GEOMETRICAL SCALING DUE TO CRITICAL BEHAVIOR NEAR THE LIGHT CONE*

H.J. Pirner

Institut für Theoretische Physik der Universität Heidelberg
Philosophenweg 19, D-69120 Heidelberg, Germany
e-mail: pir@tphys.uni-heidelberg.de
and
Max-Planck-Institut für Kernphysik, Heidelberg, Germany

(Received November 17, 2003)

At low x a transition from a dilute parton gas to a dense parton liquid takes place. We derive geometrical scaling for the structure function in deep inelastic scattering at low x from a diverging correlation length $\xi(x)$ of Wilson lines near the light cone. QCD (SU(3)) in $2+1$ space–time dimensions near the light cone becomes a critical theory in the limit of $x \to 0$ with a diverging correlation length $\xi(x) \propto x^{-1/2\lambda_2}$ where the exponent $\lambda_2 = 2.52$ is obtained from the center group Z(3) of SU(3).

PACS numbers: 12.38.Lg

High energy electron proton scattering has presented exciting new experimental results in QCD. The behavior of cross sections with energy, a long standing issue in hadronic physics, has gained in interest with the availability of small size probes. The color dipole in the photon can be made small by increasing the virtuality Q of the photon. In the course of x-evolution the photon wave function develops many additional dipoles which in general diffuse into distance scales beyond the original size $1/Q$. This increase in dipole density and/or size of the photon wave function is generally believed to be the origin of the increasing high energy cross section. Perturbative QCD has been partially successful to explain low x physics. In a recent paper [1] we have shown how DGLAP-evolution of a non perturbatively obtained gluon distribution can lead to a successful description of the structure function data at HERA.

* Presented at the XXXIII International Symposium on Multiparticle Dynamics, Kraków, Poland, September 5–11, 2003.

In this note we follow the very promising Wilson line method outlined in Ref. [4] in a Hamiltonian framework near the light cone [3]. Our approach is all the way nonperturbative in contrast to the standard perturbative approach in this field. We do not share the assumption that even at high dipole density when the average transverse distance between dipoles becomes small the hadronic state is accessible to a simple perturbative treatment, since the overall size scale is still large. For total cross sections the momentum transfer is zero and this overall size matters.

We consider it as an advantage that experiment helps us to unravel the badly understood dynamics of partons near the light cone. For years there has been a considerable effort to model and investigate QCD in light cone coordinates. But the current status only gives a moderately encouraging outlook in this field. Therefore we have followed another approach with near light cone coordinates which smoothly interpolate between the Lorentz and light front coordinates :

$$\begin{aligned} x^t = x^+ &= \frac{1}{\sqrt{2}} \left\{ \left(1 + \frac{\eta^2}{2}\right) x^0 + \left(1 - \frac{\eta^2}{2}\right) x^3 \right\}, \\ x^- &= \frac{1}{\sqrt{2}} \left(x^0 - x^3\right) \ . \end{aligned} \tag{1}$$

In this approach the question of quantum constraint equations does not arise, since we treat the negative fermion energy states and transverse electric fields as independent degrees of freedom. The price to pay for this treatment is high. The arbitrary constant η which labels the nearness to the light cone appears in the Hamiltonian. For spectrum calculations it is cumbersome to have such a parameter, since in QCD we have to extrapolate to the continuum limit with the help of the renormalization group which becomes difficult in the presence of the extra parameter η. For the discussion of scattering the parameter η presents an advantage since it allows to follow the evolution of the physics with increasing energy. Consider high energy photon–proton scattering at small $x = Q^2/s$, where $s = W^2$ is the cm energy squared. Using the photon vector $q, q^2 = -Q^2$ and the proton vector $p, p^2 = m^2 \approx 0$ we can define two light-like vectors

$$\begin{aligned} e_1 &= q - \frac{q^2}{2pq} p \\ &= q + xp \,, \qquad (2) \\ e_2 &= p \,. \qquad (3) \end{aligned}$$

For finite energies the vector of the photon q can be calculated as linear combination of the light-like vector e_1 with a small amount of e_2 admixed

$$\begin{aligned} e_\eta &= q + xp - \frac{\eta^2}{2}p \\ &= e_1 - \frac{\eta^2}{2}e_2\,. \end{aligned} \tag{4}$$

One sees that in the limit of infinite energies the mixing η is related to the Bjorken variable x and vanishes as $\eta^2/2 = x$. Therefore it is natural to formulate high energy scattering in near light cone coordinates. For small x the eikonal phases acquired by the quarks/antiquarks are the relevant collective variables. The light cone Hamiltonian on the finite light-like x^- interval of length L has Wilson line or Polyakov operators similarly to QCD formulated on a finite interval in imaginary time at finite temperature

$$\mathcal{U}(x_\perp) = P\exp\left[ig\int_0^L dx^- A_-(x_\perp, x^-)\right], \tag{5}$$

$$P(\vec{x}_\perp) = \frac{1}{N_\mathrm{c}}\mathrm{tr}\,\mathcal{U}\,. \tag{6}$$

The dynamics of these Polyakov operators is determined by the near light cone Hamiltonian H [3]. In the $x \to 0$ limit, those pieces $\mathcal{H}^\eta$ of the Hamiltonian $\mathcal{H}$ dominate which are most singular at $\eta = 0$ and do not couple to the three-dimensional gauge fields $A_\perp$ and ψ_+. These are the terms which are proportional to $1/\eta^2$ and of course also the parts where the conjugate momenta enter. This reduced Hamiltonian contains all terms with the collective variables $a_-^{c_0}$ with the color indices $c_0 = 3, 8$

$$a_-^{c_0} = \frac{1}{L}\int_0^L dx^- A_-^{c_0}(\vec{x}_\perp, x^-) \tag{7}$$

which determine the Wilson lines and live in a $2+1$ dimensional space. We consider the dynamics of the fields a_- in vacuum, *i.e.* without the source term $e_\perp$. The $\eta-$ coordinates correspond to the physics in a fast moving frame. Therefore, we factorize the reduced energy from the Lorentz boost factor $\propto 1/\eta$ and the transverse lattice cut off a

$$\begin{aligned} h_\mathrm{red} &= 2\eta a\int dx^- dx_\perp \mathcal{H}^\eta \\ &= \int dx^- dx_\perp \sum_{c^0=3,8}\left(\frac{2a}{\eta}\mathrm{tr}\left(\frac{1}{L}e_\perp^{c_0} - \nabla_\perp a_-^{c_0}\right)^2 + \frac{2a\eta}{2L^2}p_-^{c_0\dagger}p_-^{c_0}\right. \qquad (8) \\ &\quad \left. - \frac{4a}{\eta}\psi_-^\dagger g a_-^{c_0}\frac{\lambda^{c_0}}{2}\psi_-\right)\,. \qquad (9) \end{aligned}$$

This Hamiltonian is accessible to a lattice treatment in a similar way as the Hamiltonian in SU(2) Ref. [4]. It can be rewritten in terms of zero mode fields φ^{c_0}:

$$\varphi^{c_0}(\vec{b}_\perp) = \frac{1}{2} g L a_-^{c_0}(b_\perp)\,. \tag{10}$$

Large gauge transformations lead from one part of the fundamential domain to another. The effective coupling constant of the Hamiltonian coupling the zero mode fields φ^{c_0} is $g^2_{\rm eff} = g^2 \eta L/4a$.

In Ref. [4] we have done a Finite Size Scaling (FSS) analysis for SU(2) QCD obtaining a second order transition as a function of the coupling $g^2_{\rm eff}$ between a phase with massive excitations at strong coupling and a phase with mass less excitations at weak coupling. In SU(3) the reduced Hamiltonian has rather different symmetry properties than the SU(2) Hamiltonian. We think that the universality class of the reduced Hamiltonian in SU(3) is the three-state Potts model Z(3). In each subregion of the fundamental domain the zero mode variables φ^3, φ^8 are represented by one-spin orientation. The relevant center group Z(3) has a weak first order transition whose critical line ends in a second order point in the presence of an external field. We conjecture that this external field is provided by the fermion zero mode density near the light cone. To match the Hamiltonian lattice with scattering the lattice constant a is chosen to coincide with the photon resolution $\approx 1/Q$. The longitudinal lattice extension L must be larger than the color coherence length of the $q\bar{q}$ state in the photon–proton c.m. system. For details we refer to Ref. [2]. The final conclusion is that near the critical point the Wilson lines experience long range correlations which means that dipoles in the photon wave function are correlated over large distances. The correlation length ξ increases with $x \to 0$ as

$$\xi \propto \left(\frac{x}{x_0}\right)^{-\frac{1}{2\lambda_2}} f_h(0)\,. \tag{11}$$

For finite correlation length there exists an intermediate range in transverse space $1/Q < x_\perp < \xi$ for which the correlation function of Wilson lines is power behaved:

$$\langle P(x_\perp) P(0)\rangle \approx \frac{1}{x_\perp^{1+n}}\,, \tag{12}$$

where $n = 0.04$ in Ising-like systems. This scaling region is responsible for the well-known effect of critical opalescence in the gas liquid transition. For larger distances $x_\perp > \xi$ the correlation function decreases exponentially

$$\langle P(x_\perp) P(0)\rangle \approx \mathrm{e}^{-x_\perp/\xi}\,. \tag{13}$$

We do not follow the small x evolution of the photon dipole state, instead we give a qualitative description of the effective photon size as a function of x using the results of the $2+1$ dimensional critical QCD SU(3) theory as a guiding principle. We parameterize the dipole probability densities for the longitudinal and transverse photons,

$$\begin{aligned}
\rho_\gamma^{\mathrm{T}} &= \frac{6\alpha}{4\pi^2}\sum_{\mathrm{f}}\hat{e}_{\mathrm{f}}^2\varepsilon^2\left[z^2+(1-z)^2\right]F_{\mathrm{T}}(\varepsilon x_\perp)\,,\\
\rho_\gamma^{\mathrm{L}} &= \frac{6\alpha}{4\pi^2}\sum_{\mathrm{f}}\hat{e}_{\mathrm{f}}^2 4Q^2z^2(1-z)^2F_{\mathrm{L}}(\varepsilon x_\perp)\,, \qquad (14)\\
\varepsilon &= \sqrt{Q^2z(1-z)}\,. \qquad (15)
\end{aligned}$$

The perturbative scale for the dipole density is given by $1/\varepsilon$. We modify the photon wave function depending on the relation of the correlation length ξ of the Wilson loops to the perturbative scale $1/\varepsilon$. We set

$$\xi = \frac{1}{\varepsilon}\left(\frac{x}{x_0}\right)^{-\frac{1}{2\lambda_2}}\,. \qquad (16)$$

For the reference Bjorken parameter $x_0 = 10^{-2}$ the correlation length is fixed at the perturbative scale. The critical exponent $1/(2\lambda_2) = 0.2$ determines the Wilson line correlations for $x < x_0$. If the transverse size of the dipole is smaller than the perturbative length scale $x_\perp < 1/\varepsilon$ we use the perturbative dipole densities $F_{\mathrm{T}}(\varepsilon x_\perp) = K_1(\varepsilon x_\perp)^2$ and $F_{\mathrm{L}}(\varepsilon x_\perp) = K_0(\varepsilon x_\perp)^2$. For $1/\varepsilon < x_\perp < \xi$ we modify the perturbative dipole densities using the correlation functions of the critical theory, Eqs. (12),(13),

$$\begin{aligned}
F_{\mathrm{T/L}}(\varepsilon x_\perp) &= K_{1/0}(\varepsilon x_\perp)^2 && \text{for } x_\perp < \frac{1}{\varepsilon},\\
&= K_{1/0}(1)^2\left(\frac{1}{\varepsilon x_\perp}\right)^{2+2n} && \text{for } \frac{1}{\varepsilon} < x_\perp < \xi\,,\\
&= K_{1/0}(x_\perp/\xi)^2\left(\frac{1}{\xi\varepsilon}\right)^{2+2n} && \text{for } x_\perp > \xi\,. \qquad (17)
\end{aligned}$$

The current discussion about geometrical scaling evolves around the concept of saturation which has been carried over from the traditional unitarity behaviour of profile functions in impact parameter space. Recall that the integral over the profile function gives the total cross section. When the profile function reaches the unitarity limit, the target/projectile becomes black, a further increase of the cross section can only be reached by an increase in transverse size of the profile function. Now the concept of saturation is also

applied to a dipole nucleon cross section which becomes flat with increasing dipole size. The Golec-Biernat cross section becomes flat at $r_\perp \approx 1.0$ fm for $x = 10^{-2}$. Since we consider all the evolution to happen in the photon, this dipole nucleon cross section is kept fixed as a function of energy, *i.e.* as a function of x. In our picture the increasing cross section comes from an increasing size of the largest dipole state in the photon which is determined by the Wilson line correlation function as given in the previous chapter. The fact that this effective density obeys approximate scaling behaviour in the region $r_\perp < \xi$ is in our opinion the fundamental reason for geometric scaling. The photon–proton cross section can only depend on the ratio $R_0/\xi(x)$. We see this as follows: The effective dipole density combined with an energy independent dipole–proton cross section determines the structure function F_2 and the photon–proton cross section.

$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2\alpha}\left(\sigma_{\gamma p}^{\mathrm{T,tot}} + \sigma_{\gamma p}^{\mathrm{L,tot}}\right), \tag{18}$$

$$\sigma_{\gamma p}^{\mathrm{T/L,tot}} = \int d^2x_\perp \int_0^1 dz \rho_\gamma^{\mathrm{T/L}}(x_\perp, z)\sigma(x_\perp). \tag{19}$$

The Golec–Biernat–Wuesthoff [5] dipole–proton cross section is approximately equal to a simple quadratic function at small distances $r < 2R_0$ and a constant function for $r > 2R_0$,

$$\sigma(r) \approx \sigma_0\left(\frac{r^2}{4R_0^2}\Theta(2R_0 - r) + \Theta(r - 2R_0)\right). \tag{20}$$

The numerical values $R_0 = 0.33$ fm at $x_0 = 10^{-2}$ is independent of x. One can demonstrate geometrical scaling of the photon–proton cross section rather simply. We neglect the exponentially suppressed part of the dipole density in the integral over large transverse distances and set the anomalous dimension $n \to 0$. Then one gets the dominant transverse cross section in F_2 by integrating up to the correlation length ξ, using the fact that $K_1(r\varepsilon) = 1/(r\varepsilon)$ for $r\varepsilon < 1$,

$$\sigma_{\mathrm{T}} = \frac{3\alpha}{\pi}\sum e_{\mathrm{f}}^2 Q^2\sigma_0 \int_0^1 dz\left(z^2 + (1-z)^2\right)\int_0^\infty r dr \frac{z(1-z)}{r^2\varepsilon^2}\Theta\left(1 - \frac{r^2}{\xi^2}\right)\frac{\sigma(r)}{\sigma_0}. \tag{21}$$

Redefining the integration variable as $r' = r(x/x_0)^{1/(2\lambda_2)}$ one obtains that the γ^*–p cross section obeys geometrical scaling and depends only on

$R_0/\xi(x) = R(x)Q$ where

$$R(x) \;=\; R_0 \left(\frac{x}{x_0}\right)^{\frac{1}{2\lambda_2}} , \tag{22}$$

$$\frac{1}{2\lambda_2} \;=\; 0.2\,. \tag{23}$$

Therefore, the agreement with the data *cf.* Ref. [2] can be understood analytically. The critical theory gives the phenomenologically obtained power dependence of R with x. The favored x-dependence of GBW is in the range 0.145 and 0.20. The critical behavior gives the power $1/(2\lambda_2) = 0.2$. Without a model for the proton source, it is not possible to obtain the absolute length R_0. The structure function scales as a function of $Q^2 R_0^2 (x/x_0)^{1/\lambda_2}$ as can be easily derived for the simplified dipole cross section.

This work has been partially funded through the European TMR Contract HPRN-CT-2000-00130 "Electron Scattering off Confined Partons" and supported by INTAS contract "Nonperturbative QCD".

REFERENCES

[1] H.J. Pirner, A.I. Shoshi, G. Soyez, `hep-ph/0309155`.

[2] H.J. Pirner, F. Yuan, *Phys. Rev.* **D66**, 034020 (2002).

[3] R. Naus, H.J. Pirner, T.J. Fields, J.P. Vary, *Phys. Rev.* **D56**, 8062 (1997).

[4] E. M. Ilgenfritz, Y.P. Ivanov, H.J. Pirner, *Phys. Rev.* **D62**, 054006 (2000).

[5] K. Golec-Biernat, M. Wuesthoff, *Phys. Rev.* **D59**, 014017 (1999); **D60**, 114023 (1999).