INVERSION OF THE PHOTON NUMBER INTEGRAL

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We consider the behavior of the photon number integral under inversion, concentrating on Euclidean space. The discussion may be framed in terms of an additive differential I which arises under inversions. The quantity $\int\int I$ is an interesting integral invariant whose value characterizes different configurations under inversion.

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1. Introduction

The "photon number integral" [1], called n, is a construction whereby one can define the number of photons radiated by a charged particle following a prescribed trajectory in space-time (Minkowski space). For n to be finite the trajectory must obey two conditions: smoothness and equality of initial and final trajectories. The integral then returns a real number for any such curve. The idea can be extended from Minkowski to Euclidean space where it also becomes possible to consider closed curves. The identification of initial and final velocities (tangents) is then automatic.

An important question left open in [1] concerns the symmetries of n. Given $n(C)$ for a curve C for what other curves C' do we have $n(C) = n(C')$? There are the evident invariances under rotations (Lorentz transformations), translations, and reflections. Furthermore, since there are no dimensional quantities except the path itself involved in the problem, there is also an evident invariance under scaling, that is rescaling of all coordinates simultaneously, $x_\mu \to \lambda x_\mu$, where λ is a constant.

Nevertheless the question remains as to further invariances, in particular with respect to conformal transformations. These are closely related to scale transformations, and indeed conformal invariance originally entered physics as a property of electrodynamics, and ought to be expected to apply to photons. Also, in somewhat different contexts the properties of integral expressions like ours have been considered in connection with inversion [2], or "Mobius" invariance [3], which is tantamount to conformal invariance. Here we would like present a discussion of inversions for n.

The expression in question, in Minkowski space, is

$$n = \int\int dx_\mu \frac{1}{S_{i\varepsilon}^2} dx'_\mu \,. \tag{1}$$

Since our question here is essentially a mathematical one, we have dropped the (dimensionless) electromagnetic coupling constant appearing in the original expression for photons.

$S_{i\varepsilon}$ is the four-distance between the points x, x' in the following way

$$S_{i\varepsilon}^2 = (t - t' + i\varepsilon)^2 - (\boldsymbol{x} - \boldsymbol{x}')^2 \,. \tag{2}$$

The charm of the expression (1) is that despite the possible singularities it is actually finite. The possible singularity at $S^2 = 0$ is handled by the $i\varepsilon$ and the possible divergence at infinity by the "straight line condition" for the equality of initial and final paths, as may be verified from the fact that n is zero for the simple straight line [1]. This means it is well defined as it stands and needs no "regularizations" or "subtractions". In Ref. [2] a regularization is introduced by supersymmetry where a scalar particle cancels the singularity of a vector particle ("gluon") propagator. In Ref. [3] the singularity is canceled by subtracting a second term where the straight line distance S between (x, x') is replaced by the arc length along the curve. However as we anticipate from its physical origin, Eq. (1) is finite as it stands. Indeed using the identity

$$\partial_\tau \partial_{\tau'} \ln(S^2) = +\frac{4}{S^4}\left(\Delta_\mu \frac{dx_\mu}{d\tau}\right)\left(\Delta_\nu \frac{dx_\nu}{d\tau'}\right) - \frac{2}{S^2}\frac{dx_\mu}{d\tau}\frac{dx_\mu}{d\tau'} \tag{3}$$

we were able to rewrite the integral as

$$\int\int dx_\mu \frac{1}{S_{i\varepsilon}^2} dx'_\mu = 2\int\int dx_\mu \frac{\delta_{\mu\nu} - \frac{\Delta_\mu \Delta_\nu}{S^2}}{S^2} dx'_\nu \,. \tag{4}$$

Delta is the vectorial distance between the two points, $\Delta_\mu = x_\mu - x'_\mu$. We can rewrite this in a suggestive form if we introduce the "transverse vector"

$$dx_\mu^{\mathrm{T}} = dx_\mu - \frac{\Delta_\mu(\Delta \cdot dx)}{S^2} \,, \tag{5}$$

which is dx with the "longitudinal part" removed, *i.e.* has the property $dx^{\mathrm{T}} \cdot \Delta = 0$. This is suggestive of the transversality property of physical photons and it might be said that our expression is finite because it only contains the radiated and not the "coulomb" photons. In any event, the expression is now manifestly non-singular, as may be seen by expanding the numerator, see Ref. [1].

Observe that this procedure does not introduce any dimensional quantities, so the expression is still scale invariant. Apparently the $i\varepsilon$ is harmless in this respect. This manifestly non-singular expression, without the $i\varepsilon$, may be used to define n in case of doubt and we will use it in the following (mostly in its Euclidean version) taking n as:

$$n = 2\int\int dx_\mu^{\rm T}\frac{1}{S^2}dx_\mu'^{\rm T}\,. \tag{6}$$

2. Inversion

We shall concentrate on the inversion operation

$$x_i \to \frac{a^2}{x^2}x_i\,. \tag{7}$$

The full conformal group is generated by adding these inversions or "Mobius transformations" to the usual translations and rotations. To keep the physical dimensions in order we have introduced a length constant a parameterizing the operation. The inversion is around a point $\mathcal{O}$, which we take as the origin. As we shall see below, the cases where the center of inversion is on the curve C itself is of special interest. The index i means any of the coordinates, and in Minkowski space includes an inversion of the time coordinate x_0. We shall mainly focus however on the simpler case of Euclidean space. There we use boldface notation for vectors, so inversion is $\boldsymbol{x} \to \frac{a^2}{x^2}\boldsymbol{x}$.

We thus begin by considering the formal properties of the Euclidean expression

$$n = -\int\int\frac{d\boldsymbol{x}d\boldsymbol{x}'}{S^2} = -\int\int\frac{d\boldsymbol{x}d\boldsymbol{x}'}{(\boldsymbol{x}-\boldsymbol{x}')^2} \tag{8}$$

under inversion. The $(-)$ sign is the natural choice that makes n positive in Euclidean space. The curve over which the integrations are performed may either be a smooth closed curve, or a smooth infinite curve which becomes the same straight line at $\pm\infty$.

3. Formal inversion

It is illuminating to start with the relation Eq. (3). We use a differential notation, where $d_x f(\boldsymbol{x},\boldsymbol{x}') = \nabla_x f\ d\boldsymbol{x}$, so that with $S^2 = (\boldsymbol{x}-\boldsymbol{x}')^2$

$$d_x \ln S^2 = \frac{2(\boldsymbol{x}-\boldsymbol{x}')\ d\boldsymbol{x}}{S^2}\,, \qquad d_{x'}\ln S^2 = -\frac{2(\boldsymbol{x}-\boldsymbol{x}')\ d\boldsymbol{x}'}{S^2}\,. \tag{9}$$

Eq. (3) then takes the form of the curious and interesting identity

$$-\frac{1}{2}[d_x d_{x'} \ln S^2 + d_x \ln S^2 \ d_{x'} \ln S^2] = \frac{\boldsymbol{dx}\ \boldsymbol{dx'}}{S^2}\,. \tag{10}$$

In this way we express the integrand of Eq. (8) in terms of certain differentials with simple transformations under inversion. Applying the inversion to S^2

$$S^2 = (\boldsymbol{x} - \boldsymbol{x'})^2 \to \frac{a^2}{x^2}\frac{a^2}{x'^2} S^2 \tag{11}$$

and so

$$\ln S^2 \to \ln \frac{a^2}{x^2} + \ln \frac{a^2}{x'^2} + \ln S^2\,. \tag{12}$$

Hence if we insert the substitution Eq. (7) in the lhs of Eq. (10) the first term is unchanged while for the second

$$\begin{aligned} -\tfrac{1}{2}\left[d_x \ln S^2 \ d_{x'} \ln S^2\right] &\to -\tfrac{1}{2}\left[d_x \ln S^2 \ d_{x'} \ln S^2\right] \\ +\tfrac{1}{2}\left[-d_x \ln x^2 \ d_{x'} \ln x'^2 + d_x \ln S^2 d_{x'} \ln x'^2 + d_{x'} \ln S^2 \ d_x \ln x^2\right]&. \end{aligned} \tag{13}$$

In other words our fundamental form transforms additively under inversion

$$\frac{\boldsymbol{dx}\ \boldsymbol{dx'}}{S^2} \to \frac{\boldsymbol{dx}\ \boldsymbol{dx'}}{S^2} + I\,, \tag{14}$$

where we call the additional quantity I. In this way we recover the results of Ref. [2] for the gluon propagator.

4. Inversion of the explicitly finite integrand

So far we have ignored the singularity in Eq. (8). We return to the explicitly finite form Eq. (6) as the definition of n. Since the explicitly non-singular forms were found (see Ref. [1]) by subtracting 1/2 of Eq. (3) from Eq. (1), we do the same here and Eq. (10) becomes

$$\begin{aligned} &-d_x d_{x'} \ln S^2 - \frac{1}{2} d_x \ln S^2 \ d_{x'} \ln S^2 \\ &= \frac{2\ \boldsymbol{dx}\ \boldsymbol{dx'}}{S^2} - \frac{2\,[\boldsymbol{dx}(\boldsymbol{x}-\boldsymbol{x'})]\ [(\boldsymbol{x}-\boldsymbol{x'})\boldsymbol{dx'}]}{S^4} = \frac{2\ \boldsymbol{dx}^{\mathrm{T}}\ \boldsymbol{dx'}^{\mathrm{T}}}{S^2}\,, \end{aligned} \tag{15}$$

where the rhs is now the explicitly non-singular integrand of Eq. (6). The difference between the singular and non-singular forms is simply the coefficient of the first term on the left. However, since this term is in any event identically invariant under inversion, Eq. (14) still holds, that is

$$\frac{2\ \boldsymbol{dx}^{\mathrm{T}}\ \boldsymbol{dx'}^{\mathrm{T}}}{S^2} \to \frac{2\ \boldsymbol{dx}^{\mathrm{T}}\ \boldsymbol{dx'}^{\mathrm{T}}}{S^2} + I\,, \tag{16}$$

with the same I as in Eq. (14).

5. Properties of I

I is defined as

$$I = \tfrac{1}{2}[-d_x \ln x^2 \; d_{x'} \ln x'^2 + d_x \ln S^2 d_{x'} \ln x'^2 + d_{x'} \ln S^2 \; d_x \ln x^2] \,, \qquad (17)$$

where the origin is at the center of inversion $\mathcal{O}$. If the origin is placed elsewhere, x is the distance to $\mathcal{O}$. Evidently I is symmetric

$$I(x, x') = I(x', x) \,. \qquad (18)$$

Furthermore we note an important property arising from the fact that two successive applications of the inversion is the identity operation. Applying an inversion to the rhs of Eq. (14) or Eq. (16) again, we should get the original expression. We thus conclude that under inversion

$$I \to -I \,, \qquad (19)$$

which one can also check by directly inserting Eq. (7) into Eq. (17).

I is a scalar under rotations, but not under translations (holding $\mathcal{O}$ fixed) because of the presence of x^2, the distance from $\mathcal{O}$. It is invariant under rescalings $\boldsymbol{x} \to \lambda \boldsymbol{x}$ because of the presence of the differentials. For the same reason it is independent of the parameter a giving the radius of the sphere of inversion. That I does not contain the parameter of the inversion suggests that it can only depend on some global properties of the operation.

6. Integration

To study the behavior of $n(C)$ under inversion we proceed according to the following steps. We carry out a change of variables according to Eq. (7) in the integral for $n(C)$. This results simply in an identity with a new curve C_{inv} in the new variables. The integrand receives an additional term, I. which is also to be integrated over the new curve C_{inv}. That is, we have one term with an integral of the desired expression and one with I, so that

$$n(C) = n(C_{\text{inv}}) + \int\!\!\int_{C_{\text{inv}}} I \,. \qquad (20)$$

If we can show that $\int\!\int_{C_{\text{inv}}} I$ is zero then $n(C) = n(C_{\text{inv}})$.

As another expression of the fact that two successive operations with Eq. (7) are the identity we can invert $n(C_{\text{inv}})$ once more to return to the original curve, giving the relation

$$\int\!\!\int_{C} I + \int\!\!\int_{C_{\text{inv}}} I = 0 \,. \qquad (21)$$

This of course just amounts to Eq. (19) if we change variables again to make $C_{\rm inv} \to C$. An evident property following from Eq. (21) is that if $C_{\rm inv} \equiv C$, as for a circle with the center of inversion in the center, or a closed curve in 3-space on the surface of the sphere of inversion, then

$$\int\int_C I = 0\,, \qquad C \equiv C_{\rm inv}\,. \tag{22}$$

As opposed to Eq. (19) this is not merely an algebraic identity but involves of course the nature of the curves. It is evidently true by Eq. (19) if C and $C_{\rm inv}$ are identically the same curve. But it is clearly also true in some more general sense, say if one curve is simply a rotation, translation or rescaling of the other. The wider meaning of "$\equiv$" in $C \equiv C_{\rm inv}$ is an interesting question and will be further discussed below (Section 11.5).

7. Two curves

An amusing generalization of these properties of the integration suggests itself. We mention it although it lies somewhat outside our main topic. The quantity $\int\int I$ is a functional of one curve C. But actually it could be regarded as a functional of two curves C and C'. We might have *different* curves in the x and x' spaces. That is, we consider the integral $\int_C \int_{C'} I$.

Because of Eq. (18) it does not matter to which variable the curve is assigned:$\int_C \int_{C'} I = \int_{C'} \int_C I$. Now let C' be the inversion of C so $C' = C_{\rm inv}$. Then under inversion according to Eq. (19) the integral goes to minus itself. Therefore,

$$\int_C \int_{C'} I = 0\,, \qquad C' = C_{\rm inv}\,. \tag{23}$$

Eq. (22) may then be read as the special case where $C = C_{\rm inv}$.

A place where Eq. (23) might be useful concerns the question of the additivity of the parts of a curve. In general of course $\int\int I$ for a single curve cannot broken into two parts such that the total integral is the sum of the integrals for the two parts. I is a bilocal object, and there will usually be cross terms between the two parts. However, if the two parts in question are the inversion of each other, then according to Eq. (23), the cross terms are zero and the integrals for the two parts may be simply added.

To develop this idea properly a study of the possible singularities, which may be different than in the single-curve case, would be necessary. We stress that our integrals are always meant over just one curve in the original sense unless explicitly indicated as in Eq. (23).

8. Possible singularities of I

There are singularities or potential singularities of I which need to be understood. We will come to the conclusion that, despite appearances, I is a non-singular object — for a given curve.

We would like to argue that for the investigation of possible singularities at a point it suffices to consider the behavior of I for a straight line through that point. Consider the smooth curve $\boldsymbol{x}(\tau)$ in the neighborhood of the point $\boldsymbol{x}(0)$. It is convenient to introduce the "natural" parameterization where τ is the length along the curve (proper time, in Minkowski space). Then $\boldsymbol{x} \approx \boldsymbol{x}(0) + \dot{\boldsymbol{x}}\tau + \ddot{\boldsymbol{x}}(1/2)\tau^2 + \ldots$, where $\dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}, \ldots$ are the first, second,... derivatives with respect to τ. Due to the choice of τ as the path length $(\dot{\boldsymbol{x}})^2 = 1$, $\dot{\boldsymbol{x}}\ddot{\boldsymbol{x}} = 0$, and so forth. Now $\boldsymbol{x}(\tau) - \boldsymbol{x}(\tau')$ is odd with respect to interchange of τ, τ'. This leads to $(\boldsymbol{x}(\tau) - \boldsymbol{x}(\tau'))^2 \approx (\tau - \tau')^2[(\dot{\boldsymbol{x}})^2 + (\ddot{\boldsymbol{x}})^2 b(\tau, \tau') + \ldots]$, where b is a bilinear expression in τ, τ'.

Since, with $(\dot{\boldsymbol{x}})^2 = 1$

$$\ln[(\tau-\tau')^2(\dot{\boldsymbol{x}}^2+(\ddot{\boldsymbol{x}})^2 b(\tau,\tau')+\ldots)] = \ln(\tau-\tau')^2+\ln[1+(\ddot{\boldsymbol{x}})^2 b(\tau,\tau')+\ldots] \quad (24)$$

we see that the possibly singular behavior of the logarithm results from the "velocity" or tangent term $\dot{\boldsymbol{x}}$, while the dependence on the "acceleration" or curvature is non-singular for $\tau, \tau' \to 0$. Similarly for $\ln x^2$ near $x^2 = 0$ we have $\ln x^2 \approx \ln \tau^2 + \ln(1 + (\ddot{\boldsymbol{x}})^2 b(\tau) + \ldots)$. We may thus investigate the possible singularities by looking at the behavior of I for a straight line.

We first consider the possible singularity for $S^2 = 0$, while assuming $x^2 \neq 0$, that the center of inversion $\mathcal{O}$ is not on the curve. Choose the "1" axis such that it is parallel to the tangent $\dot{\boldsymbol{x}}$ at the point. Let the distance of the projected tangent from $\mathcal{O}$ *i.e.* its "impact parameter", be δ (Fig. 1). Now, for the investigation of the possible singularity we treat the curve as

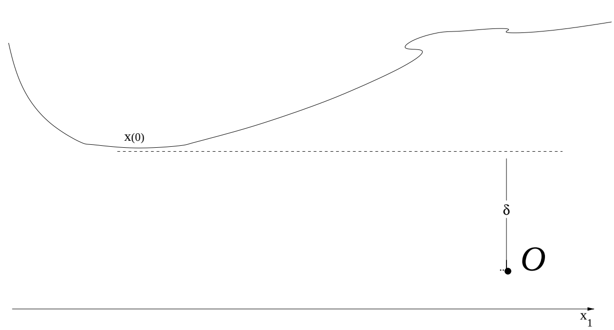

Fig. 1. Construction for studying the possible singularity due to $S^2 = 0$ at a point $x(0)$. The direction of the tangent (dotted line) is used to determine the "1" axis and has "impact parameter" δ with respect to the center of inversion.

this straight line, as just explained. We then have $S^2 = (x_1 - x'_1)^2$ and $d\boldsymbol{x} = dx_1$. After some algebra following the manipulations of Ref. [2] for the straight line we have

$$I_{\text{straight line}} = -2\ dx_1 dx'_1 \frac{\delta^2}{({x_1}^2 + \delta^2)({x'_1}^2 + \delta^2)}\,, \tag{25}$$

which is in fact non-singular at $\boldsymbol{x} \approx \boldsymbol{x'}$. The absence of the singularity may be traced to the fact that I is symmetric in x, x' and so must contain only even powers in $\boldsymbol{x} - \boldsymbol{x'}$. The potential singularity from Eq. (9), however would be odd in this variable, and so is in fact absent.

Concerning the possible singularity at $x^2 = 0$, as occurs for a curve through $\mathcal{O}$, first consider $I(0, x')$, *i.e.* one variable at some ordinary point and the other near zero. In this case the first and third term of I, Eq. (17) cancel since $d_{x'} \ln S^2 = d_{x'} \ln x'^2$ for $x = 0$. On the other hand the second term of I is non-singular, so there is no singularity.

As for $I(0, 0)$, we examine the straight line through the origin. By direct inspection of I one finds

$$I_{\text{straight line through } \mathcal{O}} = 0\,. \tag{26}$$

Therefore there are no singularities connected with $x = 0$.

Eq. (26) may be viewed as the consequence of a general symmetry property since two points along a ray may be interchanged by an inversion. Therefore by Eq. (19) $I(x, x') = -I(x', x)$. On the other hand, by Eq. (18) we also have $I(x, x') = +I(x', x)$, so I is zero.

Since in Eq. (26) we have zero for the straight line through $\mathcal{O}$, we look at the simplest curvature or "acceleration" contribution, that of the circle. Consider a circle passing through $\mathcal{O}$ and use the relation for the length of a chord $l = 2R \sin\phi$, where ϕ is the half-angle subtended at the center of the circle. This leads to

$$\begin{aligned} I \;=\; & \tfrac{1}{2}[-d \ln \sin^2\phi \; d \ln \sin^2\phi' + d \ln \sin^2(\phi - \phi')\; d \ln \sin^2\phi' \\ & + d \ln \sin^2(\phi - \phi')\; d \ln \sin^2\phi] \end{aligned} \tag{27}$$

that is $I = 2d\phi d\phi'[-\cot\phi\ \cot\phi' + \cot(\phi - \phi')(\cot\phi' - \cot\phi)]$. Using the identity $\cot(\phi - \phi') = \frac{\cot\phi \cot\phi' + 1}{\cot\phi' - \cot\phi}$ we have finally

$$I_{\text{circle through } \mathcal{O}} = 2d\phi d\phi'\,, \tag{28}$$

where $0 \geq \phi \geq \pi$.

If we have only a segment of a circle passing through $\mathcal{O}$ the integration will only be over the corresponding angle, and Eq. (28) may be said to approach Eq. (26) in the sense that as the radius of the circle R becomes very large this angular segment becomes very small. The very simple form of Eq. (28) suggests that it may sometimes be preferable to use coordinates not centered on $\mathcal{O}$ but rather on the center of curvature for the curve at $\mathcal{O}$.

9. Simplest or "reference" cases

There are four configurations to discuss, according to whether the curve is closed or infinite and whether $\mathcal{O}$ is on or off the curve. For orientation we discuss a simplest or "reference" curve for each case.

9.1. Circle, not through $\mathcal{O}$

This is the simplest case of the finite closed curve where the center of inversion is not on the curve. Inversion of a circle produces another circle, and similarly inversion of a general closed curve will produce another closed curve, as long as $\mathcal{O}$ is not on the curve.

For the circle one finds n by directly integrating Eq. (6)

$$n_{\text{circle}} = 2\pi^2 \,. \tag{29}$$

Since n is purely a property of the geometric figure, this holds for any circle.

We therefore conclude from Eq. (20)

$$\int\int I_{\text{circle not through } \mathcal{O}} = 0 \,. \tag{30}$$

As on check on this argument we can show directly that the integral of I is zero by Eq. (22), that is by Eq. (19), for those circles going identically into themselves under inversion. This occurs for inversion through the center of a circle for example, or when $\mathcal{O}$ is in the plane but outside the circle, take $a^2 = dD$ in Eq. (7) with d the closest point of the circle to $\mathcal{O}$ and D the furthest.

9.2. Infinite straight line not through $\mathcal{O}$, or circle through $\mathcal{O}$

We now turn to the consideration of curves involving infinities. The simplest case is the infinite straight line, not through $\mathcal{O}$. Under inversion this becomes a circle through $\mathcal{O}$. The general case here refers to curves, which although they become the same straight line [4] at large distances, have some arbitrary form at finite distances (Fig. 3). As with the straight line/circle pair, under inversion these become a closed curve passing through $\mathcal{O}$; and*vice-versa*. Since they are the images of each other under inversion, we consider the two cases together.

That some subtlety is involved is evident from the fact that on the one hand we have

$$n_{\text{straight line}} = 0\,. \tag{31}$$

But for the inversion of the straight line, a circle crossing the origin (or for that matter any circle), we have not zero but rather

$$n_{\text{circle}} = 2\pi^2\,, \tag{32}$$

as is found by integrating the definition Eq. (6).

That different values for n result is perhaps not entirely surprising since the inversion has produced a basic change in the figure, an infinite curve becoming a finite curve. In Ref. [2] this difference was associated with the "anomaly".

According to Eq. (20), when starting from the straight line we must integrate I over a circle through $\mathcal{O}$. Or when beginning with a circle we integrate I along an infinite straight line. To be consistent with Eq. (31) and Eq. (29), these integrations ought to produce non-zero and opposite sign contributions.

Indeed, in Eq. (25) we already have I for the straight line. Carrying out the integral we see that it is independent of δ and gives

$$\int\int_{-\infty}^{+\infty} I_{\text{straight line not through } \mathcal{O}} = -2\pi^2\,. \tag{33}$$

Similarly, we use Eq. (28) to integrate over a circle through $\mathcal{O}$

$$\int\int I_{\text{circle through } \mathcal{O}} = 2\pi^2\,. \tag{34}$$

These results are in agreement with Eq. (20), Eq. (31) and Eq. (32) and of course Ref. [2]. We note the contrast of Eq. (34) with Eq. (30); or between Eq. (33) and Eq. (35) below. Apparently while I is a non-singular object along a *given* curve, an infinitesimal change in that curve can produce a finite effect in the integral.

9.3. Straight line through $\mathcal{O}$

For closed finite curves, as typified by the circle not through $\mathcal{O}$, we had neither points at infinity nor points at zero. In the previous subsection we had either one or the other, mapping into each other under inversion. Finally, we consider having both at the same time: curves which are both infinite and pass through $\mathcal{O}$. The simplest or reference case is the straight line through $\mathcal{O}$. An inversion returns the same straight line through the $\mathcal{O}$. Observe that also for the general case of this class, namely an arbitrary curve passing through $\mathcal{O}$ and becoming the same straight line at $\pm\infty$, inversion produces a curve of this same type (Fig. 2).

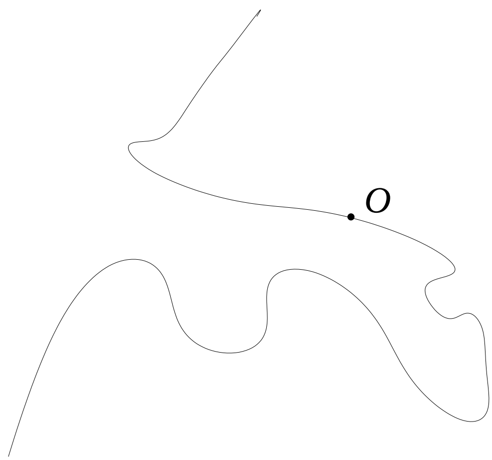

Fig. 2. Example of a general curve of the fourth type of configuration "straight line through $\mathcal{O}$". The lines continue to $\pm\infty$ along a common straight line, and an arbitrary number of Euclidean dimensions is implied.

In Eq. (26) we had $I_{\text{straight line through } \mathcal{O}} = 0$. Evidently then

$$\int\int I_{\text{straight line through } \mathcal{O}} = 0\,. \tag{35}$$

This is of course in agreement with the arguments of Section 6 since under inversion the straight line through $\mathcal{O}$, being on a ray, goes identically into itself.

This case is most liable to be of interest physically in Minkowski space where it will correspond to the path of a charge, and the inversion also produces a possible path for a charge.

10. Once-integrated expression

We now turn to the explicit integration of I. To integrate I, note that it consists of product pairs where one member of each product does not contain both variables. Hence one member of each pair, given as a total derivative, can be explicitly integrated. The limits of integration are also the same in both variables. Therefore it appears useful to interchange the names of x, x' say in the last term of I, Eq. (17) to arrive at the expression

$(1/2)\int\int d_x[-\ln x^2 + 2\ln(\boldsymbol{x}-\boldsymbol{x}')^2]d\,\ln x'^2$, where the x integration can be done immediately. However, this has the disadvantage that it is no longer explicitly symmetric in x, x', which introduces a singularity of the integrand for $\boldsymbol{x} \approx \boldsymbol{x'}$. Although this is only apparent and integrates to zero, it is perhaps more comfortable to let $\ln[S^2] \to \ln[S^2+\eta^2]$ so as to make the integrand explicitly finite and then let $\eta \to 0$ at the end of the calculation. It may be verified that I is completely regular under this procedure. After the relabeling of variables we have an explicitly finite expression (for $\mathcal{O}$ not on the curve) and can now perform the x integration along an arbitrary curve between limits $\boldsymbol{A}$ and $\boldsymbol{B}$ to obtain

$$\int\int_{\boldsymbol{B}}^{\boldsymbol{A}} I = \int_{\boldsymbol{B}}^{\boldsymbol{A}} d\,\ln x^2\left(-\frac{1}{2}\,\ln\frac{A^2}{B^2}+\ln\frac{(\boldsymbol{x}-\boldsymbol{A})^2}{(\boldsymbol{x}-\boldsymbol{B})^2}\right) \qquad C \text{ not through } \mathcal{O}\,. \tag{36}$$

Observe that there are no singularities connected with this expression. At an endpoint, for $\boldsymbol{x} \to A$ we have $\int^{\boldsymbol{A}} d\,\ln x^2 \ln(\boldsymbol{x}-\boldsymbol{A})^2 \sim \int^1 dx\,(\ln A^2 + \ln(x-1))$ which is non-singular; while $x \to 0$ is excluded by assumption.

However, we shall also need a once-integrated expression when $\mathcal{O}$ is on the curve, where $x \to 0$ must be considered. To see what replaces Eq. (36) note that Eq. (28) or Eq. (26) tell us that there is in fact no singularity associated with $\mathcal{O}$. Therefore we can excise an infinitesimal region along the curve around $x = 0$ without affecting the value of the integral:

$$\int\int_{\boldsymbol{B}}^{\boldsymbol{A}} I = \left(\int_{\boldsymbol{B}}^{-\boldsymbol{\varepsilon}} + \int_{\boldsymbol{\varepsilon}}^{\boldsymbol{A}}\right)\left(\int_{\boldsymbol{B}}^{-\boldsymbol{\varepsilon}} + \int_{\boldsymbol{\varepsilon}}^{\boldsymbol{A}}\right) I\,, \qquad \varepsilon \to 0\,. \tag{37}$$

Now again writing $\int\int I = (1/2)\int\int d_x[-\ln x^2 + 2\ln(\boldsymbol{x}-\boldsymbol{x}')^2]d\,\ln x'^2$ and carrying out the $\int d_x$ we obtain

$$\int\int_{\boldsymbol{B}}^{\boldsymbol{A}} I_{C \text{ through } \mathcal{O}} = \left(\int_{\boldsymbol{B}}^{-\boldsymbol{\varepsilon}} + \int_{\boldsymbol{\varepsilon}}^{\boldsymbol{A}}\right) d\,\ln x^2\left(-\frac{1}{2}\ln\frac{A^2}{B^2}+\ln\frac{(\boldsymbol{x}-\boldsymbol{A})^2}{(\boldsymbol{x}-\boldsymbol{B})^2}+\ln\frac{(\boldsymbol{x}+\boldsymbol{\varepsilon})^2}{(\boldsymbol{x}-\boldsymbol{\varepsilon})^2}\right). \tag{38}$$

We first consider the last term, $\int_{\boldsymbol{B}}^{-\boldsymbol{\varepsilon}} + \int_{\boldsymbol{\varepsilon}}^{\boldsymbol{A}} d\ln x^2\,\ln\frac{(\boldsymbol{x}+\boldsymbol{\varepsilon})^2}{(\boldsymbol{x}-\boldsymbol{\varepsilon})^2}$. As $\varepsilon \to 0$, the integrand vanishes for any non-infinitesimal value of x. Thus it suffices to evaluate the integral for a straight line in the vicinity of $\mathcal{O}$. Introducing the variable $y = x/\varepsilon$, this leads to the integral $\int_{-\infty}^{-1} + \int_1^\infty \frac{2dy}{y}\ln(\frac{y+1}{y-1})^2 = 8\int_1^\infty \frac{dy}{y}\ln(\frac{y+1}{y-1}) = 2\pi^2$ [5], so we may write

$$\int_{\boldsymbol{B}}^{-\boldsymbol{\varepsilon}} + \int_{\boldsymbol{\varepsilon}}^{\boldsymbol{A}} d\,\ln x^2\,\ln\frac{(\boldsymbol{x}+\boldsymbol{\varepsilon})^2}{(\boldsymbol{x}-\boldsymbol{\varepsilon})^2} = 2\pi^2 \qquad \varepsilon \to 0\,, \tag{39}$$

for any smooth curve through $\mathcal{O}$, with $\boldsymbol{A}$ and $\boldsymbol{B}$ on opposite sides of $\mathcal{O}$ (compare Eq. (2.15) of Ref. [2]).

As for the remaining part of Eq. (38), we verify that it is non-singular. For the endpoints, say $\boldsymbol{x} \to A$, it is finite for the same reason given above with regard to Eq. (36). For $x \to 0$, we write $\ln \frac{(\boldsymbol{x}-\boldsymbol{A})^2}{(\boldsymbol{x}-\boldsymbol{B})^2} = \ln \frac{A^2}{B^2} + \ln(1-2\frac{\boldsymbol{Ax}}{A^2}+\frac{x^2}{A^2}) - \ln(1-2\frac{\boldsymbol{Bx}}{B^2}+\frac{x^2}{B^2})$. The constant terms $\int_{\boldsymbol{B}}^{-\varepsilon} + \int_{\varepsilon}^{\boldsymbol{A}} (1/2) \ln \frac{A^2}{B^2} = (1/2) \ln \frac{A^2}{B^2} (\ln \frac{\varepsilon^2}{B^2} + \ln \frac{A^2}{\varepsilon^2}) = (1/2)(\ln \frac{A^2}{B^2})^2$ are nonsingular. For the x-dependent terms we can write $\ln(1-2\frac{\boldsymbol{Ax}}{A^2}+\frac{x^2}{A^2}) \approx -2\frac{\boldsymbol{Ax}}{A^2}$, and similarly for the B term, leading to an integral of the type $\int x \, d \ln x$ which is also non-singular at $x = 0$.

Thus the expression is ε independent and well defined, and we introduce the symbol $\mathcal{P}$ for this principal value-like integral: $\mathcal{P}\int_{\boldsymbol{B}}^{\boldsymbol{A}} = \int_{\boldsymbol{B}}^{-\varepsilon} + \int_{\varepsilon}^{\boldsymbol{A}}$ and can thus finally write for Eq. (38)

$$\int \int_{\boldsymbol{B}}^{\boldsymbol{A}} I = 2\pi^2 + \mathcal{P} \int_{\boldsymbol{B}}^{\boldsymbol{A}} d \ln x^2 \left(-\frac{1}{2} \ln \frac{A^2}{B^2} + \ln \frac{(\boldsymbol{x}-\boldsymbol{A})^2}{(\boldsymbol{x}-\boldsymbol{B})^2} \right) \qquad C \text{ through } \mathcal{O}\,. \tag{40}$$

We wish to use Eq. (36) and Eq. (40) in the next Section to extend the simple results found for straight lines and circles to general curves, but first we check that these formulas give the expected results in these simple cases:

1. For the circle, not through $\mathcal{O}$, we expect zero, according to Eq. (30), which is indeed what results from setting $\boldsymbol{A} = \boldsymbol{B}$ in Eq. (36).

2. For the infinite straight line, not through $\mathcal{O}$, we expect $-2\pi^2$ according to Eq. (33). Using Eq. (36), we consider the straight line at a distance δ from $\mathcal{O}$, as in Eq. (25) and introduce the variable $y = x/\sqrt{AB}$. In the limit $A, B \to \infty$ such that $A/B \to 1$ and $\delta/\sqrt{AB} \to 0$ one obtains the integral $4\int_{-1}^{1} \frac{dy}{y} \ln \frac{1-y}{1+y}$, which is indeed $-2\pi^2$.

3. Turning now to $\mathcal{O}$ on the curve and Eq. (40), for the circle we expect $2\pi^2$, according to Eq. (34). This is what we obtain upon setting $\boldsymbol{A} = \boldsymbol{B}$ in Eq. (40).

4. For the final example, the straight line through $\mathcal{O}$, we expect zero according to Eq. (35). To evaluate the principal value integral, we repeat the arguments just given for the infinite straight line not through $\mathcal{O}$, with the difference that in place of δ, we now have $\varepsilon/\sqrt{AB} \to 0$. This leads to $8\int_0^1 \frac{dy}{y} \ln \frac{1-y}{1+y} = -2\pi^2$ again and Eq. (40) is zero.

We emphasize that the complications of this Section arise from our desire to bring $\int\int I$ into the once-integrated form and the resulting asymmetric treatment of the variables; I itself is perfectly well behaved for a given curve. Perhaps if another method could be found for the problem of general curves as discussed in the next Section these complications could be avoided.

11. Integral of *I* as an invariant

Perhaps the most remarkable property of $\int\int I$ is that it is a type of invariant, having the same value for all curves of a given configuration. In our language, this was what was essentially concluded in Ref. [2] for certain field theoretic amplitudes (for our cases 1–3). We shall show this using Eq. (36) and Eq. (40) of the previous Section. Since the quantity in parenthesis in Eq. (36) and Eq. (40) does not depend on the particular curve, the nature of the actual curve in question appears only in the remaining single integration over x. This feature that makes the once-integrated expressions useful for the examination of general curves.

11.1. Closed curves not through $\mathcal{O}$

For our first case we take the generalization for Section 9.1: inversion of a general, finite closed Euclidean curve, with $\mathcal{O}$ not on the curve. The inversion operation produces another finite closed curve. Therefore we can set $\boldsymbol{A} = \boldsymbol{B}$ and the integrand of Eq. (36) is zero.

$$\int\int I_{\text{closed not through } \mathcal{O}} = 0 \,. \tag{41}$$

This is naturally in agreement with the example of Eq. (30), and in view of Eq. (20) we can finally conclude that for this case n is inversion invariant:

$$n(C) = n(C_{\text{inv}}) \,, \tag{42}$$

where C_{inv} is the inversion of any finite closed Euclidean curve C, with the center of inversion not on the curve C.

11.2. Infinite curves not through $\mathcal{O}$

We now consider the generalization of the infinite straight line. By these we mean curves which although they become the (same) straight line at large distances, have some arbitrary form at finite distances (Fig. 3).

We employ the once-integrated Eq. (36) again, where the limits $\boldsymbol{A}$ and $\boldsymbol{B}$ limits are to be sent to $\pm\infty$. The quantity in parenthesis in Eq. (36) would be the same for any curve and in particular for the straight line. Since

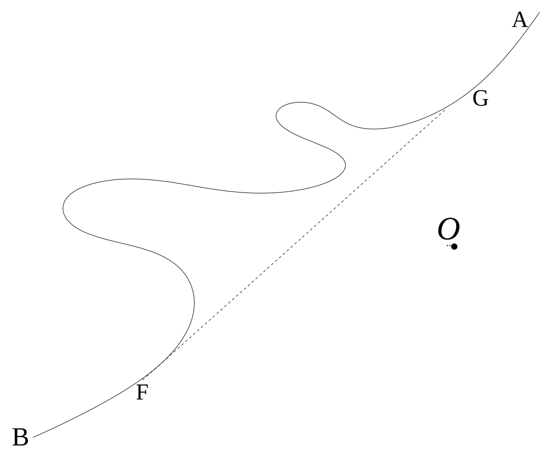

Fig. 3. Arrangement used for the infinite curve, $\mathcal{O}$ off the curve. The sketch is to be understood in an arbitrary number of Euclidean dimensions. The solid line represents the curve under consideration. The contribution from the dashed line is to be added and subtracted so as to produce the integral for the infinite straight line as $\boldsymbol{A}, \boldsymbol{B} \to \infty$ plus a contribution between $\boldsymbol{F}$ and $\boldsymbol{G}$ which contains only finite values of x.

the curve under consideration only differs from the straight line in a finite region, say between points $\boldsymbol{F}$ and $\boldsymbol{G}$, let us add and subtract a straight line contribution from $\boldsymbol{F}$ to $\boldsymbol{G}$ in Eq. (36). This leads to

$$\int_{\boldsymbol{B}}^{\boldsymbol{A}} d\,\ln x^2(\ldots) = \int_{\text{straight line}} d\,\ln x^2(\ldots) + \int_{\text{finite}} d\,\ln x^2(\ldots)\,. \qquad (43)$$

The $(\ldots)$ stands for the parenthesis in Eq. (36). $\int_{\text{straight line}}$ is from $\boldsymbol{B}$ to $\boldsymbol{A}$, and as $\boldsymbol{A}, \boldsymbol{B}$ are sent to $\pm\infty$ it becomes the integral for the infinite straight line, $-2\pi^2$. The second term, $\int_{\text{finite}}$, stands for the integral from $\boldsymbol{F}$ to $\boldsymbol{G}$ along the actual curve in question minus the integral $\boldsymbol{F}$ to $\boldsymbol{G}$ along a straight line.

We now argue that $\int_{\text{finite}}$ goes to zero as $\boldsymbol{A}, \boldsymbol{B} \to \pm\infty$. This is because x in $\int_{\text{finite}}$ is confined to finite values as say $\boldsymbol{A} \to \infty$. Thus we can write $\ln[(\boldsymbol{x} - \boldsymbol{A})^2] = \ln A^2 + \ln\left[1 - 2\frac{\boldsymbol{Ax}}{A^2} + \frac{x^2}{A^2}\right] \approx \ln A^2 - 2\frac{\boldsymbol{Ax}}{A^2}$ to leading order in $1/A$. The $2\frac{\boldsymbol{Ax}}{A^2}$ will lead to a contribution to the integral vanishing as $1/A$. Then $\int_{\text{finite}}$ becomes the difference of the integral of a total derivative over two paths between the same endpoints and so is zero.

(There might appear to be some difficulty with the argument when $\mathcal{O}$ is located such that the straight line from $\boldsymbol{F}$ to $\boldsymbol{G}$ passes through it, *i.e.*

when the straight lines at infinity lie along a ray from $\mathcal{O}$. We can deal with this by replacing the straight line as the reference curve by a curve where a semi-circle avoids $\mathcal{O}$. In this case the semi- circle contribution goes to zero as $\boldsymbol{A} \to \infty$, while the straight line integrals can be evaluated in this limit to again give $-2\pi^2$.)

We thus conclude that the integral $\int\int I$ over any curve, not passing through $\mathcal{O}$, differing from the straight line in a finite region is the same as that for the simple straight line, not passing through $\mathcal{O}$:

$$\int\int I_{\text{infinite not through } \mathcal{O}} = -2\pi^2 \,. \tag{44}$$

11.3. Closed curves through $\mathcal{O}$

Inversion now leads to the generalization of the circle through $\mathcal{O}$, the arbitrary closed curve through $\mathcal{O}$. We use Eq. (40), and setting $\boldsymbol{A} = \boldsymbol{B}$, $\int\int I$ is the same as for the circle:

$$\int\int I_{\text{closed through} \mathcal{O}} = 2\pi^2 \,. \tag{45}$$

And for n we can say

$$n(C) = n(C_{\text{inv}}) + 2\pi^2 \,. \tag{46}$$

Where C is a generalized circle and C_{inv} is its inversion, a generalized infinite straight line.

11.4. Infinite curve through $\mathcal{O}$

We come to our last case, the generalization of the straight line through $\mathcal{O}$. By this we mean an infinite curve with our usual condition that it becomes the same straight line at infinity, but now also passing through $\mathcal{O}$ at some finite point (Fig. 2). This class inverts into itself.

We now use Eq. (40) where we must evaluate $\mathcal{P}\int_{\boldsymbol{B}}^{\boldsymbol{A}} d\,\ln x^2(\ldots)$. In this integral, adding and subtracting a straight line piece between the finite points $\boldsymbol{F}$ and $\boldsymbol{G}$ (as in Fig. 3 but with $\mathcal{O}$ on the solid curve), we have again

$$\mathcal{P}\int_{\boldsymbol{B}}^{\boldsymbol{A}} d\,\ln x^2(\ldots) = \int_{\text{straight line}} d\,\ln x^2(\ldots) + \mathcal{P}\int_{\text{finite}} d\,\ln x^2(\ldots)\,, \tag{47}$$

where we have dropped the $\mathcal{P}$ in $\int_{\text{straight line}}$ since it does not pass through $\mathcal{O}$. Since $\int_{\text{straight line}} = -2\pi^2$ as $\boldsymbol{A}, \boldsymbol{B} \to \infty$, the $2\pi^2$ from Eq. (40) is canceled and we are left with

$$\mathcal{P}\int_{\text{finite}} d\,\ln x^2(\dots) = \left(-\int_{\boldsymbol{F}}^{\boldsymbol{G}} + \mathcal{P}\int_{\boldsymbol{F}}^{\boldsymbol{G}}\right) d\,\ln x^2\left(-\frac{1}{2}\,\ln\frac{A^2}{B^2} + \ln\frac{(\boldsymbol{x}-\boldsymbol{A})^2}{(\boldsymbol{x}-\boldsymbol{B})^2}\right), \tag{48}$$

where the first integral is along the straight line piece and the second integral is along the curve in question.

As before, for large $\boldsymbol{A}, \boldsymbol{B}$ the $(\dots)$ goes to an x independent piece and terms vanishing as $\boldsymbol{A}, \boldsymbol{B} \to \infty$. Since $\int_{\boldsymbol{F}}^{\boldsymbol{G}} d\,\ln x^2 = \mathcal{P}\int_{\boldsymbol{F}}^{\boldsymbol{G}} d\,\ln x^2$ for the two different paths, Eq. (48) goes to zero and we can conclude

$$\int\int I_{\text{infinite through } \mathcal{O}} = 0 \tag{49}$$

for arbitrary curves. This implies finally for n

$$n(C) = n(C_{\text{inv}})\,, \tag{50}$$

where C_{inv} is the inversion of an infinite Euclidean curve C, with the center of inversion on the curve.

11.5. Summary

We can summarize the results of this Section as follows. We have two types of curves: "finite" curves and "infinite" curves. The first are closed curves, the second open curves becoming the same straight line at large distances. There are also two possibilities for the placement of the origin of inversion $\mathcal{O}$: "on" and "off" the curve. Labeling the configurations from 1 to 4 we can exhibit their properties in a table:

Configuration	Curve	$\mathcal{O}$	Result	$\int\int I$
1	finite	off	1	0
2	finite	on	3	$2\pi^2$
3	infinite	off	2	$-2\pi^2$
4	infinite	on	4	0

The column "Result" refers to the configuration resulting from the inversion. Thus: configuration 2 is a closed finite curve with $\mathcal{O}$ on the curve. The integral of I over this curve has the value $2\pi^2$. The inversion produces configuration 3, which is an infinite curve with $\mathcal{O}$ off the curve.

A notable feature of the table is that those configurations, namely 1 and 4, which invert into themselves have $\int\int I = 0$. This is a type of generalization of Eq. (22), and appears to answer the question raised in

Section 6 as to the wider meaning of "≡". Apparently two curves should be considered "equivalent" when they belong to the same configuration in the sense of the table. Depending on the position of $\mathcal{O}$, a curve may map into its own configuration or not. When it does, the "self conjugate" property $\int\int I = 0$ obtains.

12. Infrared/ultraviolet duality

We remarked in [1] that it appeared as if the finiteness of n both at short distances (ultraviolet) and at long distances (infrared) could in a sense be attributed to the same thing, namely that the average velocity $U(x, x')$ and the instantaneous velocity u become equal. In Euclidean space U and u refer to the chord and tangent of the curve respectively. At short distances as $x \to x'$we have $U \to u$ because the curve is taken to be smooth, so for small enough intervals the chord and the tangent become the same. At large distances the curve has by assumption a constant and equal slope at $\pm\infty$, so that $U \approx u$ again.

It is interesting to note how the configurations of the table where $\mathcal{O}$ is "on", that is where "Result" is an infinite curve, represents just this situation. In inversion a point on the curve is projected along the (directed) ray connecting it to the origin. The point goes to large distances if it was close to the origin and goes to small distances if it was far from the origin. If $\mathcal{O}$ is directly on the curve, the points approaching the origin along the curve from one side will be sent to a straight line at $+\infty$, while points approaching $\mathcal{O}$ from the other side will be sent to a straight line at $-\infty$ in the opposite direction. That the *same, straight* line at $\pm\infty$ results is a consequence of the presumed smoothness of the curve at $\mathcal{O}$.

Similarly, a curve coming from large distances and finally going to large distances as the same straight line will be mapped into a curve smooth at the origin. If the infinite slopes had been different there would be a kink at the origin.

Inversion thus makes it clear how our two assumptions needed to make n finite both in the infrared and the ultraviolet are related: under an inversion whose origin is on the curve, curves which are locally smooth become straight and parallel at ∞ and *vice-versa*.

13. Minkowski space

In Minkowski space a new aspect enters in that Eq. (7) can make points that are time-like separated into points that are space-like separated. That is, in Minkowski space, where $S^2 = (x_0 - x_0')^2 - (\boldsymbol{x} - \boldsymbol{x'})^2$ the purely algebraic relation Eq. (11) is still true

$$S^2 \to \frac{a^2}{x^2}\frac{a^2}{x'^2}S^2\,, \tag{51}$$

but for an initially physical time-like curve, with $\mathcal{O}$ "off" there will be points on it space-like to $\mathcal{O}$. This gives x^2 negative in Eq. (51) and a time-like separation $S^2 > 0$ may be mapped into a space-like separation $S^2 < 0$ [6].

On the other hand for a time-like path with $\mathcal{O}$ on the curve another possible physical path results. For $\mathcal{O}$ on the path, x^2 is always positive and S^2 in Eq. (51) cannot change sign and all relatively time-like pairs of points remain time-like. This operation is like configuration 4 of the Euclidean problem where, according to the table, n is inversion invariant. It might be interesting to investigate if the inversion method could be helpful in analyzing certain practical radiation problems.

I am grateful to David Gross for stressing the interest of studying the inversion, for several discussions, and for bringing the references [2] and [3] to my attention. I would also like to thank E. De Rafael for conversations concerning dilogarithms and related problems, as well as E. Seiler for discussions and a reading of the manuscript.

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[4] We assume in our treatment that C becomes identically the straight line beyond some point. Presumably in a more sophisticated analysis this could be replaced by a sufficiently rapid approach to a straight line, to define more precisely the class of curves included in the argument.

[5] Written out explicitly, the integral is a type of Spence function or dilogarithm, as comes up in the study of radiative corrections. See for example the appendix of B.E. Lautrup, E. De Rafael, *Phys. Rev.* **D174**, 1835 (1968), or the book *Dilogarithms* by L. Lewin, Macdonald, London 1958. The integral in question here is no. 371 of the *Handbook of Chemistry and Physics*, 34th Edition. It is intriguing that the construction for the straight line Eq. (25) gives an immediate evaluation of such an integral.

[6] An examination of the resulting paths with the associated crossings of the light cone might suggest an interpretation in terms of pairs of particles.