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DENSITY MATRIX CONSTRAINTS ON SPIN OBSERVABLES IN $\overline{p}p \to \overline{\Lambda}\Lambda$

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We write down the spin density matrix of the reaction $\overline{p}p \to \overline{\Lambda}\Lambda$ in the usual matrix form, its elements are simply given as combinations of the spin observables, which have been measured at CERN with a polarized proton target. Then, we show that the standard properties of any density matrix applied to the matrix obtained allow to carry out a number of interesting, model independent and non-trivial inequalities on spin observables.

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1. Introduction

The strangeness exchange reaction $\overline{p}p \to \overline{\Lambda}\Lambda$ at low energies has been studied by the PS185 Collaboration at CERN and the experimental data on spin observables with a transverse polarized proton target have been published recently [1, 2]. These provide more important information on the mechanism of the strangeness production. As it was recalled in Refs. [3–5], the spin observables measured are not completely independent and must satisfy a number of *inequalities*. A great number of these inequalities have been carried out by Elchikh and Richard [4] in an empirical approach, in which real and imaginary parts of the amplitudes were taken in a randomly way; thus the spin observables were generated in a randomly way too by the use of their explicit expression in terms of the amplitudes. Therefore, pairs or triplets of spin observables fulfilling certain inequalities are chosen to be checked by explicit algebraic calculus, for details see the Ref. [4]. But if one starts from the spin density matrix of the reaction, one can get directly a great number of the same inequalities just as a consequence of the properties applied to the spin density matrix elements as it was suggested in Ref. [5]. In this paper, we will briefly recall the interesting properties of any density matrix, the formalism of 1/2 spin-particles scattering density matrix and finally write down explicitly the spin density matrix of the final state in a simple manner in terms of the spin observables. Then, we can extract inequalities involving pairs or triplets of the spin observables by the use of particular relations of the density matrix properties.

2. Density matrix properties

Any density matrix will be referred to by the symbol ρ. It is well known from the quantum mechanics that any diagonal element of density matrix is positive *i.e.* $\rho_{ii} \geq 0$, the other more interesting relation which will be useful for us is: $\rho_{ii}\rho_{jj} \geq \mid \rho_{ij} \mid^2$. We restrict ourselves to these two types of relations which are special cases of the more general positivity conditions as it was discussed in Ref. [6].

2.1. The density matrix of the initial spin state $\overline{p}p$

The density matrix for a polarized set of 1/2 spin-particles along a polarization vector $\overrightarrow{P}$ is given by

$$\rho_p = \tfrac{1}{2}\left(I + \overrightarrow{P} \cdot \overrightarrow{\sigma}\right), \tag{1a}$$

where $\overrightarrow{\sigma}$ being the vector formed by the Pauli matrices: σ_1, σ_2 and σ_3. The vector polarization $\overrightarrow{P}$ is a transverse to the antiproton beam direction unit-vector $\widehat{z}$, it belongs to the plan which contains the orthogonal unit-vectors $\widehat{n}$ and $\widehat{x}$. Then, it can be written as:

$$\overrightarrow{P} = \sin\phi\ \widehat{x} + \cos\phi\ \widehat{n}\,, \tag{1b}$$

where we used the fact that the set of proton targets is completely polarized so $|\overrightarrow{P}| = 1$. The antiproton beam $\overline{p}$ is not polarized then its spin matrix is simply:

$$\rho_{\overline{p}} = \tfrac{1}{2} I\,. \tag{2}$$

The spin density matrix for $\overline{p}p$ initial state, defined as $\rho_{\overline{p}} \otimes \rho_p$, is then given by

$$\begin{aligned} \rho_{\overline{p}p} &= \tfrac{1}{2} I \otimes \tfrac{1}{2}(I + \overrightarrow{P} \cdot \overrightarrow{\sigma}) && \text{(3a)} \\ &= \tfrac{1}{4}[I \otimes I + \sin\phi\ (I \otimes \widehat{x} \cdot \overrightarrow{\sigma}) + \cos\phi\ (I \otimes \widehat{n} \cdot \overrightarrow{\sigma})]. && \text{(3b)} \end{aligned}$$

The spins, *i.e.* the Pauli matrices, are projected in the frame of each particle with the prescription that the $\widehat{x}$ and $\widehat{z}$ axis of the proton are opposed to the antiproton's which have been chosen in the positive direction, the $\widehat{n}$ axis being the same in each frame. It follows that we have these abbreviations for p (and also for Λ) spin projections:

$$\sigma_x \equiv \widehat{x}_p \cdot \overrightarrow{\sigma} \ = -\widehat{x} \cdot \overrightarrow{\sigma} = -\sigma_1 \,, \tag{4a}$$

$$\sigma_n \equiv \widehat{n}_p \cdot \overrightarrow{\sigma} \ = +\widehat{n} \cdot \overrightarrow{\sigma} = +\sigma_2 \,, \tag{4b}$$

$$\sigma_z \equiv \widehat{z}_p \cdot \overrightarrow{\sigma} \ = -\widehat{z} \cdot \overrightarrow{\sigma} = -\sigma_3 \,. \tag{4c}$$

The spin density matrix for the initial $\overline{p}p$ state given by Eq. (3b) is rewritten in a usual matrix form:

$$\rho_{\overline{p}p} = \frac{1}{4} \begin{pmatrix} 1 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 1 & 0 & 0 \\ 0 & 0 & 1 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 1 \end{pmatrix} \tag{5a}$$

which can be given in a condensed manner as:

$$\rho_{\overline{p}p} = \frac{1}{4} \sum_{i=0,x,n} P_i (I \otimes \sigma_i) \,, \tag{5b}$$

where $P_0 \equiv 1$, $\sigma_0 \equiv I$ (the identity matrix). We recall that the σ_i in Eq. (5b) are the proton's spin projections.

2.2. *The density matrix of the final spin state* $\overline{\Lambda}\Lambda$

Let M be the transition matrix (amplitude) of the reaction $\overline{p}p \to \overline{\Lambda}\Lambda$ which can be given in an independent model parametrization, see for instance Ref. [4]. Then, the density matrix of the final state $\overline{\Lambda}\Lambda$ is:

$$\rho_{\overline{\Lambda}\Lambda} = M \rho_{\overline{p}p} M^\dagger = \frac{1}{4} \sum_{i=0,x,n} P_i \ M \ (I \otimes \sigma_i) \ M^\dagger \tag{6a}$$

which can also be decomposed in terms of the crossed projected Pauli matrices of $\overline{\Lambda}$ (the first one) and Λ (the second one) respectively as:

$$\rho_{\overline{\Lambda}\Lambda} = \frac{1}{4} I_0 \sum_{j,k=0,x,n,z} \left[\sum_{i=0,x,n} P_i \ O_{ijk} \ (\sigma_j \otimes \sigma_k) \right] \tag{6b}$$

with these definitions:

- $I_0 \equiv (1/4)\,\mathrm{Tr}(M M^\dagger)$ which is nothing but the differential cross section and also correspond to an unpolarized initial state.

- $O_{ijk} \equiv \mathrm{Tr}[M\ (I \otimes \sigma_i)\ M^\dagger\ (\sigma_j \otimes \sigma_k]/\,\mathrm{Tr}(MM^\dagger)$ which are the spin observables.

And we get an expression of the final density matrix as for the initial one given by Eq. (3b) (without the global factor $I_0/4$):

$$\rho_{\overline{\Lambda}\Lambda} = \sum_{j,k} O_{0jk}\ (\sigma_j \otimes \sigma_k) + \sin\phi \sum_{j,k} O_{xjk}\ (\sigma_j \otimes \sigma_k) + \cos\phi \sum_{j,k} O_{njk}\ (\sigma_j \otimes \sigma_k). \tag{6c}$$

2.3. The symmetries

The strong interaction is the dominating mechanism underlying the reaction $\overline{p}p \rightarrow \overline{\Lambda}\Lambda$. It conserves many discrete symmetries as parity and charge conjugation, see Appendix A. There is the other geometric symmetry by which a rotation of the scattering plane around the $\widehat{n}$ axis lets the matrix transition be invariant, it follows the so-called Bohr-identity $M = \sigma_n \otimes \sigma_n\ M\ \sigma_n \otimes \sigma_n$. We benefit from these symmetries reducing the great number of the O_{ijk} observables to a simple set of 21 observables by imposing many to be identically null or simply related to another observable. Here, we use these familiar notations: P (polarization), A (asymmetry), C_{jk} (correlation), D_{jk} (spin depolarization) and K_{jk} (spin transfer) instead of the O_{ijk} for one or two indices observables, we conserve the symbol O_{ijk} when the three indices subsist, see Appendix A for more details. Then, the O_{ijk} spin observables are filled in three symbolic matrices:

- C_0 which contains five O_{0jk} observables corresponding to an unpolarized proton target (0 for the first index).
- C_x which contains eight O_{xjk} observables corresponding to a polarized proton target in the $\widehat{x}$ direction (x for the first index).
- C_n which contains eight O_{njk} observables corresponding to a polarized proton target in the $\widehat{n}$ direction (n for the first index).

The above three matrices are written in the usual matrix form:

$$\begin{aligned} C_0 &\equiv \sum_{j,k} O_{0jk}\ (\sigma_j \otimes \sigma_k) \\ &= \begin{pmatrix} 1 - C_{zz} & -C_{xz} - iP_n & -C_{xz} - iP_n & -C_{nn} - C_{xx} \\ -C_{xz} + iP_n & 1 + C_{zz} & C_{nn} - C_{xx} & C_{xz} - iP_n \\ -C_{xz} + iP_n & C_{nn} - C_{xx} & 1 + C_{zz} & C_{xz} - iP_n \\ -C_{nn} - C_{xx} & C_{xz} + iP_n & C_{xz} + iP_n & 1 - C_{zz} \end{pmatrix}, \end{aligned} \tag{7a}$$

$$C_x \equiv \sum_{j,k} O_{xjk}(\sigma_j \otimes \sigma_k)$$
$$= \begin{pmatrix} -D_{xz}+K_{xz} & -iO_{xzn}-D_{xx} & K_{xx}+iO_{xnz} & i(O_{xnx}-O_{xxn}) \\ -D_{xx}+iO_{xzn} & D_{xz}+K_{xz} & i(O_{xnx}+O_{xxn}) & K_{xx}-iO_{xnz} \\ K_{xx}-iO_{xnz} & -i(O_{xnx}+O_{xxn}) & -D_{xz}-K_{xz} & -D_{xx}+iO_{xzn} \\ i(O_{xxn}-O_{xnx}) & K_{xx}+iO_{xnz} & -D_{xx}-iO_{xzn} & D_{xz}-K_{xz} \end{pmatrix}, \tag{7b}$$

$$C_n \equiv \sum_{j,k} O_{njk}\ (\sigma_j \otimes \sigma_k)$$
$$= \begin{pmatrix} A_n+O_{nxx} & -O_{nzx}-iD_{nn} & -O_{nxz}-iK_{nn} & -A_n-O_{nxx} \\ -O_{nzx}+iD_{nn} & A_n-O_{nxx} & A_n-O_{nxx} & O_{nxz}-iK_{nn} \\ -O_{nxz}+iK_{nn} & A_n-O_{nxx} & A_n-O_{nxx} & O_{nzx}-iD_{nn} \\ -A_n-O_{nxx} & O_{nxz}+iK_{nn} & O_{nzx}+iD_{nn} & A_n+O_{nxx} \end{pmatrix}. \tag{7c}$$

Then, we may write down explicitly the final density matrix given by Eq. (6c) also in the usual matrix form, which depends obviously on the ϕ angle as:

$$\rho_{\overline{\Lambda}\Lambda}(\phi) = \tfrac{1}{4} I_0 (C_0 + \cos\phi C_n + \sin\phi\ C_x)\,. \tag{8}$$

In particular, we get the matrix form for the unpolarized density:

$$\rho_{\overline{\Lambda}\Lambda}(\mathrm{unpol}) \equiv \frac{1}{2\pi}\int_0^{2\pi} \rho_{\overline{\Lambda}\Lambda}(\phi)\mathrm{d}\phi = \frac{1}{4} I_0 \sum_{j,k} O_{0jk}\ \sigma_j \otimes \sigma_k = \frac{1}{4} I_0 C_0$$

and for different values of ϕ as 0, π and $\pm\pi/2$ just by setting the corresponding value of ϕ in Eq. (8). We rewrite the explicit matrix forms (without the global factor $I_0/4$) as for $\rho_{\overline{\Lambda}\Lambda}(\mathrm{unpol})$:

$$\rho_{\overline{\Lambda}\Lambda}(\mathrm{unpol}) = C_0\,. \tag{9}$$

Then, we apply the properties of the general density matrix, which have been recalled at the top of this section, to the above special final density matrices, which hold for any value of ϕ.

2.4. Deduction of the inequalities

From the density matrix $\rho_{\overline{\Lambda}\Lambda}(\mathrm{unpol})$ given by the matrix form of Eq. (9), the relation $\rho_{11}\rho_{22} \geq |\rho_{12}^2|$ gives:

$$(1-C_{zz})(1+C_{zz}) \geq |-C_{xz} - iP_n|^2 \Longrightarrow C_{xz}^2 + P_n^2 + C_{zz}^2 \leq 1 \tag{10a}$$

which leads to these three inequalities:

$$C_{xz}^2 + P_n^2 \leq 1, \quad C_{zz}^2 + P_n^2 \leq 1 \quad \text{and} \quad C_{xz}^2 + C_{zz}^2 \leq 1\,. \tag{10b}$$

Another inequality deduced is

$$(1 + C_{zz})^2 \geq (P_n + C_{xz})^2\,. \tag{10c}$$

The inequalities given by Eqs. (10) are very general and do not depend on the amplitude parametrization of the reaction $\overline{p}p \to \overline{\Lambda}\Lambda$ so, Eqs. (10) hold for the exchange reaction $\overline{p}p \to \overline{n}n$ as an example. Then, we do the same with the matrix forms of $\rho_{\overline{\Lambda}\Lambda}(0)$ and $\rho_{\overline{\Lambda}\Lambda}(\pi)$ which mix the elements of the two symbolic matrices C_0 and C_n, and we get these two inequalities:

$$(1 + A_n)^2 - (O_{nxx} - C_{zz})^2 \geq (O_{nzx} + C_{xz})^2 + (D_{nn} + P_n)^2, \tag{11a}$$

$$(1 - A_n)^2 - (O_{nxx} + C_{zz})^2 \geq (O_{nzx} - C_{xz})^2 + (D_{nn} - P_n)^2 \tag{11b}$$

and then deduce:

$$O_{nxx}^2 + O_{nzx}^2 + C_{xz}^2 + P_n^2 + D_{nn}^2 + C_{zz}^2 \leq 1 + A_n^2 \tag{11c}$$

so, the sum of particular pairs of the left-hand member of Eq. (11c), *i.e.* the set of $\{O_{nxx}^2, O_{nzx}^2, C_{xz}^2, P_n^2, D_{nn}^2, C_{zz}^2\}$, may fulfil an inequality such as Eq. (10b). For instance, Eq. (11c) leads to this new inequality:

$$D_{nn}^2 + C_{zz}^2 + O_{nzx}^2 \leq 1 - (P_n^2 + O_{nxx}^2 + C_{xz}^2 - A_n^2)\,. \tag{11d}$$

To prove that the right-hand member of the last inequality is less than one, we deal directly with the explicit expression of the spin observables in terms of the complex amplitude parameters[1] $\{a, b, c, d, e, g\}$ to get $O_{nxx}^2 + C_{xz}^2 + P_n^2 - A_n^2 = |de + iag|^2$ which is obviously positive or null, then we write down this new inequality $D_{nn}^2 + C_{zz}^2 + O_{nzx}^2 \leq 1$ which leads to: $D_{nn}^2 + O_{nzx}^2 \leq 1$ and $C_{zz}^2 + O_{nzx}^2 \leq 1$ and to such well-known inequality [3, 4]:

$$D_{nn}^2 + C_{zz}^2 \leq 1\,. \tag{11e}$$

Furthermore, we can see from the matrix form of $\rho(0)$ and $\rho(\pi)$ that if we take $\rho_{11}\rho_{33} \geq |\rho_{13}^2|$ we get the inequalities of the type of Eqs. (11) but with K_{nn} instead of D_{nn}.

[1] The full expression of the spin observables in terms of the amplitude parameters are given, for instance, in Ref. [4]

2.5. Discussion

The inequalities among pairs or triplets of spin observables deduced here from special positivity conditions on the spin density matrix constrain and reduce their *allowed value-domain*. The spin observables, which are not directly related to each other by the usual symmetries (C,P) are *however not completely independent* because the final density matrix which contains *all the spin quantum-information* gives via its positivity conditions a number of non-trivial and model independent inequalities among the above observables. For instance, the inequality $D_{nn}^2 + C_{zz}^2 + O_{nzx}^2 \leq 1$ means that the three observables C_{zz}, D_{nn} and O_{nzx} are restricted to be found in the inner part of a unit sphere which is smaller than the cube $[-1,1]^3$, since each observable is restricted to be between -1 and 1. The spin observables are related to the mean-values of the spin projections of the scattered particles ($\overline{\Lambda}$ and Λ), which are correlated in a non simple manner.

The inequalities deduced provide consistency checks on the experimental data. Let us recall that that was the motivation for the paper by Elchikh and Richard (Ref. [4]). In fact, the earlier data showed some inconsistency like the negative proportion measured of spin-singlet fraction (see Ref. [7]). But, the recently published data (Ref. [1]) are better. It is hoped that, in the future, the density matrix constraints would be included in the Monte Carlo simulation for a wide class of reactions as well as for phenomenological models.

3. Conclusion

We have written down the spin density matrix of the reaction $\overline{p}p \to \overline{\Lambda}\Lambda$ in a usual matrix form, as a combination of the spin observables. Then, we have shown that the general properties of any density matrix applied to the matrix form found allow to extract inequalities among two or three quadratic observables. To get simple inequalities involving two or three observables from the combination of several observables, as obtained in Eq. (11c), we may use the "empirical" approach to check which pairs or triplets of observables are fulfilling simple inequalities, then we can return to the global inequalities deduced (from the positivity conditions) to prove the "true" pairs or triplets, so, the two approaches can be viewed as complementary even the density matrix formalism is powerful and uses standard quantum assumptions.

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Appendix A

Here, we give the relations obtained from the symmetries restrictions:

- The observables elements of the symbolic matrix C_0 given by Eq. (7a):
  - $C_{00} = 1$.
  - $P_{0j} = P_{j0} \equiv C_{0j} = C_{j0}$: the polarization which is null except for $j = n$.
  - $C_{ij} = C_{ji}$ for $i \neq 0$ and $j \neq 0$: the spin correlation coefficient.
- The observables elements of the symbolic matrix C_x given by Eq. (7b):
  - $A_x \equiv O_{x00} = 0$.
  - $K_{xi} \equiv O_{xi0}$: the spin-transfer coefficient which vanishes if $i = n$.
  - $D_{xi} = O_{x0i}$: the spin-depolarization coefficient which vanishes if $i = n$.
  - For O_{xij}: the following coefficients are all null: O_{xxx}, O_{xnn}, O_{xzz}, O_{xxz} and O_{xzx}.
- The observables elements of the symbolic matrix C_x given by Eq. (7c):
  - For O_{nij}: the following coefficients are all null: $O_{nnx}, O_{nnz}, O_{nzn}$.
  - Two other relations: $O_{nxx} = -O_{nzz}$ and $O_{nnn} = A_n$
  - $A_n \equiv O_{n00}$: the asymmetry measured.
  - $K_{ni} \equiv O_{ni0}$: which vanishes except for $i = n$.
  - $D_{ni} \equiv O_{n0i}$: which vanishes except for $i = n$.

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