ON TACHYON POTENTIAL IN BOUNDARY STRING FIELD THEORY AND PROBLEMS WITH BOUNDARY FERMIONS

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A calculation of a partition function Z in a system of two coincident $D1$-$\overline{D1}$ pairs of type I superstring theory is presented. According to the well known conjecture, this partition function is identified with a tachyon potential in a case of constant tachyon fields. Properties of this potential are discussed. On the way, a peculiar features of *boundary fermions* emerge. To solve the problems ensuing, a non-standard way of operator renormalization is required.

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1. Introduction

There was a sudden twist in a development of String Field Theory ten years ago, as its Boundary — or Background Independent — version, the so called BSFT, was discovered by Witten [5, 6]. Since that time quite an achievement in this field has been noted [11]. In particular, within BSFT framework it turned out possible to find an exact form of a tachyon potential, with its certain features conjectured much earlier by Sen [12]. The knowledge of this potential undoubtedly sheds much light upon the role and nature of tachyons, which are rather infamous creatures in string theory. Because tachyons may arise as states of open strings, which appear naturally stretched between D-branes in complex D-brane configurations, it follows that interactions of D-branes may be described as processes accompanying certain behavior of tachyons. These interactions may be very dramatic in nature, including creation and annihilation of branes of various dimensions, and are associated with such a non-perturbative phenomenon as rolling down tachyon potential. Among the others, this explains how a true vacuum in string theory could be found: when a perturbative vacuum is unstable due to presence of tachyon states (arising from strings stretched between various D-branes) and corresponds to a maximum of tachyon potential, these states disappear from the spectrum as a consequence of a condensation, which leads to another vacuum, associated with minimum of the potential (with D-branes annihilated). In this paper an exact form of a tachyon potential in a certain configuration of D-branes is found, which also admits such an interpretation.

Tachyon potential mentioned above is supposed to be a special case of a low energy effective action $\mathcal{S}$ for the lightest string excitations. A search for such an action, describing those lightest modes in terms of spacetime fields, is a domain of String Field Theory. In a bosonic version of BSFT, with a help of Batalin–Vilkovisky formalism, it was shown [5] that the spacetime action is closely related to a disk partition function $Z(u)$,

$$\mathcal{S} = \left(\beta^i \frac{\partial}{\partial u^i} + 1\right) Z(u)\,, \tag{1}$$

where the partition function is weighted by a worldsheet action consisting of a bulk conformal Polyakov part and not-necessarily conformal boundary piece, parametrized by couplings u^i of open string fields $\mathcal{O}_i$,

$$S = S_{\text{bulk}} + S_{\text{bnd}}\,, \qquad\qquad S_{\text{bnd}} = \oint d\varphi\, u^i \mathcal{O}_i\,, \tag{2}$$

and β^i are worldsheet beta functions governing the flow of couplings u^i. In particular, β^i vanish in fixed points of boundary theory, which correspond to such a value of u^i for which S_{bnd} is conformally invariant. In such a case the spacetime action $\mathcal{S}$ coincides with the partition sum Z.

Finding an analogous derivation for a superstring case turned out non-trivial and by today is not known. Even though, in view of explicit results for effective actions obtained for gauge fields in superstring theory [2–4], as well as some special features of supersymmetric framework, it was conjectured in [7, 10] that in superstring case a proper connection between spacetime action and partition sum is even simpler than in bosonic theory, and reads

$$\mathcal{S} = Z(u)\,. \tag{3}$$

Moreover, the conjecture states that this relation is true even off-shell, that is for such values of u^i for which S_{bnd} does not exhibit conformal symmetry.

In general the worldsheet action (2) may depend on spacetime coordinates X^μ (and ψ^μ in superstring case) through the operators $\mathcal{O}_i$. A tachyon potential can be obtained by setting all u^i to zero except those corresponding to constant spacetime tachyon fields. Thus, for tachyons in superstring framework, a relation (3) takes the form

$$\mathcal{V}(T_i) = Z(T_i)\,, \tag{4}$$

where T_i represent constant in spacetime tachyon fields.

As mentioned above, in this paper a partition function for a certain configuration of D-branes is found. This configuration is a coincident system of two $D1$-branes and two anti-$D1$-branes of type I superstring theory, in which four different tachyon fields may arise, associated with all possible brane–antibrane pairs [13]. In the course of reasoning, some peculiar features of the so-called *boundary fermions* will emerge, which will lead to a very special prescription for operator renormalization.

The paper is organized as follows. A framework for calculations, based on the work of Kraus and Larsen [8], is reviewed in Section 2. In Section 3 the computation and analysis of a partition sum in a system of two brane–antibrane pairs of type I theory is presented. Section 4 contains summary and conclusions.

2. Boundary String Field Theory — BSFT

As explained in the introduction, to compute spacetime action for tachyon field BSFT formalism is used. It's based upon the assumption of equality of the spacetime action for background fields and the partition function associated with them,

$$\mathcal{S}[T, A_\mu, \ldots] = Z[T, A_\mu, \ldots]\,. \tag{5}$$

Generally in open string theory, apart from tachyon field $T(X)$, also gauge fields $A_\mu(X)$ and possibly some other excitations may be considered in the partition function. In this section only T and A_μ are taken into account, but as the main purpose of this work is to find tachyon potential, later on constant values of tachyon fields and vanishing gauge fields will be assumed.

In this section firstly the conventions of [8] are reviewed. Then I explain how to modify them in order to consider type I strings.

2.1. Bosonic string partition function

At first let's focus on a bosonic string partition function. The type II superstring partition function will be given as a generalization of this case. Bosonic string Euclidean partition function is

$$Z = \int\limits_{\mathrm{Disk}} \mathcal{D}X\, \mathrm{e}^{-S_{\mathrm{bulk}}}\, \mathrm{e}^{-S_{\mathrm{bnd}}}. \tag{6}$$

A domain of integration is a unit disk, parametrized by radial coordinates

$$\rho \in [\,0\,,\,1\,]\,, \qquad\qquad \varphi \in [\,0\,,\,2\pi\,]\,, \tag{7}$$

and equipped with a flat metric

$$g_{ab} = \begin{bmatrix} 1 & \\ & \rho^2 \end{bmatrix}. \tag{8}$$

Worldsheet action

$$S = S_{\rm bulk} + S_{\rm bnd} \tag{9}$$

consists of a bulk and boundary parts. The bulk action is conformally invariant Polyakov action, which in coordinates (7) takes the form

$$S_{\rm bulk} = \frac{1}{4\pi\alpha'} \int\limits_{\rm Disk} \rho\, d\rho\, d\varphi \Big(\partial_\rho X^\mu\, \partial_\rho X_\mu + \rho^{-2}\, \partial_\varphi X^\mu \partial_\varphi X_\mu \Big). \tag{10}$$

Arbitrary boundary conditions are set in terms of Fourier modes,

$$X^\mu(\rho=1,\varphi) = X_0^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \Big(X_n^\mu \mathrm{e}^{in\varphi} + X_{-n}^\mu \mathrm{e}^{-in\varphi} \Big), \tag{11}$$

for which reality of X^μ implies $X_n^\mu = \bar{X}_{-n}^\mu$. In terms of these modes, Polyakov action (10) takes the form

$$S_{\rm bulk} = \frac{1}{2} \sum_{n=1}^{\infty} n\, X_{-n}^\mu X_n^\mu . \tag{12}$$

The second ingredient of the worldsheet action (9) is a boundary term $S_{\rm bnd}$, which does not have to be conformally invariant. In case of a tachyon field this is simply

$$S_{\rm bnd} = S_T = \int\limits_0^{2\pi} d\varphi\, T(X), \tag{13}$$

whereas for gauge field an appropriate gauge invariant term is

$$S_{\rm bnd} = S_A = -i \int\limits_0^{2\pi} d\varphi\, A_\mu \dot{X}^\mu, \tag{14}$$

with a dot denoting differentiation with respect to φ.

Fixing integration measure in (6) as

$$\mathcal{D}X := \prod_{n=1}^{\infty} \frac{dX_n dX_{-n}}{4\pi}\,. \tag{15}$$

specifies completely bosonic partition function.

2.2. Type II superstring partition function

Polyakov action for a superstring reads

$$S_{\rm P} = -\frac{1}{4\pi\alpha'} \int d\tau\, d\sigma \big(\partial^a X^\mu\, \partial_a X_\mu - i \overline{\Psi}^\mu \rho^a \partial_a \Psi_\mu \big). \tag{16}$$

Because we are interested in Neveu–Schwartz sector with antiperiodic boundary conditions for ψ^μ, in radial coordinates (7) we have

$$\psi(\rho = 1, \varphi) = \sum_{r=1/2}^{\infty} \Big(\psi^\mu_r\, {\rm e}^{ir\varphi} + \psi^\mu_{-r}\, {\rm e}^{-ir\varphi} \Big). \tag{17}$$

Boundary functional for fields ψ^μ has been calculated *e.g.* in [8, 14], and is equal to

$$\boldsymbol{\Psi}_{\rm bulk} = {\rm e}^{-S_{\rm bulk}}, \tag{18}$$

where

$$S_{\rm bulk} = \frac{1}{2} \sum_{n=1}^{\infty} n\, X^\mu_{-n} X^\mu_n + i \sum_{r=1/2}^{\infty} \psi^\mu_{-r} \psi^\mu_r\,. \tag{19}$$

Supersymmetric formalism allows to write boundary interactions in supersymmetricaly invariant way. To do this we expand a boundary of a disk to a superspace with coordinates $\hat{\varphi} = (\varphi, \theta)$, θ being anticommuting. Fields on the boundary are now promoted to superfields,

$$\boldsymbol{X}^\mu = X^\mu + \sqrt{\alpha'} \theta \psi^\mu\,, \tag{20}$$

and derivative along the boundary becomes super as well,

$$D = \partial_\theta + \theta \partial_\varphi\,. \tag{21}$$

Supersymmetrized tachyon and gauge fields can be set equal to

$$T(\boldsymbol{X}) = T(X) + \partial_\mu T(X)\, \theta \sqrt{\alpha'} \psi^\mu\,, \tag{22}$$

$$A_\mu(\boldsymbol{X}) = A_\mu(X) - \theta \frac{\sqrt{\alpha'}}{2} F_{\mu\nu} \psi^\nu\,, \tag{23}$$

in the second case $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for an Abelian gauge group. From these fields invariant boundary terms $S_{\rm bnd}$ can be constructed as integrals along the boundary with respect to $d\hat{\varphi}$.

Choosing fermionic integration measure as

$$\mathcal{D}\psi := \prod_{r=1/2}^{\infty} d\psi_r d\psi_{-r}\,, \tag{24}$$

and collecting the results (15), (19), a superstring partition function can eventually be written down

$$Z = \int \mathcal{D}X\mathcal{D}\psi \cdot \mathrm{e}^{-S_{\mathrm{bulk}}}\, \mathrm{e}^{-S_{\mathrm{bnd}}}. \tag{25}$$

2.3. Boundary fermions

These are non-Abelian gauge fields which appear on stacks of D-branes in string theory. They are states of open strings connecting those branes, and as such are associated with boundaries of these strings' worldsheets. Calculating partition function in a superstring theory requires taking care of a supersymmetric boundary ordering. This is not an easy task, but fortunately there is an equivalent method of finding partition function without doing path-ordered integrals — it suffices to introduce *boundary fermions*. Moreover this enables one to consider simultaneously gauge fields and tachyons, and even tachyons alone, what is of particular interest for us. These facts simplify calculations, but the price is that it is possible to analyze only stacks consisting precisely of 2^n branes [8].

Let us demonstrate first how boundary fermions arise in the case of 2^n D-branes of type II theory. There are $\mathrm{U}(2^n)$ gauge fields A_μ living in such a configuration. These fields are Lie-algebra valued. It can be shown that each element of $u(2^n)$ algebra is a linear combination of elements of a basis of Clifford algebra of the group $\mathrm{SO}(2n)$. Clifford algebra in generated by Dirac matrices γ_I which obey

$$\{\, \gamma_I, \gamma_J \,\} = 2\, \delta_{ij} \qquad\qquad \text{for} \quad I, J = 1, \ldots, 2m\,. \tag{26}$$

Clifford algebra consists of an identity matrix, Dirac matrices γ_I, and elements of the form

$$\gamma_{I_1 \ldots I_k} := \frac{1}{k!} \sum_\sigma (-1)^{sgn(\sigma)}\, \gamma_{I_{\sigma(1)}} \cdots \gamma_{I_{\sigma(k)}}. \tag{27}$$

Thus it follows that for an arbitrary $\mathrm{U}(2^n)$ gauge field we can write

$$A_\mu^{ab} = \sum_{k=0}^{2n} A_\mu^{I_1 \ldots I_k}\, \gamma^{ab}_{I_1 \ldots I_k}. \tag{28}$$

The *boundary fermions* are boundary degrees of freedom which are introduced in (28) instead of Dirac matrices. In accordance with supersymmetric formalism these boundary superfields are defined as

$$\boldsymbol{\Gamma}_I := \eta_I + \theta F_I, \qquad\qquad \text{for } I = 1, \ldots, 2m\,, \tag{29}$$

with η_I anticommuting and auxiliary fields F_I commuting. Canonical quantization of the following action

$$S_{\boldsymbol{\Gamma}} = -\int d\varphi\, d\theta\, \frac{1}{4}\boldsymbol{\Gamma}_I D\boldsymbol{\Gamma}_I, \tag{30}$$

leads then to the same commutation relations for η_I as for Dirac matrices,

$$\{\eta_I, \eta_J\} = 2\,\delta_{IJ}. \tag{31}$$

Eventually, we treat gauge fields as

$$A_\mu = \sum_{k=0}^{2n} A_\mu^{I_1\ldots I_k}\,\boldsymbol{\Gamma}_{I_1}\cdots\boldsymbol{\Gamma}_{I_k}. \tag{32}$$

The main conclusion one can draw from the above is that when calculating partition function for gauge fields, supersymmetric path ordering on a boundary of a disk is equivalent to functional integration weighted by the exponent of

$$-S_A = \int d\varphi\, d\theta \Big[\frac{1}{4}\boldsymbol{\Gamma}_I D\boldsymbol{\Gamma}_I + i\sum_{k=0}^{2m} A_\mu^{I_1,\ldots,I_k}\,\boldsymbol{\Gamma}_{I_1}\cdots\boldsymbol{\Gamma}_{I_k}\Big]. \tag{33}$$

In this setup generalization to tachyon fields is particularly simple. Tachyons in superstrings arise on a brane–antibrane pair. In a system of 2^{m-1} D-branes and the same number of antibranes, one can find two $\mathrm{U}(2^{m-1})$ gauge fields A_μ^+ and A_μ^-, associated respectively with branes or antibranes, and a bunch of tachyon fields coming from strings stretched between all possible brane–antibrane pairs. Type II superstrings are oriented, so in fact there are two degrees of freedom associated with each brane–antibrane pair, and tachyons should be treated as complex fields. All boundary interactions of these fields can be described by one $2^m \times 2^m$ matrix

$$M(\boldsymbol{X}) := \begin{bmatrix} iA_\mu^+(\boldsymbol{X})D\boldsymbol{X}^\mu & \sqrt{\alpha'}\bar{T} \\ \sqrt{\alpha'}T & iA_\mu^-(\boldsymbol{X})D\boldsymbol{X}^\mu \end{bmatrix}. \tag{34}$$

First 2^{m-1} rows and columns of M are associated with D-branes in our system; the other correspond to antibranes.

Just as for pure gauge fields the following decomposition can be carried over

$$M^{ab} = \sum_{k=0}^{2m} \frac{1}{2k!} M^{I_1\ldots I_k}\,\gamma^{ab}_{I_1\ldots I_k}, \tag{35}$$

and consequently boundary fermions $\boldsymbol{\Gamma}_I$ can be introduced. Finally the following form of the boundary action emerges

$$S_{\rm bnd} = -\int d\varphi\, d\theta \left[\frac{1}{4} \boldsymbol{\Gamma}_I D \boldsymbol{\Gamma}_I + i \sum_{k=0}^{2m} \frac{1}{2k!} M^{I_1 \ldots I_k}\, \boldsymbol{\Gamma}_{I_1} \cdots \boldsymbol{\Gamma}_{I_k} \right]. \tag{36}$$

Our purpose in what follows will be to evaluate the partition function (25) with the above $S_{\rm bnd}$ for a particular system of D-branes.

2.4. BSFT for type I strings

In the previous subsection the case of type II superstrings has been presented. There is of course a crucial difference between type I theory [1]. Firstly, strings in type I theory are unoriented, so there is only one degree of freedom associated with each tachyon field. Thus tachyon fields become real, and parts of M matrix (35) which correspond to tachyons should be symmetric. Secondly, gauge groups associated with stacks of D-brans are different than U(n). For a stack of $n \geq 2$ D1-branes — in which case we are interested in — this group is SO(n). This means that in (35) those part of M matrix which correspond to gauge fields should be antisymmetric.

There is one more subtlety associated with (36). Coefficients $M^{I_1 \ldots I_k}$ correspond to either gauge or tachyon fields, and as such should be bosonic. As the whole expression should be a c-number, it implies that coefficients corresponding to tachyons and gauge fields should be those for which k is respectively odd and even (because of superderivative in a gauge field interaction). This is a nontrivial constraint which will have to be taken care of later. As we shall see, this will also lead to peculiar properties of boundary fermions in a system with tachyon fields.

3. Two brane–antibrane pairs in type I theory

3.1. Preliminaries

The analysis of the configuration of D-branes mentioned in the introduction will now be presented. That is, I consider a stack of two $D1$-branes and two anti-$D1$-branes of type I theory. These correspond to $m = 2$ introduced at the end of Section 2.3.

Even though the main task in this paper is to find a tachyon potential, for which it suffices to consider constant tachyon fields in the system, for a while I consider more general configuration. Generally — as mentioned in Section 2.4 — there could appear two SO(2) gauge fields A^+, A^- in our system, associated respectively with two branes and two antibranes.

One would also expect four real tachyon fields associated with each brane–antibrane pair in the system. Denoting D-branes by numbers 1 and 2, and anti-D-branes by 3 and 4, in an obvious way tachyons can be named as T_{13}, T_{14}, T_{23}, T_{24}. These conventions are schematically presented in figure 1.

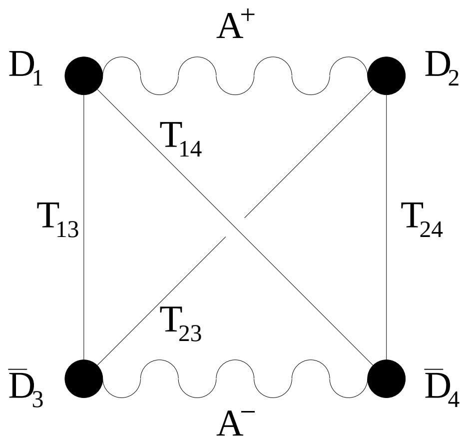

Fig. 1. Two type I $D1$-$\overline{D1}$ pairs.

The partition sum (25) should now be found, which according to the conjecture (5), would then be identified with an effective action for spacetime fields. Three ingredients of (25) are already given, *i.e.* (15), (19), (24). The fourth one is a boundary action (36). Its main ingredient is M matrix (35), which should have properties described in Section 2.4. In particular, an appropriate representation of Clifford algebra should be found, for which gauge and tachyon fields would have correct statistics. Such a representation is given explicitly in the appendix.

Decomposition of gauge fields under the particular representation given in the appendix is the following

$$\mathcal{A}_\mu = \begin{bmatrix} A^+_\mu\sigma_2 & \\ & A^-_\mu\sigma_2 \end{bmatrix} = \frac{1}{2}A_{1,\mu}\gamma_{13} + \frac{1}{2}A_{2,\mu}\gamma_{24}, \tag{37}$$

where σ_2 is a Pauli matrix and the newly introduced coefficients are

$$\begin{cases} A_{1,\mu} = i(A^-_\mu + A^+_\mu) \\ A_{2,\mu} = i(A^-_\mu - A^+_\mu) \end{cases} \tag{38}$$

which can be identified with coefficients in (35) via

$$\begin{cases} M_{13} = iA_{1,\mu}\, D\boldsymbol{X}^\mu \\ M_{24} = iA_{2,\mu}\, D\boldsymbol{X}^\mu \end{cases} \tag{39}$$

Tachyons are decomposed as

$$\begin{aligned}\sqrt{\alpha'}\,\mathcal{T} &= \sqrt{\alpha'}\begin{bmatrix} & & T_{13} & T_{14} \\ & & T_{23} & T_{24} \\ T_{13} & T_{23} & & \\ T_{14} & T_{24} & & \end{bmatrix} \\ &= \frac{1}{2}M_1\gamma_1 + \frac{1}{2}M_{134}\gamma_{134} + \frac{1}{2}M_3\gamma_3 + \frac{1}{2}M_{123}\gamma_{123}\,, \end{aligned} \tag{40}$$

what leads to

$$\left\{\begin{array}{l} M_1 = (T_{14}+T_{23})\sqrt{\alpha'}\,, \\ M_{134} = i(T_{23}-T_{14})\sqrt{\alpha'}\,, \\ M_3 = (T_{13}-T_{24})\sqrt{\alpha'}\,, \\ M_{123} = -i(T_{13}+T_{24})\sqrt{\alpha'} \end{array}\right. . \tag{41}$$

Thus, in our representation the gauge and tachyon fields indeed have the correct statistics. Above results combine to the interaction matrix (34)

$$M(\boldsymbol{X}) = i\mathcal{A}_\mu D\boldsymbol{X}^\mu + \sqrt{\alpha'}\mathcal{T}. \tag{42}$$

3.2. Calculation of the four-tachyon potential

To find the tachyon potential, it suffices to consider vanishing gauge fields, $A_\mu^\pm = 0$, and constant tachyon fields. Then the only non-zero components $M_{I_1 \ldots I_k}$ are those given in (41), which are constant. Integrating out coordinate θ in (36) under these assumptions leads to the following boundary action

$$S_{\rm bnd} = -\int d\varphi\,\left[\frac{1}{4}\dot{\eta}_I\eta_I + \frac{1}{4}F_IF_I + \frac{1}{2}M_IF_I + \frac{1}{4}M_{IJK}F_I\eta_J\eta_K\right], \tag{43}$$

where auxiliary fields F_I are superpartners of η_I, according to (29). The fields F_I can be integrated out using their equations of motion, which are

$$F_I = -M_I - \frac{1}{2}M_{IJK}\eta_J\eta_K\,. \tag{44}$$

It is tempting to insert this back into (43). However, one should now be particularly careful. In fact there is no obvious way how to treat products of η_I in the same point φ, which occur in the boundary action. On the classical level, one would think of fields η_I as mutually commuting. In fact this is how they effectively behave when considered in (43), because they are contracted with antisymmetric objects M_{IJK}. Nonetheless, as shown in [9], there may appear additional contributions from contractions in products of η_I, after we insert (44) back to (43). The most general form of such contractions is

$$\langle \eta_I(\varphi)\, \eta_J(\varphi) \rangle = \lambda \delta_{IJ}, \tag{45}$$

with a constant factor λ. In [9] it was shown, that for a boundary action with gauge fields only — what in our case would correspond to non-zero fields in (39) and vanishing those in (41) — boundary fermions η_I have anticommuting properties of gamma matrices, what is equivalent to $\lambda = 1$. We shall see that in our case such a value of λ would lead to an unsatisfactory result, and certain symmetries require λ to take another value. Thus, let us now just proceed in the most general setting. Inserting (44) into (43), both of these understood as normal ordered expressions, and performing contractions according to (45), leads to a renormalized product of operators

$$\begin{aligned} :\eta_M\eta_N::\eta_J\eta_K: \;=\; & :\eta_M\eta_N\eta_J\eta_K: \\ & +\lambda\Big(:\eta_N\eta_J:\delta_{MK} + :\eta_M\eta_K:\delta_{NJ} \\ & - :\eta_N\eta_K:\delta_{MJ} - :\eta_M\eta_J:\delta_{NK}\Big) \\ & +\lambda^2\Big(\delta_{NJ}\delta_{MK} - \delta_{MJ}\delta_{NK}\Big), \end{aligned} \tag{46}$$

where all η's under normal-ordering symbol :: mutually anticommute. In consequence, the following form of the boundary action emerges

$$S_{\rm bnd} = -\frac{1}{4}\int d\varphi \,\Big[\, \dot\eta_I\eta_I - M_IM_I - M_IM_{IJK}\eta_J\eta_K + \frac{1}{2}\lambda^2 M_{IJK}M_{IJK}\,\Big]. \tag{47}$$

In our particular situation, when the only non-zero fields are those in (41), the component form of this action is

$$\begin{aligned} S_{\rm bnd} \;=\; -\int d\varphi \Big[& \frac{1}{4}\dot\eta_I\eta_I - \frac{1}{4}\big(\, M_1^2 + M_3^2\,\big) + \frac{3}{4}\lambda^2\big(\, M_{123}^2 + M_{134}^2\,\big) \\ & -\frac{1}{2}\big(\, M_1M_{134}\eta_3\eta_4 + M_{134}M_3\eta_4\eta_1 \\ & +M_1M_{123}\eta_2\eta_3 + M_3M_{123}\eta_1\eta_2\,\big)\,\Big]. \end{aligned} \tag{48}$$

The last step leading to the partition function (25) is to integrate out fields η_I. This can be done with a help of a quantum mechanical formula

$$\int \mathcal{D}\eta\, \mathrm{e}^{-S} = \mathrm{Tr}\,\mathrm{e}^{\boldsymbol{H}}, \tag{49}$$

with H a Hamiltonian obtained from the action by a Legendre transformation. We should note that in this expression all products of η_I are multiplied by an antisymmetric quantity, thus once again they can be dealt with as classical anticommuting fields. Thus, to compute trace, it suffices simply to change η_I to a gamma matrix γ_I, and perform an usual trace. In order to do that, it is convenient to write the Hamiltonian as

$$\boldsymbol{H} = h \cdot \mathbf{1}_{4\times 4} + \boldsymbol{H}_\gamma, \tag{50}$$

where

$$h = -\frac{\pi}{2}\left(M_1^2 + M_3^2 \right) + \frac{3\pi}{2}\lambda^2\left(M_{123}^2 + M_{134}^2 \right) \tag{51}$$

$$\boldsymbol{H}_\gamma = -\pi\left(M_1 M_{134}\gamma_3\gamma_4 + M_{134}M_3\gamma_4\gamma_1 + M_1 M_{123}\gamma_2\gamma_3 + M_3 M_{123}\gamma_1\gamma_2 \right). \tag{52}$$

Then the partition function takes the form

$$Z(\mathcal{T}) = \mathcal{N}\, \mathrm{e}^h \cdot \mathrm{Tr}\, \mathrm{e}^{\boldsymbol{H}_\gamma}, \tag{53}$$

with $\mathcal{N}$ a normalization factor. To calculate the last trace we note that

$$\boldsymbol{H}_\gamma = -\pi \begin{bmatrix} U & \\ & W \end{bmatrix}, \tag{54}$$

where

$$U = \begin{bmatrix} i(M_1 M_{134} + M_3 M_{123}) & i(M_1 M_{123} - M_{134}M_3) \\ i(M_1 M_{123} - M_{134}M_3) & -i(M_1 M_{134} + M_3 M_{123}) \end{bmatrix}, \tag{55}$$

$$W = \begin{bmatrix} i(M_3 M_{123} - M_1 M_{134}) & i(M_{134}M_3 + M_1 M_{123}) \\ i(M_{134}M_3 + M_1 M_{123}) & i(M_1 M_{134} - M_3 M_{123}) \end{bmatrix}. \tag{56}$$

Thus we obtain

$$\mathrm{Tr}\, \mathrm{e}^{\boldsymbol{H}_\gamma} = \mathrm{Tr}\, \mathrm{e}^{-\pi U} + \mathrm{Tr}\, \mathrm{e}^{-\pi W}. \tag{57}$$

To find each of these traces we take advantage of the formula

$$\mathrm{Tr}\ \exp \begin{bmatrix} p & q \\ q & -p \end{bmatrix} = 2\cosh\sqrt{p^2+q^2}. \tag{58}$$

It then turns out somewhat surprisingly that both traces on the right side of (57) are equal

$$\mathrm{Tr}\, \mathrm{e}^{-\pi U} = \mathrm{Tr}\, \mathrm{e}^{-\pi W} = 2\cosh\left(\pi\alpha' \sqrt{w} \right), \tag{59}$$

where, taking (41) into account,

$$\begin{aligned} w &:= -\left(M_1^2 + M_3^2 \right)\frac{\left(M_{123}^2 + M_{134}^2 \right)}{\alpha'^2} \\ &= \left(-T_{13}^2 - T_{14}^2 + T_{23}^2 + T_{24}^2 \right)^2 + 4\left(T_{13}T_{23} + T_{14}T_{24} \right)^2 \\ &= \left(-T_{13}^2 + T_{14}^2 - T_{23}^2 + T_{24}^2 \right)^2 + 4\left(T_{13}T_{14} + T_{23}T_{24} \right)^2 \\ &= \left(T_{13}^2 + T_{14}^2 + T_{23}^2 + T_{24}^2 \right)^2 - 4\left(T_{13}T_{24} - T_{14}T_{23} \right)^2 . \end{aligned} \tag{60}$$

To sum up, the partition function (53) for constant tachyon fields should be interpreted as a tachyon effective potential according to (5). Moreover, the normalization factor $\mathcal{N}$ should be such, that for vanishing tachyon fields the value of the effective potential is equal to the sum of tensions T_{D1} of four branes in the system. Eventually we obtain the tachyon effective potential

$$\mathcal{V}(T_{13}, T_{14}, T_{23}, T_{24}) = 4T_{D1}\, \mathrm{e}^{h} \cosh\left(\pi\alpha' \sqrt{w}\,\right), \tag{61}$$

with $w \geq 0$, as implied by (60).

The effective tachyonic potential (61) still depends, through h, on the factor λ coming from contractions of boundary fermions. To determine its value we rewrite (51), using (41), as

$$h = -\frac{\alpha'\pi}{2}\Big[\left(T_{13}^2+T_{14}^2+T_{23}^2+T_{24}^2\right)(1+3\lambda^2)+2\left(T_{14}T_{23}-T_{13}T_{24}\right)(1-3\lambda^2)\Big]. \tag{62}$$

We should note now, that there are two pairs of *separated tachyons* in our system, T_{13} together with T_{24}, and T_{14} with T_{23}. There should be no distinction between these two pairs; this is equivalent to the statement, that our final effective potential should not change its form, if we rename branes or antibranes in the system according to $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$. But the expression $\left(T_{14}T_{23}-T_{13}T_{24}\right)$ in (62) is not invariant under each of these two renamings — it changes its sign! The only way to keep the (62) invariant is thus to set

$$\lambda^2 = \frac{1}{3}\,. \tag{63}$$

It should be noted that this result is non-standard, and rather difficult to foresee on the level of (45). In fact this is a consequence of the particular structure of the boundary action (36), that was stressed at the end of Section 2.4. As mentioned below (45), in [9] a value $\lambda = 1$ was obtained for a system with gauge fields only. It then led to a gauge-invariant commutator term after integrating fermions out. The difference between the present case is in the statistics: gauge fields correspond to anticommuting component fields $M_{I_i\ldots I_k}$ with k even, as in (39), and these are multiplied by an even number of boundary fermions in the action (36). On the other hand, for tachyons these are commuting components $M_{I_1\ldots I_k}$ emerging with k odd (41) — and in such a case the analysis takes another route, leading to (63).

As the dust has settled, we can write down the final form of the tachyon potential in a system of two brane–antibrane pairs in type I theory. Inserting the result (63) into (61), we find

$$\mathcal{V}(\mathcal{T}) = 4T_{D1}\, \mathrm{e}^{-\alpha'\pi\left(T_{13}^2+T_{14}^2+T_{23}^2+T_{24}^2\right)} \cosh(\alpha'\pi\sqrt{w}), \tag{64}$$

where w is given by (60),

$$w = \left(T_{13}^2 + T_{14}^2 + T_{23}^2 + T_{24}^2\right)^2 - 4\left(T_{13}T_{24} - T_{14}T_{23}\right)^2. \tag{65}$$

The method we followed, being rigorous and exact on one hand, on the other leads through rather lengthy calculations to the result (64) which is quite involved. A cautious look on the formula (50) reveals that this is equal to $-2\alpha'\pi\mathcal{T}^2$ (after decomposition in the basis of Clifford algebra) plus some additional terms, which vanish precisely when $\lambda^2 = \frac{1}{3}$. Thus a concise form of writing (64) is just

$$\mathcal{V}(\mathcal{T}) = \mathcal{N}\,\mathrm{Tr}\,\mathrm{e}^{-2\alpha'\pi\mathcal{T}^2}, \tag{66}$$

what agrees with the formula for a tachyon potential with nontrivial Chan-Paton factors conjectured in [7]. It is interesting to recover that an emergence of the form (66) is consistent with the constraint (63).

3.3. Analysis of $\mathcal{V}(\mathcal{T})$

Analysis of the result (64) reveals it has the properties one would intuitively expect.

First of all, as a special case it may be assumed only one of the fields — say T_{13} — is a varying variable, and all the others are kept fixed $T_{14} = T_{23} = T_{24} = 0$. Then (61) reduces to

$$\mathcal{V}(T_{13}) = 2T_{D1} + 2T_{D1}\mathrm{e}^{-2\pi\alpha' T_{13}^2}. \tag{67}$$

The tensions of branes D_2 and $\bar{D}_4$ are represented by the constant $2T_{D1}$, and the second term is a well known formula for a potential of a single tachyon [8]. After the tachyon condenses, $T_{13} \to \infty$, branes D_1 and $\bar{D}_3$ disappear from the system, so that the energy after condensation is equal just $2T_{D1}$.

Another special case are *separated tachyons*, which we mentioned already before. First we consider a configuration with only one pair of separated tachyons present, which corresponds to some constraints setting $T_{14} = T_{23} = 0$, as in the figure 2. Of course, it would be exactly equivalent to consider another pair of separated tachyons, with T_{13} and T_{24} held fixed.

In this situation

$$w = (T_{13}^2 - T_{24}^2)^2, \tag{68}$$

what yields

$$\mathcal{V}(T_{13}, T_{24}) = 2T_{D1}\left(\mathrm{e}^{-2\pi\alpha' T_{13}^2} + \mathrm{e}^{-2\pi\alpha' T_{24}^2}\right). \tag{69}$$

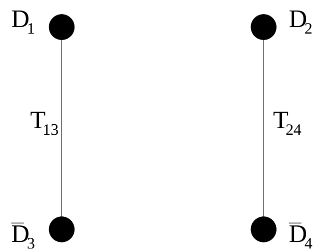

Fig. 2. Separated tachyons.

To reach minimum of this potential, which equals to 0, two fields must simultaneously condense and become infinite, what corresponds to annihilation of all four branes.

Also a little more general statement is true: if two separated tachyons — say T_{13} and T_{24} — condense, and two others (in this case T_{14} and T_{23}) are kept finite, the energy of the system after condensation will also be equal to 0. Thus a sufficient condition for an annihilation of all four branes is condensation of any two separated tachyons in the system.

To prove this statement we can write

$$\mathcal{V} = 2T_{D1}\mathrm{e}^{-\alpha'\pi(T_{14}^2+T_{23}^2)} \cdot \left(\mathrm{e}^{-\alpha'\pi(T_{13}^2+T_{24}^2+\sqrt{w})} + \mathrm{e}^{-\alpha'\pi(T_{13}^2+T_{24}^2-\sqrt{w})}\right). \quad (70)$$

The first term in the bracket decreases to 0 as $T_{13},\ T_{24} \to \infty$. The second term would also vanish in this limit providing

$$T_{13}^2 + T_{24}^2 - \sqrt{w} > 0\,. \quad (71)$$

This is indeed true, as the above expression after is equivalent to the following one,

$$4 > \frac{(T_{14}^2 - T_{23}^2)^2}{T_{13}^2 T_{24}^2} + \frac{8T_{14}T_{23}}{T_{13}T_{24}} + 2(T_{14}^2 + T_{23}^2)\left(\frac{1}{T_{13}^2} + \frac{1}{T_{24}^2}\right), \quad (72)$$

the right hand side of which explicitly vanishes in the limit $T_{13}, T_{24} \to \infty$.

4. Conclusions

In this paper a partition function for constant tachyon fields $\mathcal{T}$ in a system of two $D1$-$\overline{D1}$ pairs has been found, along the lines of Boundary String Field Theory. According to the conjecture (4), it is identified with an effective potential for tachyon fields, as stated in (61). Even though its form is quite involved (64), it turns out it is just equal to $\exp(-2\alpha'\pi\mathcal{T}^2)$, up to a normalization factor. This form is consistent with other conjectures in literature, thus confirming both validity of those conjectures, as well as particular scheme of BSFT we have followed.

Meanwhile, an interesting problem concerning boundary fermions emerged. It turns out, that to obtain a tachyon potential with a certain symmetry concerning separated tachyons, a particular prescription for operator renormalization of the form

$$\langle \eta_I(\varphi)\, \eta_J(\varphi)\rangle = \frac{1}{3}\delta_{IJ} \tag{73}$$

should be applied. This is a non-standard prescription, and in the case being considered results from a bosonic statistics of tachyons fields appearing in the boundary action (43). It should be contrasted with an analogous formula derived in literature for gauge fields, in which case the proportionality factor in the above formula would be just 1.

Basic features of the potential found has also been presented. It has been shown that achieving its minima in a process of tachyon condensation corresponds to annihilation of all the branes in the system. Moreover, it has been confirmed that condensation of only one pair of separated tachyons also leads to annihilation of all the branes.

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Appendix

SO(4) Clifford algebra

In this appendix a certain representation of SO(4) Clifford algebra is presented, which is a basis of the calculations presented in Section 3.

SO(4) Dirac matrices are of the size 4×4, and anticommute according to

$$\{\, \gamma_I,\ \gamma_J \,\} = 2\, \delta_{IJ} \qquad \text{for} \quad I, J = 1, \ldots, 4\,.$$

Denoting ordinary Pauli matrices by $\sigma_{1,2,3}$, and two-dimensional identity matrix by $\mathbb{I}_{2\times 2}$, our choice of Dirac matrices is

$$\gamma_1 = \sigma_1 \otimes \sigma_1 \qquad \gamma_2 = \sigma_1 \otimes \sigma_2$$

$$\gamma_3 = \sigma_1 \otimes \sigma_3 \qquad \gamma_4 = \sigma_2 \otimes \mathbb{I}_{2\times 2}\,.$$

It follows that the second rank elements $\gamma_{ij} = \gamma_{[i}\gamma_{j]} = \frac{1}{2}\big(\,\gamma_i\gamma_j - \gamma_j\gamma_i\,\big) = \gamma_i\gamma_j$ are

$$\gamma_{12} = i\, \mathbb{I}_{2\times 2} \otimes \sigma_3 \qquad \gamma_{13} = -i\, \mathbb{I}_{2\times 2} \otimes \sigma_2$$

$$\gamma_{14} = i\, \sigma_3 \otimes \sigma_1 \qquad \gamma_{23} = i\, \mathbb{I}_{2\times 2} \otimes \sigma_1$$

$$\gamma_{24} = i\,\sigma_3 \otimes \sigma_2 \qquad \gamma_{34} = i\,\sigma_3 \otimes \sigma_3\,.$$

The third rank elements $\gamma_{ijk} = \gamma_i \gamma_j \gamma_k$:

$$\gamma_{123} = i\,\sigma_1 \otimes \mathbb{I}_{2\times 2} \qquad \gamma_{124} = i\,\sigma_2 \otimes \sigma_3$$

$$\gamma_{134} = -i\,\sigma_2 \otimes \sigma_2 \qquad \gamma_{234} = i\,\sigma_2 \otimes \sigma_1\,.$$

The last two elements of Clifford algebra are $\gamma_{1234} = \gamma_1\gamma_2\gamma_3\gamma_4 = -\sigma_3 \otimes \mathbb{I}_{2\times 2}$ and identity matrix $\mathbb{I}_{4\times 4}$.

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