IS THERE A DYNAMICAL GROUP STRUCTURE BEHIND THE BILARGE FORM OF NEUTRINO MIXING MATRIX?*

Wojciech Królikowski

Institute of Theoretical Physics, Warsaw University
Hoża 69, 00-681 Warszawa, Poland

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We observe that the *invariance* of neutrino mixing matrix under the simultaneous discrete transformations $\nu_1\,,\nu_2\,,\nu_3 \;\rightarrow\; -\nu_1\,,-\nu_2\,,\nu_3$ and $\nu_e,\nu_\mu\,,\,\nu_\tau \rightarrow -\nu_e\,,\,\nu_\tau\,,\,\nu_\mu$ (neutrino "horizontal conjugation") *characterizes* (as a sufficient condition for it) the familiar bilarge form of neutrino mixing matrix, favored experimentally at present. Thus, the mass neutrinos ν_1,ν_2,ν_3 get a new quantum number, *covariant* with respect to their mixings into the flavor neutrinos ν_e,ν_μ,ν_τ (neutrino "horizontal parity" equal to -1, -1,1, respectively). The "horizontal parity" turns out to be embedded in a group structure consisting of some Hermitian and real 3×3 matrices μ_1,μ_2,μ_3 and $\varphi_1,\varphi_2,\varphi_3$, forming pairs interconnected through neutrino mixings. They generate some discrete transformations of mass and flavor neutrinos, respectively, in such a way that the group relations $\mu_1\mu_2=\mu_3$ (cyclic) and $\varphi_1\varphi_2=\varphi_3$ (cyclic) hold, while $\mu_a\mu_b=\mu_b\mu_a$ and $\varphi_a\varphi_b=\varphi_b\varphi_a$. Then, for instance, the μ_3 matrix may be chosen equal to the "horizontal parity".

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As is well known, the bilarge form of neutrino mixing matrix,

$$U=\begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{1}{\sqrt{2}}s_{12} & \frac{1}{\sqrt{2}}c_{12} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}s_{12} & -\frac{1}{\sqrt{2}}c_{12} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1}$$

(where $c_{23}=1/\sqrt{2}=s_{23}$ and $s_{13}=0$, while c_{12} and s_{12} are estimated to correspond to $\theta_{12}\sim 33^\circ$), is globally consistent with all present neutrino-oscillation experiments for solar ν_e's and atmospheric ν_μ's as well as with

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the negative result of Chooz experiment (giving $s_{13}^2 < 0.03$) [1] and successful KamLAND experiment [2,3–7] both for reactor $\bar{\nu}_e$'s. However, it cannot explain the possible LSND effect [8] for accelerator $\bar{\nu}_\mu$'s (and ν_μ's) whose existence is expected to be clarified soon in the MiniBOONE experiment. Its negative result would exclude mixings of active neutrinos with hypothetical light sterile neutrinos [9], leaving us with the minimal mixing unitary transformation

$$\nu_\alpha = \sum_i U_{\alpha i}\, \nu_i \,, \tag{2}$$

where $\nu_\alpha = \nu_e, \nu_\mu, \nu_\tau$ and $\nu_i = \nu_1, \nu_2, \nu_3$ represent the flavor and mass active neutrinos, respectively.

In the flavor representation, where the mass matrix for charged leptons is diagonal, the neutrino mixing matrix $U = (U_{\alpha i})$ is at the same time the diagonalizing matrix for neutrino effective mass matrix $M = (M_{\alpha\beta})$. Then,

$$M_{\alpha\beta} = \sum_i U_{\alpha i}\, m_i\, U^*_{\beta i} \,. \tag{3}$$

In the case of bilarge form (1) of U, the formula (3) gives

$$\begin{aligned}
M_{ee} &= m_1 c_{12}^2 + m_2 s_{12}^2 \,, \\
M_{\mu\mu} &= M_{\tau\tau} = \tfrac{1}{2}(m_1 s_{12}^2 + m_2 c_{12}^2 + m_3) \,, \\
M_{e\mu} &= -M_{e\tau} = \tfrac{1}{\sqrt{2}}(-m_1 + m_2) c_{12} s_{12} \,, \\
M_{\mu\tau} &= \tfrac{1}{2}(-m_1 s_{12}^2 - m_2 c_{12}^2 + m_3) \,.
\end{aligned} \tag{4}$$

Here, $M_{\beta\alpha} = M_{\alpha\beta} = M^*_{\alpha\beta}$. Making use of Eqs. (4) we can write the neutrino effective mass matrix in the form

$$\begin{aligned}
M = \; & \frac{m_1+m_2}{4}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \frac{m_3}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\
& + \frac{m_2-m_1}{4}\left[c\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \sqrt{2}\, s\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right],
\end{aligned} \tag{5}$$

where $c \equiv c_{12}^2 - s_{12}^2 = \cos 2\theta_{12}$ and $s \equiv 2c_{12}s_{12} = \sin 2\theta_{12}$. Here, all three terms, proportional to m_1+m_2, m_3 and m_2-m_1, commute (while two terms proportional to m_2-m_1, anticommute). Diagonalizing M given in Eq. (5), we obtain consistently

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = U^\dagger M U = \frac{m_1+m_2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+m_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_2 - m_1}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{6}$$

The present solar and atmospheric experimental estimates are $\Delta m_{21}^2 \equiv m_2^2 - m_1^2 \sim 7 \times 10^{-5}$ eV^2 and $\Delta m_{32}^2 \equiv m_3^2 - m_2^2 \sim 2.5 \times 10^{-3}$ eV^2, respectively, when the case of normal hierarchy $m_1 < m_2 < m_3$ is considered. Note that M gets here the form

$$M = \begin{pmatrix} A & D & -D \\ D & B & C \\ -D & C & B \end{pmatrix} , \tag{7}$$

where $A \equiv M_{ee}$, $B \equiv M_{\mu\mu} = M_{\tau\tau}$, $C \equiv M_{\mu\tau}$ and $D \equiv M_{e\mu} = -M_{e\tau}$ are given in Eqs. (4).

The bilarge mixing matrix U presented in Eq. (1) is not bimaximal as $\theta \sim 33^\circ$ and so,

$$c_{12} \sim 0.84 > \frac{1}{\sqrt{2}} > s_{12} \sim 0.54 \, . \tag{8}$$

But, since both values c_{12} and s_{12} are still large and not very distant from $1/\sqrt{2} \simeq 0.71$, one may ask the question, if and to what extent the rough approximation $c_{12} \simeq 1/\sqrt{2} \simeq s_{12}$ may work, leading through Eq. (1) to the approximate bimaximal form of the neutrino mixing matrix

$$U \simeq \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} . \tag{9}$$

It can be easily seen that in the *approximation* (9) for U three discrete transformations of mass neutrinos

$$\begin{aligned} \nu_1, \nu_2, \nu_3 &\to -\nu_2, -\nu_1, -\nu_3 \ , \\ \nu_1, \nu_2, \nu_3 &\to \ \ \nu_2, \ \ \nu_1, -\nu_3 \ , \\ \nu_1, \nu_2, \nu_3 &\to -\nu_1, -\nu_2, \ \ \nu_3 \end{aligned} \tag{10}$$

induce through the mixing unitary transformation (2) three following discrete transformations of flavor neutrinos:

$$\begin{aligned} \nu_e, \nu_\mu, \nu_\tau &\to -\nu_e, -\nu_\tau, -\nu_\mu \ , \\ \nu_e, \nu_\mu, \nu_\tau &\to \ \ \nu_e, -\nu_\mu, -\nu_\tau \ , \\ \nu_e, \nu_\mu, \nu_\tau &\to -\nu_e, \ \ \nu_\tau, \ \ \nu_\mu \ , \end{aligned} \tag{11}$$

respectively [10]. Moreover, the third Eq. (10) induces the third Eq. (11) *strictly*, if the exact form of U defined in Eq. (1) is applied in Eq. (2) [10].

Let us denote the Hermitian and real 3×3 matrices realizing the discrete transformations (10) as

$$\mu_1 \equiv \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mu_2 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mu_3 \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

and those realizing the discrete transformations (11) as

$$\varphi_1 \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \varphi_2 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \varphi_3 \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{13}$$

Then, we can readily show that in the *approximation* (9) for U the three equivalent relations [10]

$$\varphi_a U \mu_a = U \;\text{ or }\; U\mu_a = \varphi_a U \text{ or } \varphi_a = U\mu_a U^\dagger \tag{14}$$

hold for any $a = 1, 2, 3$. Moreover, for $a = 3$ these three relations are valid *strictly*, when the exact form of U given in Eq. (1) is used, since then the third Eq. (10) induces *strictly* the third Eq. (11). The first relation (14) tells us that the mixing matrix U is *invariant* under the simultaneous discrete transformations (10) and (11) (*approximately* for $a = 1, 2$ and *strictly* for $a = 3$), while the third relation (14) shows that μ_a matrices are *covariant* under the mixing unitary transformation (2), leading to φ_a matrices (again *approximately* for $a = 1, 2$ and *strictly* for $a = 3$). In particular, the matrix

$$P^{(\mathrm{H})} \equiv \mu_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{e}^{i\,2\pi\, I_2^{(\mathrm{H})}} = \mathrm{e}^{i\,\pi\,\lambda_2} \tag{15}$$

with

$$I_2^{(\mathrm{H})} \equiv \frac{1}{2}\lambda_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

may be called the "horizontal parity", getting the eigenvalues $-1,\ -1,\ 1$ for the mass neutrinos ν_1, ν_2, ν_3, respectively, when the discrete transformation

$$\begin{pmatrix} \nu_1' \\ \nu_2' \\ \nu_3' \end{pmatrix} = P^{(\mathrm{H})} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} -\nu_1 \\ -\nu_2 \\ \nu_3 \end{pmatrix} \tag{17}$$

— the "horizontal conjugation" — is performed [10]. According to Eq. (15) this conjugation is equivalent to a rotation by the angle 2π around 2-axis in the formal 8-dimensional "horizontal space", where $\lambda_1, \ldots, \lambda_8$ are Gell-Mann matrices acting on the triplet $(\nu_1, \nu_2, \nu_3)^T$ (then $I_2^{(\mathrm{H})}$ is the 2-component of the "horizontal isospin" $\vec{I}^{(\mathrm{H})} = \frac{1}{2}\vec{\lambda}$ with $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, while $Y^{(\mathrm{H})} = (1/\sqrt{3})\lambda_8$ is the "horizontal hypercharge"). In consequence,

$$\begin{pmatrix} \nu'_e \\ \nu'_\mu \\ \nu'_\tau \end{pmatrix} = U P^{(\mathrm{H})} U^\dagger \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} -\nu_e \\ \nu_\tau \\ \nu_\mu \end{pmatrix}, \tag{18}$$

where $P^{(\mathrm{H})\prime} = U P^{(\mathrm{H})} U^\dagger = \varphi_3$ and so, the "horizontal parity" is *covariant* with respect to neutrino mixings.

From Eqs. (3) and (14) we infer for any $a = 1, 2, 3$ that

$$\varphi_a M \varphi_a = M \quad \text{or} \quad M \varphi_a = \varphi_a M \tag{19}$$

i.e., the effective mass matrix M is *invariant* under the discrete transformations (11) (*approximately* for $a = 1, 2$ if in addition $m_1 \simeq m_2$, and *strictly* for $a = 3$). In fact,

$$\varphi_a M \varphi_a = \varphi_a U \,\mathrm{diag}\,(m_1, m_2, m_3)\, U^\dagger \varphi_a = U\, \mu_a \,\mathrm{diag}\,(m_1, m_2, m_3)\, \mu_a\, U^\dagger\,,$$

where

$$\mu_a \,\mathrm{diag}\,(m_1, m_2, m_3)\mu_a = \left\{ \begin{array}{lll} \mathrm{diag}\,(m_2, m_1, m_3) & \text{for} & a = 1, 2 \\ \mathrm{diag}\,(m_1, m_2, m_3) & \text{for} & a = 3 \end{array} \right. .$$

Thus, $\varphi_a M \varphi_a \simeq M$ for $a = 1, 2$ if in addition $m_1 \simeq m_2$, and $\varphi_a M \varphi_a = M$ for $a = 3$).

It is worthwhile to point out that the rough approximation $m_1 \simeq m_2$ goes in the direction shown by the experimental situation, where $\Delta m^2_{21} \sim 7 \times 10^{-5}$ eV^2 is considerably smaller than $\Delta m^2_{32} \sim 2.5 \times 10^{-3}$ eV^2.

Now, it is important to observe that the matrices (12) and (13) satisfy for $a, b = 1, 2, 3$ the following algebraic relations:

$$\mu_1 \mu_2 = \mu_3 \text{ (cyclic)}\,, \quad \mu_a \mu_b = \mu_b \mu_a\,, \quad \mu_a^2 = \mathbf{1}\,, \quad \mu_1 + \mu_2 + \mu_3 = -\mathbf{1} \tag{20}$$

and

$$\varphi_1 \varphi_2 = \varphi_3 \text{ (cyclic)}\,, \quad \varphi_a \varphi_b = \varphi_b \varphi_a\,, \quad \varphi_a^2 = \mathbf{1}\,, \quad \varphi_1 + \varphi_2 + \varphi_3 = -\mathbf{1} \tag{21}$$

(but $\mu_a \varphi_b \neq \varphi_b \mu_a$, except for $\mu_3 \varphi_2 = \varphi_2 \mu_3$).

It is easy to see that the matrices μ_1, μ_2, μ_3 and $\varphi_1, \varphi_2, \varphi_3$ given in Eqs. (12) and (13) can be used as bases for 3×3 symmetric block matrices of the types

$$\left(\begin{array}{ccc} A_1 & B_1 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & C_1 \end{array} \right) \quad \text{and} \quad \left(\begin{array}{ccc} A_2 & 0 & 0 \\ 0 & B_2 & C_2 \\ 0 & C_2 & B_2 \end{array} \right), \tag{22}$$

respectively. The sets of such matrices form two Abelian groups with respect to matrix multiplication, if the inverse of their four blocks exists. They are isomorphic, being related through the unitary transformation generated by the bimaximal mixing matrix U given on the rhs of Eq. (9): $U\,\{1\}\,U^\dagger = \{2\}$, where $\{1\}$ and $\{2\}$ symbolize the sets of matrices of the first and second type (22). The group character of these sets is reflected in the group relations $\mu_1\mu_2 = \mu_3$ (cyclic) and $\varphi_1\varphi_2 = \varphi_3$ (cyclic) for their bases, while their isomorphism corresponds to the unitary transformation $\varphi_a = U\mu_a U^\dagger$ between both bases. Of course, these two groups are Abelian subgroups of the group of all 3×3 nonsingular matrices that can be spun by the basis consisting of $\mathbf{1}$ and Gell-Mann matrices $\lambda_1, \ldots, \lambda_8$.

In terms of the matrices (12) and (13) the effective mass matrix presented in Eq. (5) can be rewritten as

$$\begin{aligned} M &= \frac{m_1 + m_2}{4}\,(\mathbf{1} - \varphi_3) + \frac{m_3}{2}\,(\mathbf{1} + \varphi_3) \\ &\quad + \frac{m_2 - m_1}{4}\left[c\,(\varphi_1 - \varphi_2) + \sqrt{2}\,s\,(\lambda_1 - \lambda_4)\right], \end{aligned} \tag{23}$$

where

$$\lambda_1 - \lambda_4 = \left(\begin{array}{rrr} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right) = \tfrac{1}{2}\,\{\varphi_3, \mu_1 - \mu_2\}\,. \tag{24}$$

When $c_{12} \simeq 1/\sqrt{2} \simeq s_{12}$, then $c \simeq 0$ and $s \simeq 1$. If $m_1 \simeq m_2$, Eq. (23) gives

$$M \simeq \frac{m_1 + m_2}{4}\,(\mathbf{1} - \varphi_3) + \frac{m_3}{2}\,(\mathbf{1} + \varphi_3)\,. \tag{25}$$

In this case, $D \simeq 0$ in Eq. (7). Then, approximately, M is a matrix of the second type (22).

One may speculate in connection with the formula (23) that the 3×3 matrices φ_a and μ_a ($a = 1, 2, 3$), where $\varphi_1\varphi_2 = \varphi_3$ (cyclic) and $\mu_1\mu_2 = \mu_3$ (cyclic), can help us to find the desired *dynamical variables* solving hopefully the basic problem of fermion masses. In such a case there may appear a more or less instructive analogy with Pauli matrices, where $\sigma_1\sigma_2 = i\sigma_3$ (cyclic), which have led to Dirac matrices solving the problem of fermion spins.

The discrete transformations generated by φ_a and μ_a matrices and the related discrete symmetries may play an important role in Nature because of the absence for neutrinos of electromagnetic and strong interactions. Otherwise, these interactions could largely suppress such fragile, discrete horizontal symmetries that, in contrast to the Standard Model gauge interactions, do not treat equally three fermion generations.

Finally, we should like to point out that both sets of algebraic relations (20) and (21) would still hold, if $\varphi_1, \varphi_2, \varphi_3$ matrices were defined not by Eqs. (13), but through the relations

$$\begin{aligned}
\varphi_1 \equiv U\mu_1 U^\dagger &= \begin{pmatrix} -s & -\frac{1}{\sqrt{2}}c & \frac{1}{\sqrt{2}}c \\ -\frac{1}{\sqrt{2}}c & -\frac{1}{2}(1-s) & -\frac{1}{2}(1+s) \\ \frac{1}{\sqrt{2}}c & -\frac{1}{2}(1+s) & -\frac{1}{2}(1-s) \end{pmatrix} \stackrel{s\to 1}{\to} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \\
\varphi_2 \equiv U\mu_2 U^\dagger &= \begin{pmatrix} s & \frac{1}{\sqrt{2}}c & -\frac{1}{\sqrt{2}}c \\ \frac{1}{\sqrt{2}}c & -\frac{1}{2}(1+s) & -\frac{1}{2}(1-s) \\ -\frac{1}{\sqrt{2}}c & -\frac{1}{2}(1-s) & -\frac{1}{2}(1+s) \end{pmatrix} \stackrel{s\to 1}{\to} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
\varphi_3 \equiv U\mu_3 U^\dagger &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\end{aligned} \tag{26}$$

where U was of its exact form (1) and μ_1, μ_2, μ_3 matrices were given as before in Eqs. (12). In this case, our relations (14) would be valid *strictly also* for $a = 1, 2$, not only for $a = 3$ as before in the case of Eqs. (13). Of course, in the limit of $c_{12} \to 1/\sqrt{2} \leftarrow s_{12}$ *i.e.*, $c \to 0$ and $s \to 1$, Eqs. (26) would tend to Eqs. (13). Note that, generically, the relations (20) and (21) as well as (14) would hold, if $\varphi_a = \tilde{U}\mu_a\tilde{U}^\dagger$ with $\tilde{U}$ being any 3×3 unitary matrix and μ_a were given in Eqs. (12) ($a = 1, 2, 3$). However, in such a case, one would get Eqs. (26) only for the unitary matrices $\tilde{U}$ equal to $V_\varphi U V_\mu$, where U would have the form (1), while V_φ and V_μ would be any unitary matrix commuting with φ_a and μ_a, respectively [say, $V_\varphi = f_\varphi(\varphi_1, \varphi_2, \varphi_3)$ and $V_\mu = f_\mu(\mu_1, \mu_2, \mu_3)$]. Then,

$$\begin{aligned}
V_\varphi \varphi_a V_\varphi^\dagger = \varphi_a \;&\equiv\; V_\varphi U V_\mu \,\mu_a\, V_\mu^\dagger U^\dagger V_\varphi^\dagger\,, \\
\varphi_a \;&=\; U V_\mu \mu_a V_\mu^\dagger U^\dagger = U\mu_a U^\dagger\,,
\end{aligned} \tag{27}$$

where

$$[V_\varphi\,,\, \varphi_a] = 0\,, \quad [V_\mu\,,\, \mu_a] = 0\,. \tag{28}$$

Thus, in the class of $V_\varphi U V_\mu$ matrices, one might restrict oneself to the U matrix of the form (1), putting $V_\varphi = \mathbf{1}$ and $V_\mu = \mathbf{1}$. The form (1) of U is a sufficient condition for the invariances $\varphi_a U \mu_a = U$ with μ_a and φ_a given as in Eqs. (12) and (26), while the form $U \rightarrow V_\varphi U V_\mu$ is also their necessary condition.

In conclusion, we have introduced two Abelian algebras of Hermitian and real 3×3 matrices μ_1, μ_2, μ_3 and $\varphi_1, \varphi_2, \varphi_3$ satisfying the group relations $\mu_1 \mu_2 = \mu_3$ (cyclic) and $\varphi_1 \varphi_2 = \varphi_3$ (cyclic) as well as the constraints $\mu_1 + \mu_2 + \mu_3 = -\mathbf{1}$ and $\varphi_1 + \varphi_2 + \varphi_3 = -\mathbf{1}$. These two algebras are isomorphic, as being related through the unitary transformation $\varphi_a = U \mu_a U^\dagger$ $(a = 1, 2, 3)$, where U is the neutrino mixing matrix. Thus, μ_a are *covariant* with respect to neutrino mixings, leading to φ_a. Such a unitary transformation implies the *invariances* $\varphi_a M \varphi_a = M$ of the neutrino effective mass matrix M: strictly for $a = 3$ and, if $m_1 \simeq m_2$, approximately for $a = 1, 2$.

The unitary transformation $\varphi_a = U \, \mu_a \, U^\dagger$ $(a = 1, 2, 3)$ is equivalent to the *invariances* $\varphi_a \, U \, \mu_a = U$ of the neutrino mixing matrix U. With given μ_a and φ_a matrices as in Eqs. (12) and (26), respectively, these invariances *characterize* (as a sufficient condition for them) the monomaximal form $(\theta_{23} = 45^\circ)$ of the bilarge mixing matrix U that for $\theta_{12} \simeq 45^\circ$ should be approximately bimaximal $(\theta_{12} \sim 33^\circ$ is the actual experimental estimate). On the other hand, the charged-current weak interactions may violate maximally the "horizontal parity" $P^{(\mathrm{H})} \equiv \mu_3$. This is the case, if $\nu \rightarrow \varphi_3 \nu$ and, as the mass neutrinos, $e \rightarrow \mu_3 e$ (not $e \rightarrow \varphi_3 e$) with $\nu = (\nu_e, \nu_\mu, \nu_\tau)^{\mathrm{T}}$ and $e = (e, \mu, \tau)^{\mathrm{T}}$. Then, the neutrino effective and charged-lepton mass terms conserve this parity.

Summarizing, the algebraic properties of μ_a matrices can be expressed by the relations

$$\{\mu_1, \mu_2\} = 2\mu_3 \, (\text{cyclic}) \, , \quad [\mu_a, \mu_b] = \mathbf{0} \, , \quad \mu_a^2 = \mathbf{1} \, , \quad \mu_1 + \mu_2 + \mu_3 = -\mathbf{1} \tag{29}$$

$(a, b = 1, 2, 3)$. The identical relations hold also for φ_a matrices equal to $U \mu_a U^\dagger$. We suggest that φ_a and μ_a matrices $(a = 1, 2, 3)$ play the role of *dynamical variables* in the problem of neutrino masses (and, hopefully, of other fermion masses). In fact, according to Eqs. (23) and (24) the neutrino effective mass matrix M can be expressed by means of the matrices $\mathbf{1}$, φ_3 and μ_1, μ_2 $(\mathbf{1} = -\varphi_1 - \varphi_2 - \varphi_3 = -\mu_1 - \mu_2 - \mu_3)$ and the parameters m_1, m_2, m_3 and s, the number of the latter should be certainly reduced, say, by the conjecture that $m_1{:}m_2{:}m_3 \simeq m_e{:}m_\mu{:}m_\tau$ [10]. Here, we have $\varphi_1 - \varphi_2 = \mathbf{1} - \varphi_3 - 2\{\varphi_3, \mu_3\} = \mathbf{1} + 3\varphi_3 + 2\{\varphi_3, \mu_1 + \mu_2\}$, beside $\lambda_1 - \lambda_4 = \frac{1}{2}\{\varphi_3, \mu_1 - \mu_2\}$, expressing $\varphi_1 - \varphi_2$ and $\lambda_1 - \lambda_4$ through φ_3 and μ_1, μ_2.

From the mathematical viewpoint, four 3×3 matrices $\mathbf{1}, \mu_1, \mu_2, \mu_3$ satisfying Eqs. (29) form a matrix representation of the $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ group of four elements (1,1), $(1,-1)$, $(-1,1)$, $(-1,-1)$, where $\boldsymbol{Z}_2$ is the group of two square roots 1,-1 of 1. The same is true for four 3×3 matrices $\mathbf{1}, \varphi_1, \varphi_2, \varphi_3$.

On the other hand, for the 3×3 matrices μ_a and φ_a $(a=1,2,3)$ given in Eqs. (12) and (13) the formulae

$$\exp\left(-i\sum_a \mu_a\theta_a\right) = \exp\left(i\frac{1}{3}\sum_a \theta_a\right)\exp\left[i\frac{1}{\sqrt{3}}\lambda_8\left(-\theta_1-\theta_2+2\theta_3\right)\right] \times \exp\left[i\lambda_1\left(\theta_1-\theta_2\right)\right] \tag{30}$$

and

$$\exp\left(-i\sum_a \varphi_a\theta_a\right) = \exp\left(i\frac{1}{3}\sum_a \theta_a\right)\exp\left[i\frac{1}{2}\left(\lambda_3+\frac{1}{\sqrt{3}}\lambda_8\right)\left(\theta_1-2\theta_2+\theta_3\right)\right] \times \exp\left[i\lambda_6\left(\theta_1-\theta_3\right)\right], \tag{31}$$

hold, where $[\lambda_1,\lambda_8]=0$ and $[\lambda_6, \lambda_3+\frac{1}{\sqrt{3}}\lambda_8]=0$. Thus, μ_a and φ_a $(a=1,2,3)$ generate two subgroups (30) and (31) [having the forms $\exp\left(i\,\frac{1}{3}\sum_a\theta_a\right)\ \mathrm{SU}_1(1)\times \mathrm{SU}_2(1)$] of the horizontal unitary group $U(3)=\exp\left(i\,\frac{1}{3}\sum_a\theta_a\right)\mathrm{SU}(3)$, where SU(3) is generated by $\lambda_1,\ldots,\lambda_8$. Here, $\det U(3)=\exp\left(i\sum_a\theta_a\right)$.

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