NESTED MULTI-SOLITON SOLUTIONS WITH ARBITRARY HOPF INDEX

A. Wereszczyński

Marian Smoluchowski Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland
wereszcz@alphas.if.uj.edu.pl

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Generalized Aratyn–Ferreira–Zimerman O(3) nonlinear sigma model with a particular symmetry breaking term, so-called dielectric function, is discussed. Static multi-soliton configurations with finite energy and nontrivial Hopf index are found. We show that such configurations consist of nested toroidal solitons. Moreover, nontrivial sphaleron-like solutions *i.e.* configurations with zero total topological charge are also presented.

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1. Introduction

It is widely believed that toroidal knotted topological solitons so-called hopfions play a very important role in the temporary physics. In fact, they seem to give a natural language of describing particle excitations in the low energy sector of the quantum gluodynamics *i.e.* famous glueball states. This idea, proposed and developed by Faddeev, Niemi [1, 2] and Cho [3], provides an elegant framework where masses as well as other physical properties can be understood in terms of topological characteristics of solutions[1]. In particular, the well-known Vakulenko–Kapitansky inequality [4] gives the following spectrum of the glueballs $E \sim 1500$ MeV$|Q_{\mathrm{H}}|^{\frac{3}{4}}$, where $Q_{\mathrm{H}} = 1, 2...$ is Hopf index. Moreover, it has been also observed that the spectrum should possess right–left degeneracy [5]. It follows from the observation that energy of the topological solutions does not depend on the sign of the topological charge.

Unfortunately, in case of the Faddeev–Niemi model only numerical solutions have been found [6–8]. However, there exist models which allow

[1] For other applications of knotted solitons see [17].

us to learn something more about the mathematical structure and behavior of toroidal solutions [9]. They can be regarded as a toy models where we can test some ideas borrowed from the standard soliton theory in two space dimensions. In fact, the Aratyn–Ferreira–Zimerman [10] model is a widely discussed example of a theory where exact soliton solutions with arbitrary Hopf number have been obtained. Moreover, some generalizations to N-interacting Aratyn–Ferreira–Zimerman-like models [11] or including a symmetry breaking term have been discussed and analytically solutions obtained [12]. In particular, the problem of the existence of hopfions in models with broken O(3) global symmetry seems to play the important role in the context of gluodynamics. It follows from the fact that the Faddeev–Niemi model admits massless excitations being an effect of the spontaneous O(3) symmetry breaking. This pathological behavior can be cured by adding some explicitly symmetry breaking terms in the action, see for example [13–16]. In general such modified Faddeev–Niemi Lagrangian has the form

$$\mathcal{L} = -\frac{\sigma_1(\vec{n})}{4}\Big[\vec{n}\cdot(\partial_\mu \vec{n}\times\partial_\nu \vec{n})\Big]^2 + \frac{\sigma_2(\vec{n})}{2}(\partial_\mu \vec{n})^2\,, \tag{1}$$

where two so-called dielectric functions σ_1 and σ_2 have been introduced. Moreover, one can also include a potential term for $\vec{n}$. Obviously this modification makes the original model even more complicated and, so far, no analytical calculations have been presented.

In the present work we would like to continue the investigation of hopfions in models with broken global O(3) symmetry. In order to do it, we will take advantage of the Aratyn–Ferreira–Zimerman model [10] with a particular symmetry breaking dielectric function [12].

The main aim of our work is to analyze multi-soliton configurations in this toy model. We are especially interested in construction of sphaleron-like solutions *i.e.* configurations with zero total topological charge but non-trivial local topological structure. Such solutions might be helpful in finding of a time depending topological soliton *i.e.* breather, which in general consists of one-soliton and one antisoliton component. Additionally, it would give us also a chance to investigate the decay and scattering of hopfions.

Due to the fact that the symmetry breaking is realized in the same manner as in the QCD motivated model (1) our work might be regarded us the first step in analytical investigation of the scattering and decay of glueballs as well as in finding of the breather which can change the spectrum of the glueballs.

2. Multi-soliton configuration

In this paper we will look for toroidal topologically nontrivial configurations in $(3+1)$ Minkowski space–time for the following Lagrangian density [12,18]

$$\mathcal{L} = \sigma(\vec{n}) \left[[\vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n})]^2 \right]^{\frac{3}{4}}, \tag{2}$$

where the symmetry breaking dielectric function σ, so-called dielectric function, is chosen in a very special form which provides analytical solutions

$$\sigma(\vec{n}) = \frac{1}{\left(1 - (n^3)^2\right)^{\frac{3}{4}}}. \tag{3}$$

$\vec{n} = (n^1, n^2, n^3)$ is a unit three-component vector field. As it was shown in [12] such a model belongs to a wide family of integrable theories. Here, integrability is understood in the sense that infinitely many conserved currents exist [19,20].

In order to find static soliton solutions we take advantage of the stereographic projection

$$\vec{n} = \frac{1}{1+|u|^2} \left(u + u^*, -i(u - u^*), |u|^2 - 1\right) \tag{4}$$

and introduce toroidal coordinates

$$\begin{aligned} x &= \frac{a}{q} \sinh\eta \cos\phi\,, \\ y &= \frac{a}{q} \sinh\eta \sin\phi\,, \\ z &= \frac{a}{q} \sin\xi\,, \end{aligned} \tag{5}$$

where $q = \cosh\eta - \cos\xi$ and $a > 0$ is a constant of dimension of length fixing the scale in the coordinates. Moreover, we use Aratyn–Ferreira–Zimerman Ansatz [10]

$$u(\eta, \xi, \phi) \equiv f(\eta) e^{i(m\xi + n\phi)}, \tag{6}$$

where m, n are integers.

Then the static equation of motion reads as follows [12]

$$\partial_\eta \ln \frac{\sigma^{2/3} f f'}{(1+f^2)^2} = -\frac{2m^2 \sinh^2\eta - n^2}{m^2 \sinh^2\eta + n^2} \frac{\cosh\eta}{\sinh\eta}. \tag{7}$$

It can be integrated and we find

$$\frac{\sigma^{2/3} f f'}{(1+f^2)^2} = \frac{k_1}{|m|^3} \frac{\sinh\eta}{\left(\frac{n^2-m^2}{m^2} + \cosh^2\eta\right)^{3/2}}\,, \tag{8}$$

where k_1 is a constant. In case of the previously introduced dielectric function this equation can be rewritten as

$$\int \frac{1}{(1+f^2)} df = \frac{-k_1}{|m|(m^2-n^2)} \frac{\cosh\eta}{\left(\frac{n^2-m^2}{m^2} + \cosh^2\eta\right)^{1/2}} - \frac{k_2}{2} \tag{9}$$

and integrated. We obtain the general solution

$$\arctan f = \frac{-k_1}{|m|(m^2-n^2)} \frac{\cosh\eta}{\left(\frac{n^2-m^2}{m^2} + \cosh^2\eta\right)^{1/2}} - \frac{k_2}{2}, \tag{10}$$

where k_2 is a second integration constant. To fix the value of the integration constants one has to specify the asymptotic conditions. They can be chosen as

$$\vec{n} \to (0,0,-1) \;\; i.e. \quad f \to 0 \;\text{ as }\; \eta \to 0 \tag{11}$$

and

$$\vec{n} \to (0,0,1) \;\; i.e. \quad f \to \infty \;\text{ as }\; \eta \to \infty\,. \tag{12}$$

Then, after some algebra one gets

$$\arctan f = \frac{\pi}{2} \frac{(2l+1)}{|m|-|n|} \left(|m| - |n| \frac{\cosh\eta}{\left(\frac{n^2}{m^2} + \sinh^2\eta\right)^{1/2}} \right). \tag{13}$$

In other words, we have obtained a whole family of solutions of the Eq. (9) which fulfill the assumed asymptotic conditions. This family is labelled by the positive and integer parameter $l = 0, 1, 2\ldots$. Thus, our solutions are given by the formula

$$f = \tan\left[\frac{\pi}{2} \frac{(2l+1)}{|m|-|n|} \left(|m| - |n| \frac{\cosh\eta}{\left(\frac{n^2}{m^2} + \sinh^2\eta\right)^{1/2}} \right) \right]. \tag{14}$$

This corresponds with the following formula for the n^3 component of the unit field

$$n^3 = 1 - \frac{2}{1 + \tan^2\left[\frac{\pi}{2} \frac{(2l+1)}{|m|-|n|} \left(|m| - |n| \frac{\cosh\eta}{\left(\frac{n^2}{m^2} + \sinh^2\eta\right)^{1/2}} \right) \right]}\,. \tag{15}$$

One can see that n^3 starts in -1 and tends to $+1$. Additionally, it flips $2l+1$ times between -1 and $+1$. The points, where $n^3=-1$ define the positions of the solitons. More precisely, the solution describes a soliton if n^3 increases from -1 to $+1$. Analogously, antisoliton appears when n^3 decreases from $+1$ to -1. Thus, there are $2l+1$ nested toroidal solitons.

Let us now find the energy corresponding to the solutions. One obtains

$$E \equiv \int d^3x T_{00} = (2\pi)^2 8\cdot 2^{3/4} \int\limits_0^\infty \frac{d\eta \sinh\eta}{(1+f^2)^3}\left(m^2+\frac{n^2}{\sinh^2\eta}\right)^{\frac{3}{4}} f^{\frac{3}{2}} f'^{\frac{3}{2}}\sigma(f)\,. \tag{16}$$

Inserting our solutions into (16) we find that the energy is finite and given by the expression

$$E^l_{m,n} = (2\pi)^2 4\cdot 2^{1/4}(2l+1)^{3/2}\sqrt{|m||n|(|m|+|n|)}. \tag{17}$$

The behavior of n^3 and distribution of the energy density in case of $m=2$, $n=1$ is shown in Figs. 1 and 2.

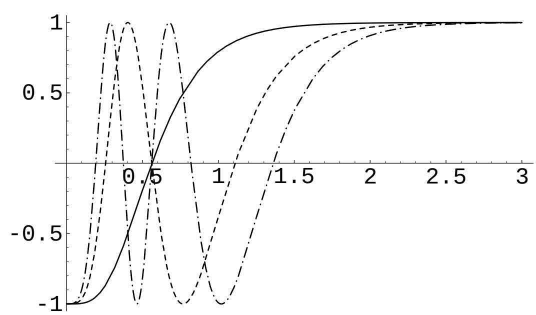

Fig. 1. $n^3(\eta)$ for $l=0,1,2$ — solid, dashed, dot-dashed lines, respectively.

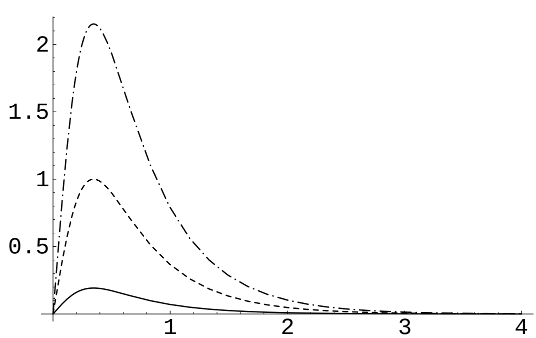

Fig. 2. Energy density for $l=0,1,2$ — solid, dashed, dot-dashed lines, respectively.

It is straightforward to see that also the following asymptotic conditions can lead to multi-soliton configurations

$$\vec{n} \rightarrow (0,0,-1) \;\; i.e. \quad f \rightarrow 0 \;\; \text{as} \;\; \eta \rightarrow 0 \tag{18}$$

and

$$\vec{n} \rightarrow (0,0,-1) \;\; i.e. \quad f \rightarrow 0 \;\; \text{as} \;\; \eta \rightarrow \infty. \tag{19}$$

Here, in contradiction to the case discussed above, the value of n^3 in the center of the torus and in the spatial infinity is identical. One can check that solutions form a family also labelled by positive integer number $k = 0, 1, 2...$

$$n^3 = 1 - \frac{2}{1 + \tan^2 \left[\frac{\pi}{2} \frac{2k}{|m|-|n|} \left(|m| - |n| \frac{\cosh \eta}{\left(\frac{n^2}{m^2} + \sinh^2 \eta \right)^{1/2}} \right) \right]}. \tag{20}$$

Now, n^3 flips $2k$ times between $+1$. It means that there are even numbers of the nested toroidal solitons and, as we prove it below, the total topological charge vanishes. Such solutions possess the following total energy

$$E_{m,n} = (2\pi)^2 4 \cdot 2^{1/4} (2k)^{3/2} \sqrt{|m||n|(|m| + |n|)}. \tag{21}$$

In Figs. 3 and 4 the energy density for $m = 2, n = 1$ and n^3 are shown.

Let us now calculate the Hopf index of the obtained solutions. It can be done using the method presented in [10]. We introduce new functions

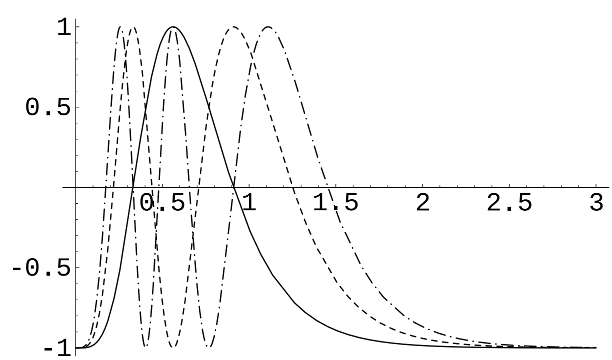

Fig. 3. $n^3(\eta)$ for $k = 0, 1, 2$ — solid, dashed, dot-dashed lines, respectively.

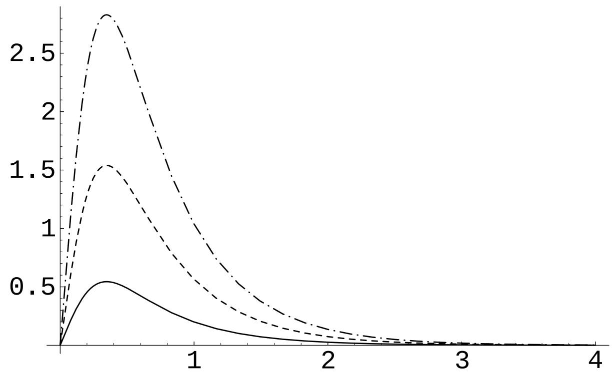

Fig. 4. Energy density for $k = 0, 1, 2$ — solid, dashed, dot-dashed lines, respectively.

$$\Phi_{\binom{1}{2}} = \left(\frac{f}{\sqrt{f^2+1}} \right) \times \binom{\cos m\xi}{\sin m\xi} \tag{22}$$

and

$$\Phi_{\binom{3}{4}} = \left(\frac{1}{\sqrt{f^2+1}} \right) \times \binom{\cos n\phi}{-\sin n\phi}, \tag{23}$$

which are connected with the unit vector field by the relation $n_i = Z^\dagger \sigma_i Z$, where $\vec{\sigma}$ are well-known Pauli matrices. Here

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \quad Z^\dagger = (Z_1^*, Z_2^*) \tag{24}$$

and

$$Z_1 = \Phi_1 + i\Phi_2, \quad Z_2 = \Phi_3 + i\Phi_4. \tag{25}$$

Because of the fact that Hopf index is defined by means of the Abelian vector field and its derivatives

$$Q_{\mathrm{H}} = \frac{1}{4\pi^2} \int d^3x \vec{A} \cdot \vec{B}, \tag{26}$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$, we have to find $\vec{A}$ as a function of the primary unit field. It can be done and we get

$$A_i = \frac{i}{2} \left(Z^\dagger \partial_i Z - \partial_i Z^\dagger Z \right). \tag{27}$$

Then, Hopf index can be evaluated and reads

$$Q_{\mathrm{H}} = \frac{nm}{2} \sum_{i=0}^{l} \left[(\Phi_1^2 + \Phi_2^2)^2 - (\Phi_3^2 + \Phi_4^2)^2 \right]_{\eta_i}^{\eta_{i+1}} , \tag{28}$$

where, for odd i η_i is i-th singular point of the function f (14) whereas for even value of i η_i is i-th zero of the function f. Of course, one can introduce Hopf index for all soliton components of the solution. Namely,

$$Q_{\mathrm{H}}^i = \frac{nm}{2} \left[(\Phi_1^2 + \Phi_2^2)^2 - (\Phi_3^2 + \Phi_4^2)^2 \right]_{\eta_i}^{\eta i+1} = (-1)^i mn \, , \tag{29}$$

where

$$\eta_{2i+1} = \mathrm{ar\ sinh} \left[\frac{1 + \left(\frac{|n|}{|m|} - 1 \right) \frac{2i+1}{2l+1}}{\sqrt{1 - \left(\left(1 - \frac{|m|}{|n|} \right) \frac{2i+1}{2l+1} + \frac{|m|}{|n|} \right)^2}} \right],$$

and

$$\eta_{2i} = \mathrm{ar\ sinh} \left[\frac{1 + \left(\frac{|n|}{|m|} - 1 \right) \frac{2i}{2l+1}}{\sqrt{1 - \left(\left(1 - \frac{|m|}{|n|} \right) \frac{2i}{2l+1} + \frac{|m|}{|n|} \right)^2}} \right],$$

with $i = 0, 1...l$.

Finally we obtain

$$Q_{\mathrm{H}} = -mn \, . \tag{30}$$

This result shows that the obtained solution (14) indeed consists of odd number of the toroidal solutions with nontrivial topological charge. Each of the solitons corresponds to the same absolute value of Hopf index, whereas the sign oscillates. The total topological charge is constant and does not depend on the number of oscillations.

Analogously, solution (20) is made of even numbers of toroidal solitons with zero total topological charge.

3. Conclusions

In the present paper, a Lorentz invariant model based on the unit, three-component vector field has been investigated. This model consists of two parts multiplying each other. Namely, the first part, symmetric under the global O(3) rotations and the second which breaks this symmetry. The violating function (dielectric function) has been chosen in the special form (3).

It has been proved that such defined model possesses not only standard toroidal solutions with arbitrary topological charge known from recent work but also multi-soliton configurations. Exact solutions, their energies and values of the Hopf index have been obtained. In general, the solutions can be divided into two classes with nonzero or zero total topological charge. The solutions with nontrivial total Hopf index consist of odd numbers nested toroidal solitons with a partial charge $\pm Q$ whereas configurations with zero total Hopf index are build of even numbers of such nested solitons. It has to be stressed that only the most nested soliton *i.e.* located at $\eta = 0$ is a line-like object. Remaining hopfions are a little bit pathological. They are two dimensional toruses with the topological charge homogeneously spread on their surface.

Of course, because of the fact that all multi-soliton configurations, with constant total Hopf index, have larger energy than the standard one-soliton solution, we can expect that they are unstable. Due to the Vakulenko–Kapitansky inequality one can immediately see that all single solitons would attract each other. Our knotted multi-soliton solution unties leading to the stable one-soliton state. Nonetheless, the multi-soliton solutions obtained (in particular the soliton–antisoliton state) may give a chance to find a breather-like state *i.e.* oscillating soliton with vanishing total topological charge.

It should be noticed that, as it was proved in [12], investigated model is very unusual. It follows from the observation that all dielectric functions σ give the same spectrum of the solitons *i.e.* their masses and total topological charges are identical and do not depend on the form of σ. The dielectric function inflects only the shape of the hopfion. It is true also in the case of here analyzed model, but only in the one-soliton sector that is $k = l = 0$. Nested hopfions differ a lot from standard Aratyn–Ferreira–Zimerman solutions. In addition to different shapes they possess different total energy. Moreover, since they consist of many solitons and antisolitons, the topological contents of obtained configurations is also dissimilar.

However, because of the previously mentioned fact that one-soliton sector is identical for all σ [12], one can suppose that such equivalence can be valid for multi-soliton sector as well. If this conjecture were true then multi-soliton solutions (and in particular sphalerons) should be observed in case of more realistic dielectric functions [16].

To conclude, the existence of the multi-hopfions (sphaleron-like) solutions, at least in the toy model, is very promising. In particular, the problem of the breathers seems to be important in the context of the Faddeev–Niemi model of glueballs where such time-depending breathers could probably influence the spectrum of the solutions *i.e.* expected spectrum of the glueball states. We would like to address this problem in the forthcoming paper.

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