BILARGE MIXING MATRIX AND ITS INVARIANCE UNDER "HORIZONTAL CONJUGATION" — A NEW DISCRETE TRANSFORMATION FOR NEUTRINOS*

Wojciech Królikowski

Institute of Theoretical Physics, Warsaw University
Hoża 69, 00–681 Warszawa, Poland

In the first part of the note, we consider a neutrino texture, where the Dirac and righthanded Majorana masses are proportional. If the former are approximately proportional also to the charged-lepton masses, then taking $\Delta m_{32}^2 \sim 3 \times 10^{-3}$ eV^2 we estimate approximately that $\Delta m_{21}^2 \sim O(10^{-5}$ eV$^2)$, what is not very different from the recent KamLAND estimation $\Delta m_{21}^2 \sim 7 \times 10^{-5}$ eV^2, consistent with the LMA solar solution. In the second part, we show generically that the *invariance* of neutrino mixing matrix under the simultaneous discrete transformations $\nu_e \to -\nu_e$, $\nu_\mu \to \nu_\tau$, $\nu_\tau \to \nu_\mu$ and $\nu_1 \to -\nu_1$, $\nu_2 \to -\nu_2$, $\nu_3 \to \nu_3$ (neutrino "horizontal conjugation") *characterizes* the familiar bilarge form of mixing matrix, favored phenomenologically at present. Then, in the case of this form, the mass neutrinos ν_1, ν_2, ν_3 get a new quantum number, *covariant* in their mixings (neutrino "horizontal parity" equal to $-1, -1, 1$, respectively). Conversely, such a covariance may be the *origin* of the bilarge mixing matrix. In Section 5, the "horizontal parity" is embedded in a group structure.

PACS numbers: 12.15.Ff, 14.60.Pq, 12.15.Hh

1. Introduction

In a recent note [1] we considered the neutrino texture, where the Dirac and righthanded components $M^{(\mathrm{D})}$ and $M^{(\mathrm{R})}$ of the generic neutrino 6×6 mass matrix

$$\begin{pmatrix} 0 & M^{(\mathrm{D})} \\ M^{(\mathrm{D})\,\mathrm{T}} & M^{(\mathrm{R})} \end{pmatrix} \tag{1}$$

commute and have (at least, approximately) proportional eigenvalues,

$$\lambda_1 : \lambda_2 : \lambda_3 = \Lambda_1 : \Lambda_2 : \Lambda_3 \,. \tag{2}$$

If, in addition, they are in a better or worse approximation proportional to the charged-lepton masses m_e, m_μ, m_τ, then in the seesaw mechanism the neutrino masses m_1, m_2, m_3 being equal to $-\lambda_1^2/\Lambda_1, -\lambda_2^2/\Lambda_2, -\lambda_3^2/\Lambda_3$, respectively, are approximately proportional to m_e, m_μ, m_τ. In fact,

$$m_1{:}m_2{:}m_3 = \frac{\lambda_1^2}{\Lambda_1}{:}\frac{\lambda_2^2}{\Lambda_2}{:}\frac{\lambda_3^2}{\Lambda_3} = \lambda_1{:}\lambda_2{:}\lambda_3 \simeq m_e{:}m_\mu{:}m_\tau \,. \tag{3}$$

In this case, we called the attention to the approximate relation

$$\frac{\Delta m_{21}^2}{\Delta m_{32}^2} \simeq \frac{m_\mu^2 - m_e^2}{m_\tau^2 - m_\mu^2} = 3.548 \times 10^{-3}\,, \tag{4}$$

predicting the value

$$\Delta m_{21}^2 \sim 1.1 \times 10^{-5}\ \text{eV}^2\,, \tag{5}$$

when the SuperKamiokande estimate $\Delta m_{32}^2 \sim 3 \times 10^{-3}$ eV^2 [2] is used (we assume that $\Delta m_{32}^2 \geq 0$). Then,

$$m_1^2 \sim 2.5 \times 10^{-10}\ \text{eV}^2,\ m_2^2 \sim 1.1 \times 10^{-5}\ \text{eV}^2,\ m_3^2 \sim 3 \times 10^{-3}\ \text{eV}^2 \tag{6}$$

and one obtains (with $m_1, m_2, m_3 > 0$ and so, $\Lambda_1, \Lambda_2, \Lambda_3 < 0$)

$$|\Lambda_1| \sim 1.7 \times 10^7\ \text{GeV},\ |\Lambda_2| \sim 3.4 \times 10^9\ \text{GeV},\ |\Lambda_3| \sim 5.8 \times 10^{10}\ \text{GeV}, \tag{7}$$

if one normalizes $\lambda_1^2 = m_e^2$ (i.e., $|\Lambda_1| = \lambda_1^2/m_1 = m_e^2/m_1$). It is convenient to put $\lambda_1, \lambda_2, \lambda_3 > 0$. Of course, if the SuperKamiokande estimate for Δm_{32}^2 decreased, the prediction for Δm_{21}^2 would also decrease.

The prediction (5) is not very different from the Large Mixing Angle MSW value supported confidently by the recent KamLAND experiment [3] and estimated as $\Delta m_{21}^2 \sim 7 \times 10^{-5}$ eV^2 (the lower LMA solution) [4–8]. In order to get this value more precisely, one may put $m_2/m_3 = \lambda_2/\lambda_3 \sim 2.6 m_\mu/m_\tau$ in place of $m_2/m_3 = \lambda_2/\lambda_3 \simeq m_\mu/m_\tau$, where $m_\tau/m_\mu = 16.82$ (here, Eq. (2) still holds).

2. Dirac and Majorana masses: the conjecture of proportionality

In the present note, we make the above considerations more operative putting (at least, approximately) $M^{(\text{R})} = -\zeta M^{(\text{D})}$ i.e., $\Lambda_{1,2,3} = -\zeta \lambda_{1,2,3}$, where $\zeta \gg 1$ holds in consistency with the seesaw mechanism. In addition to this conjecture of proportionality, we will assume the approximate proportionality $\lambda_1{:}\lambda_2{:}\lambda_3 \simeq m_e{:}m_\mu{:}m_\tau$, considered in Introduction, as normalized to the approximate equality $\lambda_{1,2,3} \simeq m_{e,\mu,\tau}$. Of course, the conjectured relation $M^{(\text{R})}_{\alpha\beta} = -\zeta M^{(\text{D})}_{\alpha\beta}$ ($\alpha, \beta = e, \mu, \tau$) is valid after the spontaneous breaking

of the electroweak symmetry $SU(2)_L \times U(1)_Y$ *i.e.*, when $M^{(D)}_{\alpha\,\beta} = Y^{(D)}_{\alpha\,\beta} \langle \phi^0 \rangle$. Then, also $M^{(R)}_{\alpha\,\beta}$ must include a mass scale. For instance, in the case of spontaneously broken left-right symmetry $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, where $Q = I^{(L)}_3 + Y/2$ and $Y/2 = I^{(R)}_3 + (B - L)/2$, this mass scale may be given by $\langle \phi^0_1 \rangle$ with $\vec{\phi}_1 = (\phi^{++}_1, \phi^+_1, \phi^0_1)$ denoting a Higgs right weak-isospin triplet (and a left weak-isospin singlet) which carries the (nonconserved) lepton number $L = -2$ (and the baryon number $B = 0$). In this case, $M^{(R)}_{\alpha\,\beta} = Y^{(R)}_{\alpha\,\beta} \langle \phi^0_1 \rangle = -\zeta Y^{(D)}_{\alpha\,\beta} \langle \phi^0 \rangle = -M^{(D)}_{\alpha\,\beta}$ according to our basic conjecture. If $Y^{(R)}_{\alpha\,\beta} = -Y^{(D)}_{\alpha\,\beta}$, then $\langle \phi^0_1 \rangle = \zeta \langle \phi^0 \rangle > 0$.

Under these conjectures, the 6×6 generic neutrino mass matrix (1) takes in the case of $M^{(D)T} = M^{(D)} = M^{(D)*}$ the form

$$
\begin{pmatrix} 0 & M^{(D)} \\ M^{(D)} & -\zeta M^{(D)} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1}^{(3)} \\ \mathbf{1}^{(3)} & -\zeta \mathbf{1}^{(3)} \end{pmatrix} \begin{pmatrix} M^{(D)} & 0 \\ 0 & M^{(D)} \end{pmatrix}, \quad (8)
$$

where $\mathbf{1}^{(3)} = \mathrm{diag}(1,1,1)$ and both matrix factors on the rhs commute. Using the diagonalizing matrix U for $M^{(D)}$ we obtain

$$
\begin{pmatrix} U^\dagger & 0 \\ 0 & U^\dagger \end{pmatrix} \begin{pmatrix} 0 & M^{(D)} \\ M^{(D)} & -\zeta M^{(D)} \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & \mathbf{1}^{(3)} \\ \mathbf{1}^{(3)} & -\zeta \mathbf{1}^{(3)} \end{pmatrix} \begin{pmatrix} \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3) & 0 \\ 0 & \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3) \end{pmatrix}, \quad (9)
$$

where both matrix factors on the rhs commute, of course. The diagonal form of the first matrix factor on the rhs of Eq. (9) turns out to be (for $\zeta > 0$):

$$
\begin{pmatrix} \left[-\frac{\zeta}{2} + \sqrt{\left(\frac{\zeta}{2}\right)^2 + 1} \right] \mathbf{1}^{(3)} & 0 \\ 0 & \left[-\frac{\zeta}{2} - \sqrt{\left(\frac{\zeta}{2}\right)^2 + 1} \right] \mathbf{1}^{(3)} \end{pmatrix}. \quad (10)
$$

Thus, the diagonal form of the mass matrix (8) is (for $\zeta > 0$):

$$
\begin{pmatrix} \left[-\frac{\zeta}{2} + \sqrt{\left(\frac{\zeta}{2}\right)^2 + 1} \right] \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) & 0 \\ 0 & \left[-\frac{\zeta}{2} - \sqrt{\left(\frac{\zeta}{2}\right)^2 + 1} \right] \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \end{pmatrix}
$$

$$
= \begin{pmatrix} \frac{1}{\zeta} \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3) & 0 \\ 0 & -\zeta \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3) \end{pmatrix}, \quad (11)
$$

where the rhs is valid for $\zeta \gg 1$. Hence, for $\zeta \gg 1$

$$
\begin{aligned}
m_{1,2,3} &= \frac{1}{\zeta}\lambda_{1,2,3} \simeq \frac{1}{\zeta}m_{e,\mu,\tau} \simeq -\frac{1}{\zeta^2}m_{4,5,6}\,, \\
m_{4,5,6} &= -\zeta\lambda_{1,2,3} \simeq -\zeta m_{e,\mu,\tau} \simeq -\zeta^2 m_{1,2,3}
\end{aligned} \tag{12}
$$

under our assumption of $\lambda_{1,2,3} \simeq m_{e,\mu,\tau}$. Using $m_\tau = 1776.99^{+0.29}_{-0.26}$ MeV and the SuperKamiokande estimate $\Delta m^2_{32} \sim 3\times10^{-3}\mathrm{eV}^2$ giving $m_3 \sim 5.5\times10^{-2}$ eV, we obtain

$$
\zeta = \frac{\lambda_3}{m_3} \simeq \frac{m_\tau}{m_3} \sim 3.2\times10^{10} \tag{13}
$$

and so, with $m_e = 0.510999$ MeV and $m_\mu = 105.658$ MeV we predict

$$
m_1 = \frac{1}{\zeta}\lambda_1 \simeq \frac{1}{\zeta}m_e \sim 1.6\times10^{-5} \text{ eV} \;,\;\; m_2 = \frac{1}{\zeta}\lambda_2 \simeq \frac{1}{\zeta}m_\mu \sim 3.3\times10^{-3} \text{ eV}. \tag{14}
$$

Thus, $m_1^2 \sim 2.5\times10^{-10}$ eV^2, $m_2^2 \sim 1.1\times10^{-5}$ eV^2 and $m_3^2 \sim 3\times10^{-3}$ eV^2.

The KamLAND estimate $\Delta m^2_{21} \sim 7\times10^{-5}$ eV^2 gives the value $m_2 \sim 8.4\times10^{-3}$ eV which lies not so far from our parameter-free prediction $m_2 \sim 3.3\times10^{-3}$ eV. Note that we may get the KamLAND value more precisely putting $m_2/m_3 = \lambda_2/\lambda_3 \sim 2.6m_\mu/m_\tau$ $i.e.$, for instance, $\zeta m_1 = \lambda_1 \sim m_e$, $\zeta m_2 = \lambda_2 \sim 0.90m_\mu$ and $\zeta m_3 = \lambda_3 \sim 0.35m_\tau$ in place of $\zeta m_2 = \lambda_2 \simeq m_\mu$ and $\zeta m_3 = \lambda_3 \simeq m_\tau$, as $0.90/0.35 = 2.6$ (here, Eq. (2) still holds, now with $\Lambda_{1,2,3} = -\zeta\lambda_{1,2,3}$). In this example, $\zeta = \lambda_3/m_3 \sim 0.35m_\tau/m_3 \sim 1.1\times10^{10}$.

It is natural that the neutrino Dirac masses $\lambda_1, \lambda_2, \lambda_3$ may be not equal to the charged-lepton masses m_e, m_μ, m_τ (in fact, somewhat smaller than these masses because of electromagnetic interactions of charged leptons). Of course, the neutrino Majorana masses m_1, m_2, m_3 are dramatically smaller than m_e, m_μ, m_τ, as $m_{1,2,3} \simeq \lambda_{1,2,3}/\zeta \ll \lambda_{1,2,3} \le m_{e,\mu,\tau}$ due to $\zeta \gg 1$. It seems also natural that the lepton Dirac masses $\lambda_1, \lambda_2, \lambda_3$ and m_e, m_μ, m_τ are smaller than the masses of respective up and down quarks, since quarks participate in strong interactions.

3. Bilarge mixing matrix: the invariance induced by $\nu_1, \nu_2, \nu_3 \rightarrow -\nu_1, -\nu_2, \nu_3$

In our texture, where

$$
M^{\mathrm{eff}} \equiv \left[-\frac{\zeta}{2} + \sqrt{\left(\frac{\zeta}{2}\right)^2 + 1} \right] M^{(\mathrm{D})} = \frac{1}{\zeta}M^{(\mathrm{D})} \tag{15}
$$

($\zeta \gg 1$) with

$$U^\dagger M^{(\mathrm{D})} U = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \,, \tag{16}$$

and in consequence

$$m_{1,2,3} = \left[-\frac{\zeta}{2} + \sqrt{\left(\frac{\zeta}{2}\right)^2 + 1} \right] \lambda_{1,2,3} = \frac{1}{\zeta} \lambda_{1,2,3} \tag{17}$$

($\zeta \gg 1$), the form of the Dirac mass matrix $M^{(\mathrm{D})}$ is unknown. In the situation, when the form of effective mass matrix M^{eff} for active neutrinos ν_e, ν_μ, ν_τ is theoretically *not known* enough, the questions of the neutrino mass spectrum m_1, m_2, m_3 and of the diagonalizing matrix U for M^{eff},

$$U^\dagger M^{\mathrm{eff}} U = \mathrm{diag}(m_1, m_2, m_3) \,, \tag{18}$$

are phenomenologically *independent*, though they lead jointly to

$$M^{\mathrm{eff}} = U \mathrm{diag}(m_1, m_2, m_3) U^\dagger \,. \tag{19}$$

This independence enables, *a priori*, a hierarchical mass spectrum to coexist with a large mixing of neutrino states by the diagonalizing matrix.

In the flavor representation, where the mass matrix for charged leptons is diagonal, the neutrino diagonalizing matrix $U = (U_{\alpha i})$ ($\alpha = e, \mu, \tau$ and $i = 1, 2, 3$) is at the same time the mixing matrix for active neutrinos according to the unitary transformation

$$\nu_\alpha = \sum_i U_{\alpha i} \nu_i \,, \tag{20}$$

where $\nu_\alpha \equiv \nu_{\alpha \mathrm{L}}$ and $\nu_i \equiv \nu_{i \mathrm{L}}$ denote the active-neutrino flavor and mass fields, respectively. As is well known, the bilarge form of the mixing matrix

$$U = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{1}{\sqrt{2}} s_{12} & \frac{1}{\sqrt{2}} c_{12} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} s_{12} & -\frac{1}{\sqrt{2}} c_{12} & \frac{1}{\sqrt{2}} \end{pmatrix} \,, \tag{21}$$

where $c_{23} = 1/\sqrt{2} = s_{23}$ and $s_{13} = 0$ (and $s_{12} < c_{12}$ with $\theta_{12} \sim 33°$ [4-8] are also large), is globally consistent with all present neutrino oscillation experiments for solar ν_e's and atmospheric ν_μ's as well as with the negative Chooz experiment [9] for reactor $\bar{\nu}_e$'s (giving $s_{13}^2 < 0.03$), but it cannot explain the possible LSND effect [10] for accelerator $\bar{\nu}_\mu$'s (and ν_μ's) whose existence is expected to be clarified soon in the MiniBOONE experiment (in Ref. [11] a "last hope" for explaining the possible LSND effect by a hypothetic sterile neutrino is considered).

In the case of the mixing matrix U as given in Eq. (21), the unitary transformation (20) gets the form

$$
\begin{aligned}
\nu_e &= c_{12}\nu_1 + s_{12}\nu_2\,,\\
\nu_\mu &= -\frac{1}{\sqrt{2}}(s_{12}\nu_1 - c_{12}\nu_2) + \frac{1}{\sqrt{2}}\nu_3\,,\\
\nu_\tau &= \frac{1}{\sqrt{2}}(s_{12}\nu_1 - c_{12}\nu_2) + \frac{1}{\sqrt{2}}\nu_3\,,
\end{aligned}
\tag{22}
$$

while the inverse transformation reads

$$
\begin{aligned}
\nu_1 &= c_{12}\nu_e - s_{12}\frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau)\,,\\
\nu_2 &= s_{12}\nu_e + c_{12}\frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau)\,,\\
\nu_3 &= \frac{1}{\sqrt{2}}(\nu_\mu + \nu_\tau)\,.
\end{aligned}
\tag{23}
$$

It can be seen that due to Eq. (22) the discrete transformation $\nu_1 \to -\nu_1$, $\nu_2 \to -\nu_2$, $\nu_3 \to \nu_3$ of mass neutrinos induces for flavor neutrinos the discrete transformation $\nu_e \to -\nu_e$, $\nu_\mu \to \nu_\tau$, $\nu_\tau \to \nu_\mu$ *i.e.*, the change of sign of ν_e and the interchange of ν_μ and ν_τ [this is a consequence of the maximal mixing of ν_μ and ν_τ in Eqs. (23)]. We can conclude that the above interplay between both discrete transformations *characterizes* the form (21) of mixing matrix and so, if conjectured, *selects* such a form (for any c_{12} and s_{12}) from its other possible forms. Formally, we infer that the above interplay is realized just in the case of U given in Eq. (21) because of the relations

$$
\begin{pmatrix} -\nu_1 \\ -\nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}\,,
$$

$$
\begin{pmatrix} -\nu_e \\ \nu_\tau \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}
\tag{24}
$$

and

$$
\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} U \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = U\,,
\tag{25}
$$

where due to Eq. (20)

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} . \tag{26}$$

Here, the relation (25) is crucial, telling us that the mixing matrix U of the form (21) *is invariant* under the simultaneous transformations $\nu_e \to -\nu_e$, $\nu_\mu \to \nu_\tau$, $\nu_\tau \to \nu_\mu$ and $\nu_1 \to -\nu_1$, $\nu_2 \to -\nu_2$, $\nu_3 \to \nu_3$. Given such a mixing matrix U, the first transformation is induced by the second through the unitary transformation between their matrices: $(I) = U(II)U^\dagger$. That is equivalent to $(I)U(II) = U$ *i.e.*, to Eq. (25).

Making use of the (formal) horizontal SU(3) group generated by $\widehat{\lambda}_a/2$ ($a = 1, 2, \ldots, 8$) with $\widehat{\lambda}_a$ being the Gell-Mann 3×3 matrices acting on the horizontal triplet $(\nu_1, \nu_2, \nu_3)^{\mathrm{T}}$, we can realize the above discrete transformations for mass and flavor neutrinos by means of the matrices

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\frac{1}{3}\widehat{1} - \frac{2}{\sqrt{3}}\widehat{\lambda}_8 ,$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = -\frac{1}{3}\widehat{1} - \frac{1}{2}\left(\widehat{\lambda}_3 + \frac{1}{\sqrt{3}}\widehat{\lambda}_8\right) + \widehat{\lambda}_6 , \tag{27}$$

where

$$\widehat{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \widehat{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\widehat{\lambda}_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad \widehat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{28}$$

Note that

$$Q^{(\mathrm{H})} \equiv \frac{1}{2}\left(\widehat{\lambda}_3 + \frac{1}{\sqrt{3}}\widehat{\lambda}_8\right) = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \tag{29}$$

plays a role of "horizontal charge", while $I_3^{(\mathrm{H})} \equiv \widehat{\lambda}_3/2$ and $Y^{(\mathrm{H})} \equiv \widehat{\lambda}_8/\sqrt{3}$ are the 3-component of the "horizontal isospin" and the "horizontal hypercharge", respectively.

Notice also that the mixing matrix (21) can be written in the form

$$U = e^{i\widehat{\lambda}_7 \, \pi/4} e^{i\widehat{\lambda}_2 \, \theta_{12}} \, , \tag{30}$$

since

$$e^{i\widehat{\lambda}_7 \theta_{23}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad \widehat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{31}$$

and

$$e^{i\widehat{\lambda}_2 \theta_{12}} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widehat{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{32}$$

where $\theta_{23} = \pi/4$.

We can see from Eq. (32) that

$$e^{i\widehat{\lambda}_2 \pi} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \tag{33}$$

and so, our discrete transformation of mass neutrinos can be realized as

$$e^{i\widehat{\lambda}_2 \pi} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} -\nu_1 \\ -\nu_2 \\ \nu_3 \end{pmatrix} \tag{34}$$

i.e., as the unitary rotation around the "horizontal" 2-axis by angle 2π, generated by the 2-component $I_2^{(\mathrm{H})} \equiv \widehat{\lambda}_2/2$ of the "horizontal isospin". The discrete transformation $\nu_1 \to -\nu_1$, $\nu_2 \to -\nu_2$, $\nu_3 \to \nu_3$ realized in Eq. (34) may be called the "horizontal conjugation", while its matrix

$$P^{(\mathrm{H})} \equiv e^{i I_2^{(\mathrm{H})} \, 2\pi} = e^{i\widehat{\lambda}_2 \, \pi} \tag{35}$$

given in Eq. (33) — the "horizontal parity". Then, the mass neutrinos ν_1, ν_2, ν_3 correspond to the eigenvalues $-1, -1, 1$ of this parity, respectively; also the flavor neutrino ν_e gets the eigenvalue -1, while ν_μ and ν_τ mix the eigenvalues -1 and 1 in such a way that $(\nu_\mu \mp \nu_\tau)/\sqrt{2}$ have the eigenvalues ∓ 1. Thus, it follows from the unitary transformation (23) that for the form (21) of mixing matrix U the "horizontal parity" (35) is an observable *covariant* in neutrino mixings: $P^{(\mathrm{H})'} = U P^{(\mathrm{H})} U^\dagger$. This is equivalent to $P^{(\mathrm{H})'} U P^{(\mathrm{H})} = U$ *i.e.*, to Eq. (25).

At the same time, we can infer from the relation (25) and Eqs. (30) with (33) that

$$
e^{i\widehat{\lambda}_7\pi/4}e^{i\widehat{\lambda}_2\pi}e^{-i\widehat{\lambda}_7\pi/4} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \tag{36}
$$

and thus, in consequence of the unitary rotation (34) for mass neutrinos, the composed unitary rotation

$$
e^{i\widehat{\lambda}_7\pi/4}e^{i\widehat{\lambda}_2\pi}e^{-i\widehat{\lambda}_7\pi/4} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} -\nu_e \\ \nu_\tau \\ \nu_\mu \end{pmatrix} \tag{37}
$$

is induced for flavor neutrinos. This interplay between the discrete transformations (34) and (37) selects the form (21) of mixing matrix U (for any c_{12} and s_{12}) as satisfying the relation (25) that now is identically fulfilled, being reduced trivially to

$$
e^{i\widehat{\lambda}_7\pi/4}e^{i\widehat{\lambda}_2\theta_{12}} = e^{i\widehat{\lambda}_7\pi/4}e^{i\widehat{\lambda}_2\theta_{12}} \tag{38}
$$

due to Eqs. (30), (33) and (36).

4. Effective mass matrix: the invariance under
$\nu_e, \nu_\mu, \nu_\tau \to -\nu_e, \nu_\tau, \nu_\mu$

Making use of the formulae (19) and (21), and parametrizing the neutrino mass spectrum as

$$
m_1 = \overset{0}{m} -\delta, \; m_2 = \overset{0}{m} +\delta, \; m_3 = \overset{0}{m} +\Delta, \tag{39}
$$

we can write the effective mass matrix for active neutrinos ν_e, ν_μ, ν_τ in the form

$$
M^{\text{eff}} = \overset{0}{m} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\Delta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}
$$
$$
+ \frac{1}{2}\delta \begin{pmatrix} -2c & \sqrt{2}\,s & -\sqrt{2}\,s \\ \sqrt{2}\,s & c & -c \\ -\sqrt{2}\,s & -c & c \end{pmatrix}, \tag{40}
$$

where $c \equiv \cos 2\theta_{12} = c_{12}^2 - s_{12}^2$ and $s \equiv \sin 2\theta_{12} = 2c_{12}s_{12}$. In Eq. (40) all three terms commute (the product of the second and third term in both orders vanishes). Thus, consistently

$$\mathrm{diag}(m_1, m_2, m_3) = U^\dagger M^{\mathrm{eff}} U$$

$$= \overset{0}{m} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \Delta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \delta \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (41)$$

In Eq. (40), the third term is equal to the sum

$$\frac{1}{2}\delta \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} c + \frac{1}{\sqrt{2}}\delta \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} s. \quad (42)$$

Formally, in deriving Eq. (41) from Eq. (40) the relations

$$U^\dagger \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} U = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U^\dagger \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} U = 2 \begin{pmatrix} -c & -s & 0 \\ -s & c & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U^\dagger \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} U = \sqrt{2} \begin{pmatrix} -s & c & 0 \\ c & s & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (43)$$

are involved.

We can easily check that the effective mass matrix (40) of flavor neutrinos ν_e, ν_μ, ν_τ *is invariant* under the discrete transformation $\nu_e \to -\nu_e$, $\nu_\mu \to \nu_\tau$, $\nu_\tau \to \nu_\mu$ induced by $\nu_1 \to -\nu_1$, $\nu_2 \to -\nu_2$, $\nu_3 \to \nu_3$ ("horizontal conjugation"):

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} M^{\mathrm{eff}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = M^{\mathrm{eff}}. \quad (44)$$

In fact, all matrices appearing in Eqs. (40) and (42) commute with the transformation matrix in Eq. (44) that squared gives the unit matrix. The invariance (44) follows also directly from the formula (19) and the relation (25).

In terms of the Gell-Mann 3×3 matrices $\widehat{\lambda}_a$ $(a = 1, 2, \ldots, 8)$ we can put in Eqs. (40) and (42)

$$
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \frac{2}{3}\widehat{1} - \frac{1}{2}(\widehat{\lambda}_3 + \frac{1}{\sqrt{3}}\widehat{\lambda}_8) + \widehat{\lambda}_6 \ ,
$$

$$
\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = -\frac{3}{2}(\widehat{\lambda}_3 + \frac{1}{\sqrt{3}}\widehat{\lambda}_8) - \widehat{\lambda}_6 \ ,
$$

$$
\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \widehat{\lambda}_1 - \widehat{\lambda}_4 \ , \tag{45}
$$

where in addition to Eqs. (28) we use

$$
\widehat{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \widehat{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} . \tag{46}
$$

Then, from Eqs. (40) and (42) we obtain the formula

$$
\begin{aligned}
M^{\text{eff}} &= \left(\overset{0}{m} + \frac{1}{3}\varDelta\right)\widehat{1} - \frac{1}{2}\left(\varDelta + 3\delta c\right)\frac{1}{2}\left(\widehat{\lambda}_3 + \frac{1}{\sqrt{3}}\widehat{\lambda}_8\right) \\
&\quad + \frac{1}{2}\left(\varDelta - \delta c\right)\widehat{\lambda}_6 + \frac{1}{\sqrt{2}}\delta s\left(\widehat{\lambda}_1 - \widehat{\lambda}_4\right) \ .
\end{aligned} \tag{47}
$$

The effective mass matrix $M^{\text{eff}} = (M_{\alpha\beta})$ $(\alpha, \beta = e, \mu, \tau)$ may be also presented as

$$
M^{\text{eff}} = \sum_{\alpha\beta} M_{\alpha\beta} \, \widehat{e}_{\alpha\beta} \tag{48}
$$

in terms of the basic matrices $\widehat{e}_{\alpha\beta} \equiv (\delta_{\alpha\gamma}\,\delta_{\beta\delta})$, where

$$
\begin{aligned}
\widehat{e}_{ee} &= \frac{1}{3}\widehat{1} + \frac{1}{2}\left(\widehat{\lambda}_3 + \frac{1}{\sqrt{3}}\widehat{\lambda}_8\right), \quad \widehat{e}_{e\mu} = \frac{1}{2}\left(\widehat{\lambda}_1 + i\widehat{\lambda}_2\right) = \widehat{e}_{\mu e}^\dagger \ , \\
\widehat{e}_{\mu\mu} &= \frac{1}{3}\widehat{1} + \frac{1}{2}\left(-\widehat{\lambda}_3 + \frac{1}{\sqrt{3}}\widehat{\lambda}_8\right), \quad \widehat{e}_{e\tau} = \frac{1}{2}\left(\widehat{\lambda}_4 + i\widehat{\lambda}_5\right) = \widehat{e}_{\tau e}^\dagger \ , \\
\widehat{e}_{\tau\tau} &= \frac{1}{3}\widehat{1} - \frac{1}{\sqrt{3}}\widehat{\lambda}_8 \ , \qquad\qquad\quad \widehat{e}_{\mu\tau} = \frac{1}{2}\left(\widehat{\lambda}_6 + i\widehat{\lambda}_7\right) = \widehat{e}_{\tau\mu}^\dagger \quad \tag{49}
\end{aligned}
$$

with $\widehat{1} = (\delta_{\gamma\delta})$ and $\widehat{\lambda}_a = (\lambda_{a\gamma\delta})$ $(a = 1, 2, \ldots, 8)$. The matrix elements $M_{\alpha\beta}$ of M^{eff} are determined by the formula (19), $M_{\alpha\beta} = \sum_i U_{\alpha i} m_i U_{\beta i}^*$, that due

to Eqs. (21) and(39) gives

$$
\begin{aligned}
M_{ee} &= \overset{0}{m} - \delta c\,, \\
M_{\mu\mu} &= M_{\tau\tau} = \overset{0}{m} + \frac{1}{2}\Delta + \frac{1}{2}\delta c\,, \\
M_{e\mu} &= -M_{e\tau} = \frac{1}{\sqrt{2}}\delta s = M_{\mu e} = -M_{\tau e}\,, \\
M_{\mu\tau} &= \frac{1}{2}\Delta - \frac{1}{2}\delta c = -M_{\tau\mu}\,.
\end{aligned}
\tag{50}
$$

It may be worthwhile to note that the imaginary matrices $\widehat{\lambda}_2, \widehat{\lambda}_7$ and

$$
\widehat{\lambda}_5 = \left(\begin{array}{ccc} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{array}\right)\,,
\tag{51}
$$

although they appear within the basic matrices $\widehat{e}_{\alpha\beta}$, are canceled out in M^{eff} due to the relations $M_{\beta\alpha} = M_{\alpha\beta}$ and $\widehat{e}_{\beta\alpha} = \widehat{e}_{\alpha\beta}^{\dagger}$ applied to Eq. (48).

Finally, it is tempting to speculate that the invariance under the simultaneous discrete transformations $\nu_e \to -\nu_e$, $\nu_\mu \to \nu_\tau$, $\nu_\tau \to \nu_\mu$ and $\nu_1 \to -\nu_1$, $\nu_2 \to -\nu_2$, $\nu_3 \to \nu_3$ (neutrino "horizontal conjugation") — that, as shown in this note, characterizes the phenomenologically favored form (21) of neutrino mixing matrix — may play an important role in Nature because of the absence for neutrinos of electromagnetic and strong interactions. Otherwise, these interactions could largely suppress such a fragile, discrete horizontal symmetry that, in contrast to the Standard Model gauge interactions, does not treat equally three fermion generations. This may be also the reason, why the quark mixing matrix does not involve large mixings, in contrast to the lepton mixing matrix (i.e., neutrino mixing matrix in the flavor representation) observed by means of neutrino oscillations.

5. Bimaximal mixing matrix as an approximation

We know from experiments for solar ν_e's that the bilarge mixing matrix U given in Eq. (21) is not bimaximal, as $\theta_{12} \sim 33° < 45°$, and so

$$
c_{12} \sim 0.84 > \frac{1}{\sqrt{2}} > s_{12} \sim 0.54\,.
\tag{52}
$$

But, since both values c_{12} and s_{12} are still large and not very distant from $1/\sqrt{2} \simeq 0.71$, one may ask the question, if and to what extent the approximation $c_{12} \simeq 1/\sqrt{2} \simeq s_{12}$ may work, leading through Eq. (21) to the

approximate bimaximal form for the mixing matrix:

$$U \simeq \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} \end{pmatrix}. \tag{53}$$

For such an approximate form of U, the unitary transformation (20) implies

$$\begin{aligned} \nu_e &\simeq \frac{1}{\sqrt{2}}(\nu_1 + \nu_2), \\ \nu_\mu &\simeq -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(\nu_1 - \nu_2) + \frac{1}{\sqrt{2}}\nu_3, \\ \nu_\tau &\simeq \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(\nu_1 - \nu_2) + \frac{1}{\sqrt{2}}\nu_3 \end{aligned} \tag{54}$$

and

$$\begin{aligned} \nu_1 &\simeq \frac{1}{\sqrt{2}}\nu_e - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau), \\ \nu_2 &\simeq \frac{1}{\sqrt{2}}\nu_e + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau), \\ \nu_3 &\simeq \frac{1}{\sqrt{2}}(\nu_\mu + \nu_\tau). \end{aligned} \tag{55}$$

It is easy to see from Eqs. (54) and (55) that now, beside the previous (strict) symmetry (25), where

$$\nu_1, \nu_2, \nu_3 \to -\nu_1, -\nu_2, \nu_3 \quad \text{induces} \quad \nu_e, \nu_\mu, \nu_\tau \to -\nu_e, \nu_\tau, \nu_\mu,$$

there exist also two (approximate) symmetries, where

$$\nu_1, \nu_2, \nu_3 \to -\nu_2, -\nu_1, -\nu_3 \quad \text{induces} \quad \nu_e, \nu_\mu, \nu_\tau \to -\nu_e, -\nu_\tau, -\nu_\mu$$

and

$$\nu_1, \nu_2, \nu_3 \to \nu_2, \nu_1, -\nu_3 \quad \text{induces} \quad \nu_e, \nu_\mu, \nu_\tau \to \nu_e, -\nu_\mu, -\nu_\tau.$$

Both are excluded if it is considered that $c_{12} \neq s_{12}$ distinctly. If it is accepted that $c_{12} \simeq s_{12}$, then — in addition to the relation (25) — two new relations

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} U \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \simeq U \tag{56}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} U \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \simeq U \qquad (57)$$

follow, respectively, expressing two new (approximate) invariances of the (approximate) form (53) of U. These two (approximate) symmetries introduce the difference between the (approximate) bimaximal form (53) of U and its (strict) monomaximal form (21) where $c_{12} \neq s_{12}$ distinctly, because they work only for the former.

Let us denote our Hermitian and real 3×3 matrices transforming the mixing matrix U in the relations (56), (57) and (25) as

$$\widehat{\varphi}_1 \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \widehat{\varphi}_2 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \widehat{\varphi}_3 \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (58)$$

and

$$\widehat{\mu}_1 \equiv \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \widehat{\mu}_2 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \widehat{\mu}_3 \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (59)$$

Then, with $i = 1, 2, 3$ the symmetries (56), (57) and (25) of the (approximate) bimaximal mixing matrix U given in Eq. (53) may be expressed in three ways:

$$\widehat{\varphi}_i U \widehat{\mu}_i = U \text{ or } U \widehat{\mu}_i = \widehat{\varphi}_i U \text{ or } \widehat{\varphi}_i = U \widehat{\mu}_i U^{\dagger}, \qquad (60)$$

where for $i = 1, 2$ the equality is (only) approximate [while for $i = 3$ it is exact with U as given in Eq. (21)]. Note that Tr $\widehat{\varphi}_i = -1$ and Tr $\widehat{\mu}_i = -1$.

From the definitions (58) and (59) we can readily show that with $i, j = 1, 2, 3$

$$\widehat{\varphi}_i^2 = \widehat{1} \ , \ \widehat{\varphi}_1 \widehat{\varphi}_2 = \widehat{\varphi}_3 \text{ (cyclic)} \ , \ \widehat{\varphi}_i \widehat{\varphi}_j = \widehat{\varphi}_j \widehat{\varphi}_i \ , \ \widehat{\varphi}_1 + \widehat{\varphi}_2 + \widehat{\varphi}_3 = -\widehat{1} \qquad (61)$$

and

$$\widehat{\mu}_i^2 = \widehat{1} \ , \ \widehat{\mu}_1 \widehat{\mu}_2 = \widehat{\mu}_3 \text{ (cyclic)} \ , \ \widehat{\mu}_i \widehat{\mu}_j = \widehat{\mu}_j \widehat{\mu}_i \ , \ \widehat{\mu}_1 + \widehat{\mu}_2 + \widehat{\mu}_3 = -\widehat{1} \qquad (62)$$

(but in general $\widehat{\varphi}_i \widehat{\mu}_j \neq \widehat{\mu}_j \widehat{\varphi}_i$). Hence, we get in a more compact notation

$$\{ \widehat{\varphi}_i , \widehat{\varphi}_j \} = 2\delta_{ij} \widehat{1} + 2 \sum_k |\varepsilon_{ijk}| \widehat{\varphi}_k \ , \ [\widehat{\varphi}_i , \widehat{\varphi}_j] = 0 \qquad (63)$$

and

$$\{ \widehat{\mu}_i , \widehat{\mu}_j \} = 2\delta_{ij} \widehat{1} + 2 \sum_k |\varepsilon_{ijk}| \widehat{\mu}_k \ , \ [\widehat{\mu}_i , \widehat{\mu}_j] = 0 \qquad (64)$$

[but $\{\widehat{\varphi}_i, \widehat{\mu}_j\} \neq 0$ and in general $[\widehat{\varphi}_i, \widehat{\mu}_j] \neq 0$, *cf.* Eq. (72)]. We can also write for U given in Eq. (53)

$$\left(\delta_{ij}\widehat{1} + \sum_k |\varepsilon_{ijk}|\widehat{\varphi}_k\right)U = \widehat{\varphi}_i U\widehat{\mu}_j = U\left(\delta_{ij}\widehat{1} + \sum_k |\varepsilon_{ijk}|\widehat{\mu}_k\right) , \qquad (65)$$

where for $j = 1, 2$ and $i = 1, 2$, respectively, the first and second equality is only approximate [while for $j = 3$ and $i = 3$ it is exact with U as given in Eq. (21)]. Note that here $(|\varepsilon_{ij1}|) = \widehat{\lambda}_6$, $(|\varepsilon_{ij2}|) = \widehat{\lambda}_4$ and $(|\varepsilon_{ij3}|) = \widehat{\lambda}_1$.

It is easy to see that the effective mass matrix $M^{\text{eff}} = U\text{diag}(m_1, m_2, m_3)U^\dagger$ with U in the form (53) reveals for $i = 1, 2, 3$ the symmetries

$$\widehat{\varphi}_i M^{\text{eff}} \widehat{\varphi}_i = M^{\text{eff}} \ \text{ or } \ M^{\text{eff}}\widehat{\varphi}_i = \widehat{\varphi}_i M^{\text{eff}} , \qquad (66)$$

where for $i = 1, 2$ the equality is (only) approximately valid, provided we can accept beside the approximation $c_{12} \simeq s_{12}$ *also* $m_1 \simeq m_2$ *i.e.*, $\delta \simeq 0$ [while for $i = 3$ it is exact with U as given in Eq. (21)]. In the case of $\delta \simeq 0$, the values of c_{12} and s_{12} become irrelevant in M^{eff} [*cf.* Eqs. (50)].

The matrices $\widehat{\varphi}_i$ and $\widehat{\mu}_i$ $(i = 1, 2, 3)$ as defined in Eqs. (58) and (59) may be used as bases for 3×3 symmetric block matrices of the types

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & C & B \end{pmatrix} \text{ and } \begin{pmatrix} D & E & 0 \\ E & D & 0 \\ 0 & 0 & F \end{pmatrix} ,$$

respectively ($\widehat{1}$ is not needed in these bases because of the form of constraints $\widehat{\varphi}_1 + \widehat{\varphi}_2 + \widehat{\varphi}_3 = -\widehat{1}$ and $\widehat{\mu}_1 + \widehat{\mu}_2 + \widehat{\mu}_3 = -\widehat{1}$). The sets of such matrices form two Abelian groups with respect to matrix multiplication, if the inverse of their four blocks exists. They are isomorphic, as they are related through the unitary transformation generated by the bimaximal mixing matrix U given by the rhs of Eq. (53), $(I) = U(II)U^\dagger$, where (I) and (II) symbolize the sets of matrices of the first and second type. The group character of these sets is reflected in the group relations $\widehat{\varphi}_1\widehat{\varphi}_2 = \widehat{\varphi}_3$ (cyclic) and $\widehat{\mu}_1\widehat{\mu}_2 = \widehat{\mu}_3$ (cyclic) for their bases [*cf.* Eqs. (61) and (62)], while their isomorphism corresponds to the unitary transformation $\widehat{\varphi}_i = U\widehat{\mu}_i U^\dagger$ $(i = 1, 2, 3)$ between both bases [*cf.* Eqs. (60)]. These two groups are, of course, subgroups of the group of all 3×3 nonsingular matrices that can be spun by the basis consisting of $\widehat{1}$ and the Gell-Mann matrices $\widehat{\lambda}_a$ $(a = 1, 2, \dots, 8)$.

It is interesting to note that the neutrino effective mass matrix M^{eff} is of the form [*cf.* Eqs. (50)] belonging to the set (I) for $d = 0$:

$$\begin{pmatrix} a & d & -d \\ d & b & c \\ -d & c & b \end{pmatrix} .$$

This form commutes exactly with $\widehat{\varphi}_3$, while with $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ (only) approximately, provided $d \equiv \delta s/\sqrt{2} \simeq 0$ and so $\delta \simeq 0$ *i.e.*, $m_1 \simeq m_2$. Here, $\widehat{\varphi}_3 = U\widehat{\mu}_3 U^\dagger \equiv UP^{(\mathrm{H})}U^\dagger$ is the unitary transform of "horizontal parity" with U as given in Eq. (21). In this way, $\widehat{\varphi}_3 M^{\mathrm{eff}} \widehat{\varphi}_3 = UP^{(\mathrm{H})}\mathrm{diag}(m_1, m_2, m_3)P^{(\mathrm{H})}U^\dagger = M^{\mathrm{eff}}$. In terms of $\widehat{\varphi}_1$, $\widehat{\varphi}_2$, $\widehat{\varphi}_3$ and $\widehat{\lambda}_1$, $\widehat{\lambda}_4$ with $(\widehat{\varphi}_1 + \widehat{\varphi}_2)(\widehat{\varphi}_1 - \widehat{\varphi}_2) = 0$ and $(\widehat{\varphi}_1 + \widehat{\varphi}_2)(\widehat{\lambda}_1 - \widehat{\lambda}_4) = 0$ we can write

$$
M^{\mathrm{eff}} = -\left(\overset{0}{m} + \frac{1}{2}\Delta\right)(\widehat{\varphi}_1 + \widehat{\varphi}_2) - \overset{0}{m}\,\widehat{\varphi}_3 + \frac{1}{2}\delta\,c\,(\widehat{\varphi}_1 - \widehat{\varphi}_2) + \frac{1}{\sqrt{2}}\delta s\left(\widehat{\lambda}_1 - \widehat{\lambda}_4\right),
$$
(67)

where $\widehat{\varphi}_3 \widehat{\lambda}_{1,4} = -\widehat{\lambda}_{4,1}\widehat{\varphi}_3$ makes $\widehat{\varphi}_3$ commute with $\widehat{\lambda}_1 - \widehat{\lambda}_4$, while $\widehat{\varphi}_1 \widehat{\lambda}_{1,4} = \widehat{\lambda}_{4,1}\widehat{\varphi}_1$ and $\widehat{\varphi}_2 \widehat{\lambda}_{1,4} = -\widehat{\lambda}_{1,4}\widehat{\varphi}_2$ imply that $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ anticommute with $\widehat{\lambda}_1 - \widehat{\lambda}_4$. Also note that $U^\dagger\left[\frac{1}{2}c\,(\widehat{\varphi}_1 - \widehat{\varphi}_2) + \frac{1}{\sqrt{2}}s\left(\widehat{\lambda}_1 - \widehat{\lambda}_4\right)\right]U = -\widehat{\lambda}_3$ for any $c \equiv c_{12}^2 - s_{12}^2$ and $s \equiv 2c_{12}s_{12}$ [*cf.* Eqs. (43)], where U is given as in Eq. (21). If $c_{12} = 1/\sqrt{2} = s_{12}$, then $c = 0$ and $s = 1$.

In connection with the formula (67) we wonder, if the 3×3 matrices $\widehat{\varphi}_i$ and $\widehat{\mu}_i$ $(i = 1, 2, 3)$ may help us to find the desired dynamical variables solving hopefully the basic problem of fermion masses (notice that $\widehat{\lambda}_1 - \widehat{\lambda}_4 = \frac{1}{2}\{\widehat{\varphi}_3, \widehat{\mu}_1 - \widehat{\mu}_2\}$). In such a case there may appear a more or less instructive analogy with Pauli matrices that have led to Dirac matrices solving the problem of fermion spins.

In terms of the Gell-Mann 3×3 matrices $\widehat{\lambda}_a$ $(a = 1, 2, \ldots, 8)$ and their basic combinations $\widehat{e}_{\alpha\beta}$ $(\alpha, \beta = e, \mu, \tau)$ presented in Eqs. (49) we obtain

$$
\widehat{\varphi}_1 = -\widehat{\lambda}_6 - \widehat{e}_{ee}\,,\ \ \widehat{\varphi}_2 = -\widehat{1} + 2\widehat{e}_{ee}\,,\ \ \widehat{\varphi}_3 = \widehat{\lambda}_6 - \widehat{e}_{ee}
$$
(68)

and

$$
\widehat{\mu}_1 = -\widehat{\lambda}_1 - \widehat{e}_{\tau\tau}\,,\ \ \widehat{\mu}_2 = \widehat{\lambda}_1 - \widehat{e}_{\tau\tau}\,,\ \ \widehat{\mu}_3 = -\widehat{1} + 2\widehat{e}_{\tau\tau}\,,
$$
(69)

where

$$
\widehat{e}_{ee} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) = \frac{1}{3}\widehat{1} + \frac{1}{2}\left(\widehat{\lambda}_3 + \frac{1}{\sqrt{3}}\widehat{\lambda}_8\right),
$$

$$
\widehat{e}_{\tau\tau} = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right) = \frac{1}{3}\widehat{1} - \frac{1}{\sqrt{3}}\widehat{\lambda}_8\,.
$$
(70)

Due to Eqs. (33) and (36) we can write for $i = 3$

$$
\widehat{\mu}_3 = \mathrm{e}^{i\widehat{\lambda}_2 \pi}\,,\ \ \widehat{\varphi}_3 = \mathrm{e}^{i\widehat{\lambda}_7 \pi/4}\mathrm{e}^{i\widehat{\lambda}_2 \pi}\mathrm{e}^{-i\widehat{\lambda}_7 \pi/4} = U\mathrm{e}^{i\widehat{\lambda}_2 \pi}U^\dagger
$$
(71)

with the mixing matrix U as presented in Eq. (30). For $i = 3$ it is not necessary to make in U the approximation $\theta_{12} \simeq 45°$ (here, this mixing angle has its actual value $\theta_{12} \sim 33°$).

Since the matrices $\widehat{\varphi}_i$ and $\widehat{\mu}_i$ ($i = 1, 2, 3$) are Hermitian and real, their commutators are antiHermitian and real, and thus, can be expressed as combinations of three imaginary matrices $\widehat{\lambda}_2$, $\widehat{\lambda}_5$ and $\widehat{\lambda}_7$ of eight Gell-Mann 3×3 matrices $\widehat{\lambda}_a$ ($a = 1, 2, \ldots, 8$) which all are Hermitian. In fact, we find

$$([\widehat{\varphi}_i, \widehat{\mu}_j]) = i \begin{pmatrix} \widehat{\lambda}_2 - \widehat{\lambda}_5 + \widehat{\lambda}_7 & -\widehat{\lambda}_2 + \widehat{\lambda}_5 + \widehat{\lambda}_7 & -2\widehat{\lambda}_7 \\ -2\widehat{\lambda}_2 & 2\widehat{\lambda}_2 & 0 \\ \widehat{\lambda}_2 + \widehat{\lambda}_5 - \widehat{\lambda}_7 & -\widehat{\lambda}_2 - \widehat{\lambda}_5 - \widehat{\lambda}_7 & 2\widehat{\lambda}_7 \end{pmatrix}. \tag{72}$$

Hence, $\sum_i [\widehat{\varphi}_i, \widehat{\mu}_j] = 0$ and $\sum_j [\widehat{\varphi}_i, \widehat{\mu}_j] = 0$, what is consistent with two constraints $\sum_i \widehat{\varphi}_i = -\widehat{1}$ and $\sum_j \widehat{\mu}_j = -\widehat{1}$. The imaginary matrices $\widehat{\lambda}_2, \widehat{\lambda}_5$ and $\widehat{\lambda}_7$ are absent from $\widehat{\varphi}_i$ and $\widehat{\mu}_i$ that are combinations of $\widehat{1}, \widehat{\lambda}_1, \widehat{\lambda}_3, \widehat{\lambda}_8$ and $\widehat{\lambda}_6$ [*cf.* Eqs. (68)–(70)].

Analogically, it is easy to evaluate the anticommutators $\{\widehat{\varphi}_i, \widehat{\mu}_j\}$ that can be expressed as combinations of $\widehat{1}$ and five real matrices $\widehat{\lambda}_1, \widehat{\lambda}_3, \widehat{\lambda}_4, \widehat{\lambda}_6$ and $\widehat{\lambda}_8$ of eight $\widehat{\lambda}_a$ ($a = 1, 2, \ldots, 8$). None of these anticommutators is zero.

Note that the matrices $\widehat{\varphi}_i$ and $\widehat{\mu}_i$ defined in Eqs. (58) and (59) have the following block structure

$$\widehat{\varphi}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & & \\ & & -\widehat{\sigma}_1^P \\ 0 & & \end{pmatrix}, \; \widehat{\varphi}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ & & -\widehat{1}^P \\ 0 & & \end{pmatrix}, \; \widehat{\varphi}_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & & \\ & & \widehat{\sigma}_1^P \\ 0 & & \end{pmatrix} \tag{73}$$

and

$$\widehat{\mu}_1 = \begin{pmatrix} & & 0 \\ -\widehat{\sigma}_1^P & & \\ & & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; \widehat{\mu}_2 = \begin{pmatrix} & & 0 \\ \widehat{\sigma}_1^P & & \\ & & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; \widehat{\mu}_3 = \begin{pmatrix} & & 0 \\ -\widehat{1}^P & & \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{74}$$

where

$$\widehat{1}^P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \widehat{\sigma}_1^P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{75}$$

6. Conclusions

We have introduced in Section 5 two algebras of commuting Hermitian 3×3 matrices $\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3$ and $\widehat{\mu}_1, \widehat{\mu}_2, \widehat{\mu}_3$, satisfying the group relations $\widehat{\varphi}_1\widehat{\varphi}_2 = \widehat{\varphi}_3$ (cyclic) and $\widehat{\mu}_1\widehat{\mu}_2 = \widehat{\mu}_3$ (cyclic) as well as the constraints $\widehat{\varphi}_1 + \widehat{\varphi}_2 + \widehat{\varphi}_3 = -\widehat{1}$ and $\widehat{\mu}_1 + \widehat{\mu}_2 + \widehat{\mu}_3 = -\widehat{1}$. These two algebras are isomorphic, as being related by the unitary transformation $\widehat{\varphi}_i = U \widehat{\mu}_i U^\dagger$ ($i = 1, 2, 3$), where U is the neutrino mixing matrix of the bilarge form (21), phenomenologically favored at present, which for $i = 1, 2$ is approximated to the nearly bimaximal form (53) with $c_{12} \simeq 1/\sqrt{2} \simeq s_{12}$ (*i.e.*, with $\theta_{12} \simeq 45°$, while the actual experimental estimate is $\theta_{12} \sim 33°$). For $i = 3$ the unitary transformation is exact, thus the resulting *invariance* $\widehat{\varphi}_3 U \widehat{\mu}_3 = U$ is also exact. It *characterizes* the monomaximal form (21) of neutrino mixing matrix U for any c_{12} and s_{12}. The resulting *approximate invariances* $\widehat{\varphi}_i U \widehat{\mu}_i = U$ for $i = 1, 2$, if accepted as working, *suggest* that this monomaximal form of U is, in fact, nearly bimaximal. Of course, the closer the experimetal estimate of θ_{12} is to $45°$, the better is this conclusion.

We have called the transformation $U' = \widehat{\varphi}_3 U \widehat{\mu}_3$ providing the exact invariance $U' = U$ the neutrino "horizontal conjugation", while its matrix $P^{(\mathrm{H})} \equiv \widehat{\mu}_3 = \exp(i\widehat{\lambda}_2\pi)$ — the neutrino "horizontal parity". The latter is a unitary rotation by the angle 2π around the "horizontal" 2-axis. Such a conjugation implies the transformation

$$\begin{pmatrix} \nu'_1 \\ \nu'_2 \\ \nu'_3 \end{pmatrix} = P^{(\mathrm{H})} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} -\nu_1 \\ -\nu_2 \\ \nu_3 \end{pmatrix} \tag{76}$$

for mass neutrinos, inducing simultaneously the transformation

$$\begin{pmatrix} \nu'_e \\ \nu'_\mu \\ \nu'_\tau \end{pmatrix} = U P^{(\mathrm{H})} U^\dagger \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} -\nu_e \\ \nu_\tau \\ \nu_\mu \end{pmatrix} \tag{77}$$

for flavor neutrinos, since $\nu'_\alpha = \sum_i U_{\alpha i}\nu'_i$ and $\nu_i = \sum_\alpha U^*_{\alpha i}\nu_\alpha$. Thus, the "horizontal parity" displays the *covariance* $P^{(\mathrm{H})'} = U P^{(\mathrm{H})} U^\dagger$ with $P^{(\mathrm{H})'} \equiv \widehat{\varphi}_3$. Here, the mass neutrinos ν_1, ν_2, ν_3 get the "horizontal parity" equal to $-1, -1, 1$, respectively, while the flavor neutrinos ν_e, ν_μ, ν_τ, except for ν_e, mix the "horizontal parity".

The corresponding transformation for the neutrino effective mass matrix $M^{\mathrm{eff}} = U \operatorname{diag}(m_1, m_2, m_3) U^\dagger$ reads $M^{\mathrm{eff}'} = \widehat{\varphi}_3 M^{\mathrm{eff}} \widehat{\varphi}_3$, implying the exact invariance $M^{\mathrm{eff}'} = M^{\mathrm{eff}}$ because of the relations $\widehat{\varphi}_3 U = U \widehat{\mu}_3$ and $\widehat{\mu}_3 \operatorname{diag}(m_1, m_2, m_3) \widehat{\mu}_3 = \operatorname{diag}(m_1, m_2, m_3)$. The invariances $\widehat{\varphi}_i M^{\mathrm{eff}} \widehat{\varphi}_i =$

M^{eff} for $i = 1, 2$ are (only) approximately valid, provided we can accept beside the approximation $c_{12} \simeq s_{12}$ *also* $m_1 \simeq m_2$.

Thus, finally, we can conclude that — as a result of there being the "horizontal parity" covariant in neutrino mixings — three active neutrinos may develop *in a natural way* the familiar bilarge form of mixing matrix, favored at present phenomenologically. The suggestion that the resulting monomaximal form is, in fact, bilarge (approximately bimaximal) comes from the group character of relations $\widehat{\mu}_1 \widehat{\mu}_2 = \widehat{\mu}_3$ (cyclic), where the "horizontal parity" $P^{(\text{H})} \equiv \widehat{\mu}_3$ (generating the symmetry under the "horizontal conjugation", $\widehat{\varphi}_3 U \widehat{\mu}_3 = U$) is embedded. Then, the symmetries $\widehat{\varphi}_i U \widehat{\mu}_i = U$ or $\widehat{\varphi}_i = U \widehat{\mu}_i U^\dagger$ with $\widehat{\varphi}_1 \widehat{\varphi}_2 = \widehat{\varphi}_3$ (cyclic) hold also for $i = 1, 2$, if U — monomaximal with some c_{12} and s_{12} — is approximated to bimaximal form with $c_{12} \to 1/\sqrt{2} \leftarrow s_{12}$, in order to include also $\widehat{\varphi}_1, \widehat{\varphi}_2$ and $\widehat{\mu}_1, \widehat{\mu}_2$ in describing potential symmetries of U. Obviously, the smaller are the experimentally estimated differences $c_{12} - 1/\sqrt{2}$ and $s_{12} - 1/\sqrt{2}$, the better are the approximated symmetries for $i = 1, 2$.

REFERENCES

[1]   W. Królikowski, `hep-ph/0301161`.

[2]   S. Fukuda *et al.* (SuperKamiokande Collaboration), *Phys. Rev. Lett.* **85**, 3999 (2000).

[3]   K. Eguchi *et al.* (KamLAND Collaboration), *Phys. Rev. Lett.* **90**, 021802 (2003).

[4]   V. Barger, D. Marfatia, `hep-ph/0212126`.

[5]   G.L. Fogli *et al.*, `hep-ph/0212127`.

[6]   M. Maltoni, T. Schwetz, J.W.F. Valle, `hep-ph/0212129`.

[7]   A. Bandyopadhyay *et al.*, `hep-ph/0212146v2`.

[8]   J.N. Bahcall, M.C. Gonzalez-Garcia, C. Peña-Garay, `hep-ph/0212147v2`.

[9]   M. Appolonio *et al.* (Chooz Collaboration), `hep-ex/0301017`.

[10]   G. Mills, *Nucl. Phys. Proc. Suppl.* **91**, 198 (2001).

[11]   C. Giunti, `hep-ph/0302173`.