GLUON SUPERPROPAGATOR AS THE ORIGIN OF QCD POTENTIAL

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We show that the confining term in the widely used Cornell form of QCD potential is derivable from the gluon superpropagator for an exponential form of gluon self-interaction, if one assumes that the gluon–gluon coupling constant has the character of a running coupling constant. We also consider a rational form of self-interaction.

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The typical form of QCD potential [1–3] which has now been in use for over two decades is:

$$V(r) = \frac{\alpha_{\rm s}}{r} - \lambda r^{\varepsilon} - C\,, \tag{1}$$

where r is the inter-quark distance, $\alpha_{\rm s}$ and λ are, respectively, the coupling constants corresponding to the 'Coulomb' and the confining parts of the potential, ε is a positive rational number and C is a constant. We note that RHS of (1) is to be multiplied by an appropriate colour factor which, for a quark–antiquark pair, for instance, is $(-4/3)$. While choices for ε have varied from 0.1 to 2.0, the most popular choices have been 1.0 and 2.0 (the linear and the harmonic oscillator potentials), and these have been employed not only within a non-relativistic framework [1], but also within a relativistic framework [2,4]. The performance of such models in accounting for the experimental hadron masses to varying degrees of precision has been impressive.

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As is well known, the form of the above potential is based on the belief in an underlying gauge theory of hadrons, salient features of which are asymptotic freedom and infrared slavery. In physical terms, the latter feature meets the phenomenological requirement that free quarks be unobservable. Nonetheless, from the point of view of field theory, potentials of the form of (1) have posed a puzzle all along. Unlike the Coulomb part of the potential, which corresponds to the Feynman diagram for one gluon exchange between the bound quarks, the field theoretic origin (if it exists) of the confining part has remained obscure. It is the purpose of this note to deal with this puzzle. The physical basis of our approach is the non-Abelian nature of QCD, which gives rise to self-interactions of the gauge field. Consequently, the gluon-exchanges between q and $\bar{q}$, for instance, are collimated into a tube-like shape, unlike in (Abelian) QED where the photons do not interact amongst themselves, and the photon-exchanges between two charges spread out to infinity in a spherical manner. We follow old custom in referring to such a tube of propagators of massless and coloured gluons as the gluon superpropagator. We now proceed to investigate if such a field-theoretic construct can yield the confining term in (1), and we choose to do so in as simple a framework as possible. Thus, we assume the gluon field, $\phi(x)$, to be a scalar field interacting with itself via a non-polynomial interaction $U[\phi(x)]$. From what has been said above, it follows that we need to consider the propagator for the "field" $U[\phi(x)]$, rather than for the field $\phi(x)$. Several choices for $U[\phi(x)]$ may be suggested:

$$\frac{1}{1+g\phi(x)}\,, \quad \frac{1}{1+g^2\phi^2(x)}\,, \quad \exp[g\phi(x)]\,, \quad etc.\,,$$

where g is the so-called minor coupling constant. The first among these choices is the simplest rational function of the gluon field; not surprisingly, it is one of the most widely studied non-polynomial interactions. The superpropagator corresponding to it has been calculated in different ways [*e.g.*, [5, 6]], and the representations so obtained — while they differ widely in form — have been shown to be equivalent [6]. Let us therefore deal with this case first. If we define

$$U[\phi(x)] = \frac{1}{1+g\phi(x)}\,, \tag{2}$$

then the gluon superpropagator for the superfield $U[\phi(x)]$ is given by

$$\begin{aligned} F(\Delta(x)) &= \langle 0|T[U(\phi(x))U(\phi(0))]|0\rangle \\ &= \sum_{n=0}^{\infty} g^{2n} n! \Delta^n(x)\,, \end{aligned} \tag{3}$$

where $\Delta(x)$ is the usual one-gluon propagator.

For the sake of completeness here, we recall from [6] the main steps in the calculation of (3). The expansion in (3) — even though formally divergent — can be summed by using the Euler–Borel formula

$$n! = \int_0^\infty \mathrm{e}^{-\zeta}\zeta^n d\zeta\,, \tag{4}$$

whence

$$F(\Delta) = \int d\zeta \mathrm{e}^{-\zeta} \sum_{n=0}^{\infty} \left(g^2\zeta\Delta\right)^n\,. \tag{5}$$

Let us denote the sum in (5) by $u(x)$; then $u(x)$ can be regarded as the iterative solution of an algebraic equation which, when transformed to momentum space, yields

$$u(p) = (2\pi)^4\delta^4(p) - \frac{ig^2\zeta}{(2\pi)^4}\int d^4q \frac{u(q)}{(p-q)^2}\,, \tag{6}$$

where $u(p)$ is the Fourier transform of $u(x)$.

Equation (6) can be easily converted into an equation for the modified Bessel function, and hence can be solved exactly. The momentum space superpropagator, $F(p^2)$, is then found to be given by

$$\begin{aligned} F(p^2) &= \int d\zeta \mathrm{e}^{-\zeta} u(p,\zeta) \\ &= (2\pi)^4\delta^4(p) + \frac{(g^2)^2}{p^2g^2}\exp\left(-\frac{p^2g^2}{32\pi^2}\right) W_{-2,1/2}\left(-\frac{p^2g^2}{16\pi^2}\right)\,, \end{aligned} \tag{7}$$

where $W_{\lambda,\mu}$ is the Whittaker function. As has been shown [6], (7) can be rewritten as

$$F(p^2) = \frac{g^4}{32\pi^2}\left[\frac{16\pi^2}{p^2g^2} + \int_0^\infty dt \frac{\sigma(t)}{(t - p^2g^2/16\pi^2)}\right]\,, \tag{8}$$

where $\sigma(t) = (2-t)\exp(-t)$.

This form of the superpropagator resembles a weighted superposition of propagators for the exchange of a group of gluon fields, as was intuitively expected. The potential implied by (8) is determined essentially by replacing

p^2 in it by $\boldsymbol{p}^2$ (instantaneous approximation) and then Fourier transforming to coordinate space. We thus have [7]

$$\begin{aligned} V(r) &\equiv \int F(p^2)\exp(i\boldsymbol{p}\cdot\boldsymbol{r})\frac{d^3p}{(2\pi)^3} = \frac{g}{2x} \\ &-\frac{g}{2x}\left[2 - x\sum_{k=0}^{\infty}(-1)^k k! x^{2k+2}/(2k+1)!\,{}_1F_1(2;1/2;-x^2/4)\right], \end{aligned} \quad (9)$$

where $x = 4\pi r/g$.

The above expression already brings out the richness of structure that a non-polynomial model of gluon self-interaction possesses. It is easy to see that, for *small* x, (9) reduces to the Cornell form of confining term, while for large x, its behaviour is ambiguous. Thus, the form of gluon self-interaction given by (2) is unacceptable. In a way, this result is not quite surprising because of the intrinsic Borel ambiguity that the theory suffers from (see also remarks at the end).

We now turn to the exponential form of self-interaction, which does not suffer from the Borel ambiguity. This model has been studied as extensively as the rational form dealt with above. The expressions for the superpropagator of this theory have been obtained by Okubo [8], Salam *et al.* [5], and by Biswas *et al.* [9]. Thus, following essentially the same procedure as outlined above, the superpropagator for the "field"

$$U[\phi(x)] = \exp[g\phi(x)] \quad (10)$$

is given by

$$F(p^2)|_{\exp} = (2\pi)^4\delta^4(p) - \frac{2\lambda}{p^2}\int_0^{\infty} d\beta \frac{J_2(\beta)}{\beta}\exp\left(-i\frac{p^2\lambda}{4\pi^2\beta^2}\right), \quad (11)$$

where $g^2 = -i\lambda$.

The QCD potential is now given by

$$\begin{aligned} V(r) &= -i\int F(p^2)|_{\exp}\exp(i\boldsymbol{p}\cdot\boldsymbol{r})\frac{d^3p}{(2\pi)^3} \\ &= -\frac{g^2}{4\pi r}\,{}_1F_1(1/2;2;g^2/4\pi^2r^2)\,. \end{aligned} \quad (12)$$

From the behaviour of ${}_1F_1(a;b;z)$ function in the limit of large z [10], it follows that the potential in (12) behaves in an unacceptable manner for $r \to 0$. At this stage one might be tempted to abandon the exponential model too. However, it seems legitimate to ask if, in some way, one could "save" the situation. Interestingly enough, if one assumes that the minor coupling constant has the character of running coupling constant, (12) becomes mathematically well-defined. Specifically, if we set

$$g \approx r \,, \tag{13}$$

we obtain

$$V(r) = -\lambda_{\mathrm{s}} r \,, \tag{14}$$

which again has the form of the confining term in the Cornell potential; however, the constant λ_{s} is now unambiguous, and is given by $(1/4\pi)\,{}_1F_1(1/2; 2; 1/4\pi^2)$. We conclude with the following remarks:

(1) One of the characteristic features of the rational theory was encountered above viz., Borel ambiguity; another such feature is its non-localizability. In contrast, the exponential theory is not only free from Borel ambiguity but is also a localizable theory. Furthermore, the theory is casual, analytic, unitary and positive-definite. These and other virtues of the theory have been discussed at length in [5]. It is in this sense that the exponential theory is spoken of as a "good" field theory.

(2) The suggestion that the strength of gluon–gluon interaction has the character of a running coupling constant may not be considered as far-fetched, if one recalls that the quark–gluon coupling constant behaves in a qualitatively similar manner. Indeed, the considerations of this note are heuristic and, possibly, indicative of the direction along which one might look for the origin of the QCD potential. It is clear that in a more realistic approach one would construct the superfield $U[\phi(x)]$ respecting the vector nature of the gluons. Even within the restricted framework employed here, the task of unearthing the origins of several other potentials *e.g.*, the logarithmic and the harmonic oscillator models remains. Possibly, the considerations of the preceding paragraph will help to rule out some of these choices.

(3) Note that the full Cornell potential, *i.e.*, the confining and the Coulomb parts, may be obtained by considering the "propagator" for the superfield $\{U[\phi(x)] - \phi(x)\}$.

(4) Finally, we would like to draw attention to an approach initiated in [11], which incorporates temperature into the dynamics of the bound state problem. For a follow up of this approach in the realm of QED, we refer the reader to references given in [12]. This approach can be adapted for QCD, irrespective of the origin or the form of the potential/kernel (in the instantaneous approximation), and brings out rather directly — as has been shown in [13–15] — the role that temperature plays in the problem.

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