to study betatron oscillations in accelerators and storage rings, is of the same type. See, for example, K. Takayama, FERMILAB-FN-0349 (http://fnalpubs.fnal.gov/archive/fn/FN-0349.pdf).

[84] About this equation see P.B. Espinoza in [83]. See also L.M. Berkovich, N.H. Rozov, *Archivum Mathematicum (Brno)* **33**, 75 (1997).

[85] D. Horn, R. Silver, *Ann. Phys.* **66**, 509 (1971).

[86] A.Z. Mekjian, *Phys. Rev.* **C60**, 067902 (1999).

[87] A simple proof can be found, for example, in the book J.D. Bjorken, S.D. Drell, *Relativistic quantum fields*, McGraw-Hill, 1965. §123.

[88] For a historical review see M.M. Nieto, `quant-ph/9708012`.

[89] V.I. Man'ko, `quant-ph/9509018`.

[90] S.Y. Khlebnikov, V.A. Rubakov, P.G. Tinyakov, *Nucl. Phys.* **B350**, 441 (1991).

[91] T.M. Gould, S.D. Hsu, E.R. Poppitz, *Nucl. Phys.* **B437**, 83 (1995).

[92] R.F. Alvarez-Estrada, A. Gomez Nicola, L.L. Sanchez-Soto, A. Luis, *J. Phys. A* **28**, 3439 (1995); G. Alvarez, R.F. Alvarez-Estrada, *J. Phys. A* **28**, 5767 (1995).

[93] A.B. Klimov, L.L. Sanchez-Soto, J. Delgado, *Opt. Commun.* **191**, 419 (2001).

[94] E. Gladysz-Dziadus, `hep-ph/0111163`.

[95] M. Rybczynski, Z. Wlodarczyk, G. Wilk, *Nuovo Cim.* **24C**, 645 (2001); *Acta Phys. Pol. B* **33**, 277 (2002); E. Gladysz-Dziadus *et al.*, `hep-ex/0209008`.

[96] J.D. Bjorken [MiniMax Collaboration], `hep-ph/9610379`.

[97] T.C. Brooks *et al.* [MiniMax Collaboration], *Phys. Rev.* **D61**, 032003 (2000).

[98] M.M. Aggarwal *et al.* [WA98 Collaboration], *Phys. Rev.* **C64**, 011901 (2001).

[99] A.L. Angelis *et al.*, `hep-ex/9901038`.

[100] X. Artru, *Nucl. Phys.* **B85**, 442 (1975); G.C. Rossi, G. Veneziano, *Nucl. Phys.* **B123**, 507 (1977).

[101] D. Kharzeev, *Phys. Lett.* **B378**, 238 (1996); G.T. Garvey, B.Z. Kopeliovich, B. Povh, *Comments Mod. Phys.* **A2**, 47 (2001).

[102] T. Csorgo, M. Gyulassy, D. Kharzeev, `hep-ph/0112066`; `hep-ph/0102282`.

[103] X. Zhang, T. Huang, R.H. Brandenberger, *Phys. Rev.* **D58**, 027702 (1998); A.P. Balachandran, S. Digal, *Int. J. Mod. Phys.* **A17**, 1149 (2002).

[104] P.D. Morley, I.A. Schmidt, *Z. Phys.* **C26**, 627 (1985); D. Kharzeev, R.D. Pisarski, M.H. Tytgat, *Phys. Rev. Lett.* **81**, 512 (1998); A.P. Balachandran, S. Digal, *Phys. Rev.* **D66**, 034018 (2002).

[105] W. Shakespeare, *Hamlet, Prince of Denmark*, 1601. Act 1, scene 5.

[106] http://www.geocities.com/televisioncity/set/4567/bakeex.htm