HYDRO-INSPIRED PARAMETERIZATIONS OF FREEZE-OUT IN RELATIVISTIC HEAVY-ION COLLISIONS* **

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Popular parameterizations of the freeze-out conditions in relativistic heavy-ion collisions are discussed. Similarities and differences between the blast-wave model and the single-freeze-out model, both used recently to interpret the RHIC data, are outlined. A non-boost-invariant extension of the single-freeze-out model is proposed and applied to describe the recent BRAHMS data.

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1. The blast-wave model of Siemens and Rasmussen

In 1979 Siemens and Rasmussen formulated a model describing the hadron production in Ne + Na F reactions at the beam energy of 800 MeV per nucleon [1]. The physical picture behind the model was that the fast hydrodynamic expansion of the produced hadronic matter leads to a sudden decoupling of hadrons and freezing of their momentum distributions, which retain their thermal character (although modified by the collective expansion effects) until the observation point. In their own words, Siemens and Rasmussen described the collision process as follows: "central collisions of heavy nuclei at kinetic energies of a few hundred MeV per nucleon produce fireballs of hot, dense nuclear matter; such fireballs explode, producing blast

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waves of nucleons and pions". In this way, with Ref. [1], the concept of the blast waves of hadrons and the blast-wave model itself entered the field of relativistic heavy-ion collisions.

Although the model of Siemens and Rasmussen was motivated by an earlier hydrodynamic calculation by Bondorf, Garpman and Zimanyi [2], the results presented in Ref. [1] were not obtained by solving the hydrodynamic equations but followed from the specific assumptions on the freeze-out conditions. The most important ingredient of the model was the spherically symmetric expansion of the shells of matter with constant radial velocity. With an additional assumption about the times when such shells disintegrate into freely streaming hadrons (this point will be discussed in a greater detail in Sec. 3) Siemens and Rasmussen obtained the formula for the momentum distribution of the emitted hadrons [1]

$$\frac{dN}{d^3p} = Z \exp\left(-\frac{\gamma E}{T}\right)\left[\left(1+\frac{T}{\gamma E}\right)\frac{\sinh a}{a} - \frac{T}{\gamma E}\cosh a\right]. \tag{1}$$

In Eq. (1) Z is a normalization factor, $E = \sqrt{m^2+p^2}$ denotes the hadron energy, T is the temperature of the fireball (the same for all fluid shells), and $\gamma = (1-v^2)^{-1/2}$ is the Lorentz gamma factor with v denoting the radial collective velocity (radial flow). A dimensionless parameter a is defined by the equation

$$a = \frac{\gamma v p}{T}. \tag{2}$$

Small values of v (and a) correspond to small expansion rate and, as expected, a simple Boltzmann factor is obtained from Eq. (1) in the limit $v \to 0$

$$\frac{dN}{d^3p} \to Z \exp\left(-\frac{E}{T}\right). \tag{3}$$

The fits to the data based on the formula (1) gave $T = 44\,\mathrm{MeV}$ and $v = 0.373$. Interestingly, the value of the radial flow v turned out to be quite large suggesting the strong collective behavior. This was an unexpected feature summarized by the authors with the statement: "Monte Carlo studies suggest that Ne + Na F system is too small for multiple collisions to be very important, thus, this evidence for a blast feature may be an indication that pion exchange is enhanced, and the effective nucleon mean free path shortened in dense nuclear matter".

2. Cooper–Frye formula

Below we shall analyze the formal steps leading to Eq. (1). Our starting point is the expression defining the momentum distribution of particles as the integral of the phase–space distribution function $f(x,p)$ over the freeze-out hypersurface Σ, *i.e.*, the renowned Cooper–Frye formula [3],

$$E\,\frac{dN}{d^3p} = \frac{dN}{dy\,d^2p_\perp} = \int d^3\Sigma_\mu(x) p^\mu f(x,p)\,. \tag{4}$$

The three-dimensional element of the freeze-out hypersurface in Eq. (4) may be obtained from the formula

$$d^3\Sigma_\mu = \varepsilon_{\mu\alpha\beta\gamma}\frac{dx^\alpha}{d\alpha}\frac{dx^\beta}{d\beta}\frac{dx^\gamma}{d\gamma}\,d\alpha\,d\beta\,d\gamma\,, \tag{5}$$

where $\varepsilon_{\mu\alpha\beta\gamma}$ is the Levi–Civita tensor and α,β,γ are the three independent coordinates introduced to parameterize the hypersurface.

We note that for systems in local thermodynamic equilibrium we have

$$E\,\frac{dN}{d^3p} = \int d^3\Sigma_\mu(x)\,p^\mu f_{\rm eq}\left(u_\mu(x)\,p^\mu\right)\,, \tag{6}$$

where the function $f_{\rm eq}$ is the equilibrium distribution function

$$f_{\rm eq}(E) = \frac{1}{(2\pi)^3}\left[\exp\left(\frac{E-\mu}{T}\right)+\epsilon\right]^{-1}\,. \tag{7}$$

Here the case $\epsilon = +1\,(-1)$ corresponds to the Fermi–Dirac (Bose–Einstein) statistics, and the limit $\epsilon \to 0$ yields the classical (Boltzmann) statistics. For a static fireball one finds

$$d^3\Sigma_\mu = (dV,0,0,0)\,, \qquad u_\mu = (1,0,0,0)\,, \tag{8}$$

and Eq. (6) is reduced to the formula

$$\frac{dN}{d^3p} = V f_{\rm eq}(E)\,, \tag{9}$$

where V is the volume of the system. Eq. (9) agrees with Eq. (3) in the classical limit if the normalization constant Z is taken as

$$Z = \frac{V}{(2\pi)^3}\exp\left(\frac{\mu}{T}\right)\,. \tag{10}$$

3. Spherically symmetric freeze-outs

For spherically symmetric freeze-outs it is convenient to introduce the following parameterization of the space–time points on the freeze-out hypersurface [4]

$$x^\mu = (t, x, y, z) = (t(\zeta)\,, r(\zeta)\sin\theta\cos\phi\,, r(\zeta)\sin\theta\sin\phi\,, r(\zeta)\cos\theta)\ . \quad (11)$$

The freeze-out hypersurface is completely defined if a curve, *i.e.*, the mapping $\zeta \longrightarrow (t(\zeta), r(\zeta))$ in the $t-r$ space is given. This curve defines the (freeze-out) times when the hadrons in the shells of radius r stop to interact, see Fig. 1. The range of ζ may be always restricted to the interval: $0 \le \zeta \le 1$. The three coordinates: $\phi \in [0, 2\pi], \theta \in [0, \pi]$, and $\zeta \in [0, 1]$ play the role of the variables α, β, γ appearing in Eq. (5). Hence, the element of the spherically symmetric hypersurface has the form

$$\begin{aligned} d^3\Sigma^\mu \ &= \ \big(r'(\zeta), t'(\zeta)\sin\theta\cos\phi, t'(\zeta)\sin\theta\sin\phi, t'(\zeta)\cos\theta\big) \\ &\quad \times r^2(\zeta)\sin\theta\, d\theta\, d\phi\, d\zeta\,, \end{aligned} \quad (12)$$

where the prime denotes the derivatives taken with respect to ζ. Besides the spherically symmetric hypersurface we introduce the spherically symmetric (hydrodynamic) flow

$$u^\mu \ = \ \gamma(\zeta)\,(1\,, v(\zeta)\sin\theta\cos\phi\,, v(\zeta)\sin\theta\sin\phi\,, v(\zeta)\cos\theta)\ , \quad (13)$$

where $\gamma(\zeta)$ is the Lorentz factor, $\gamma(\zeta) = (1 - v^2(\zeta))^{-1/2}$. In a similar way the four-momentum of a hadron is parameterized as:

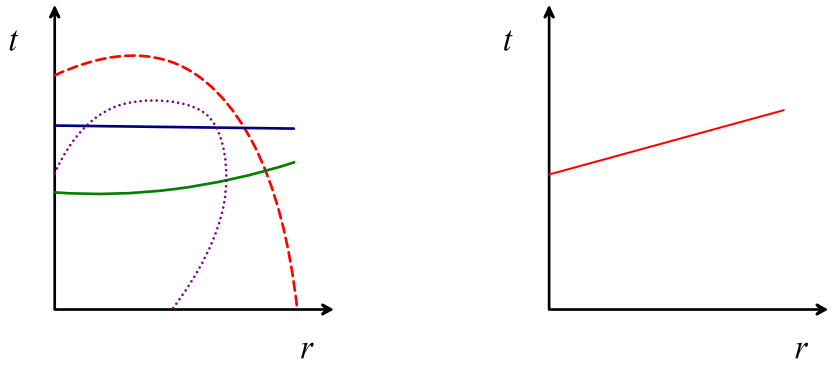


Fig. 1. Left: *A priori* possible, different freeze-out curves in Minkowski space. The dotted and dashed lines describe the cases where both the space-like and time-like parts are present. The solid lines describe the cases where only the time-like part is present. Right: The (time-like) freeze-out curve assumed in the blast-wave model of Siemens and Rasmussen, compare Eq. (19).

$$p^\mu = (E\,, p\sin\theta_p\cos\phi_p\,, p\sin\theta_p\sin\phi_p\,, p\cos\theta_p)\ , \tag{14}$$

and we find the two useful expressions:

$$p\cdot u \ = \ (E - pv(\zeta)\cos\theta)\ \gamma(\zeta)\,, \tag{15}$$

$$d^3\Sigma\cdot p \ = \ \left(Er'(\zeta) - pt'(\zeta)\cos\theta\right) r^2(\zeta)\sin\theta\, d\theta\, d\phi\, d\zeta\,. \tag{16}$$

We note that the spherical symmetry allows us to restrict our considerations to the special case $\theta_p = 0$.

In the case of the Boltzmann statistics, with the help of Eqs. (4), (15) and (16) we obtain the following form of the momentum distribution

$$E\frac{dN}{d^3p} = \int\limits_0^1 \frac{e^{-(E\gamma-\mu)/T}}{2\pi^2}\left[E\,\frac{\mathrm{sinh}a}{a}\,\frac{dr}{d\zeta} + T\,\frac{(\mathrm{sinh}a - a\mathrm{cosh}a)}{a\gamma v}\frac{dt}{d\zeta}\right] r^2(\zeta)d\zeta\,. \tag{17}$$

Here v, γ, r and t are functions of ζ, and the parameter a is defined by Eq. (2). The thermodynamic parameters T and μ may also depend on ζ. To proceed further we need to make certain assumptions about the ζ-dependence of these quantities. In particular, to obtain the model of Siemens and Rasmussen we assume that the thermodynamic parameters as well as the transverse flow velocity are constant

$$T = \mathrm{const}\,, \quad \mu = \mathrm{const}\,, \quad v = \mathrm{const} \quad (\gamma = \mathrm{const}\,, \quad a = \mathrm{const})\,. \tag{18}$$

Moreover, we should assume that the freeze-out curve in the $t - r$ space satisfies the condition

$$dt = v\, dr\,, \qquad t = t_0 + vr\,. \tag{19}$$

In this case we obtain the formula

$$\frac{dN}{d^3p} = \frac{e^{-(E\gamma-\mu)/T}}{2\pi^2}\left[\left(1 + \frac{T}{\gamma E}\right)\,\frac{\mathrm{sinh}a}{a} - \frac{T}{\gamma E}\,\mathrm{cosh}a\right]\int\limits_0^1 r^2(\zeta)\frac{dr}{d\zeta}\,d\zeta\,. \tag{20}$$

Equation (20) coincides with Eq. (1) if we use Eq. (10) and make the following identification

$$\int\limits_0^1 r^2(\zeta)\frac{dr}{d\zeta}\,d\zeta = \frac{r_{\mathrm{max}}^3}{3}, \qquad V = \frac{4}{3}\,\pi r_{\mathrm{max}}^3\,. \tag{21}$$

Note that the quantity r_{max} does not necessarily denote the maximum value of the radius of the system, see the dotted line on the left-hand side of Fig. 1.

An interesting and perhaps unexpected feature of the model proposed by Siemens and Rasmussen is the relation between the times and positions of the freeze-out points, see Eq. (19) illustrated on the right-hand side part of Fig. 1. Eq. (19) indicates that the fluid elements which are further away from the center freeze-out later. Moreover, taking into account Eq. (19) in the formula (12) we find that the four-vector describing the hypersurface is parallel to the four-vector describing the flow, compare Eqs. (12) and (13) giving $d^3\Sigma^\mu \sim u^\mu$ in this case. As we shall see the same features are assumed in the single-freeze-out model [5–7].

It is worth to emphasize that in the hydrodynamic approach the $t-r$ freeze-out curves contain the space-like and time-like parts[1]. The treatment of the space-like parts leads to conceptual problems since particles emitted from such regions of the hypersurface enter again the system and the hydrodynamic description of such regions (combined with the use of the Cooper–Frye formula) is inadequate. Recently much work has been done to develop a consistent description of the freeze-out process from the space-like parts [8, 9]. However, very often only a quantitative argument is presented [10] that the contributions from the space-like parts are small and may be neglected compared to the contributions from the time-like regions. The choice of Siemens and Rasmussen seems to have anticipated such arguments.

4. Boost-invariant blast-wave model of Schnedermann, Sollfrank and Heinz

The model presented above is appropriate for the low-energy scattering processes where the two nuclei completely merge at the initial stage of the collision and further expansion of the system is, to large extent, isotropic. At higher energies such a picture is not valid anymore and, following the famous paper by Bjorken [11], the boost-invariant and cylindrically symmetric models have been introduced to describe the collisions[2].

The boost-invariance (symmetry with respect to the Lorentz transformations) may be incorporated in the hydrodynamic equations, kinetic equations, and also in the modeling of the freeze-out process. In the latter case, the appropriate formalism was developed by Schnedermann, Sollfrank and Heinz [12]. The ansatz for the boost-invariant, cylindrically symmetric freeze-out hypersurface has the form

$$x^\mu = (t,x,y,z) = \left(\tilde{\tau}(\zeta)\cosh\,\alpha_\parallel\,,\rho(\zeta)\cos\phi\,,\rho(\zeta)\sin\phi\,,\tilde{\tau}(\zeta)\sinh\,\alpha_\parallel\right)\,. \quad (22)$$

[1] We use the convention that in the space-like (time-like) region the vector normal to the freeze-out curve is space-like (time-like).

[2] As is discussed in greater detail below, the data delivered by the BRAHMS Collaboration indicate that the systems produced at RHIC may be treated as boost-invariant only in the limited rapidity range $-1 < y < 1$. Moreover, the assumption about the cylindrical symmetry is valid only for the most central data.

Here, the parameter $\alpha_\parallel$ is the space–time rapidity. At $\alpha_\parallel = 0$ the longitudinal coordinate z is also zero and the variable $\tilde{\tau}(\zeta)$ coincides with the time coordinate t. Similarly to the spherical expansion discussed in Sec. 3, the boost-invariant freeze-out hypersurface is completely defined if the freeze-out curve $\zeta \to (\tilde{\tau}(\zeta), \rho(\zeta))$ is given. This curve defines the freeze-out times of the cylindrical shells with the radius ρ. Because of the boost-invariance it is enough to define this curve at $z = 0$, since for finite values of z the freeze-out points may be obtained by the Lorentz transformation.

The volume element of the freeze-out hypersurface is obtained from Eq. (5)

$$d^3\Sigma^\mu = \left(\frac{d\rho}{d\zeta}\cosh\,\alpha_\parallel, \frac{d\tilde{\tau}}{d\zeta}\cos\phi, \frac{d\tilde{\tau}}{d\zeta}\sin\phi, \frac{d\rho}{d\zeta}\sinh\,\alpha_\parallel\right)\rho(\zeta)\tilde{\tau}(\zeta)d\zeta d\alpha_\parallel d\phi\,. \tag{23}$$

Similarly to Eq. (22) the boost-invariant four-velocity field has the structure

$$u^\mu = \cosh\alpha_\perp(\zeta)\cosh\alpha_\parallel\left(1, \tanh\alpha_\perp(\zeta)\cos\phi, \tanh\alpha_\perp(\zeta)\sin\phi, \tanh\alpha_\parallel\right)\,. \tag{24}$$

We note that the longitudinal flow is simply $v_z = \tanh\alpha_\parallel = z/t$ (as in the one-dimensional Bjorken model), whereas the transverse flow is $v_r = \tanh\alpha_\perp(\zeta)$.

With the standard parameterization of the particle four-momentum in terms of rapidity y and the transverse mass $m_\perp$,

$$p^\mu = (m_\perp\mathrm{cosh}y\,, p_\perp\cos\varphi\,, p_\perp\sin\varphi\,, m_\perp\mathrm{sinh}y)\,, \tag{25}$$

we find

$$p\cdot u = m_\perp\cosh(\alpha_\perp)\cosh(\alpha_\parallel - y) - p_\perp\sinh(\alpha_\perp)\cos(\phi-\varphi), \tag{26}$$

and

$$d^3\Sigma\cdot p = \left[m_\perp\cosh(y-\alpha_\parallel)\frac{d\rho}{d\zeta} - p_\perp\cos(\phi-\varphi)\frac{d\tilde{\tau}}{d\zeta}\right]\rho(\zeta)\tilde{\tau}(\zeta)\,d\zeta\,d\alpha_\parallel\,d\phi. \tag{27}$$

For the Boltzmann statistics, with $\beta = 1/T$, the Cooper–Frye formalism gives the following momentum distribution

$$\begin{aligned}\frac{dN}{dyd^2p_\perp} &= \frac{e^{\beta\mu}}{(2\pi)^3}\int_0^{2\pi}d\phi\int_{-\infty}^{\infty}d\alpha_\parallel\int_0^1 d\zeta\;\rho(\zeta)\tilde{\tau}(\zeta) \\ &\times\left[m_\perp\cosh(\alpha_\parallel - y)\frac{d\rho}{d\zeta} - p_\perp\cos(\phi-\varphi)\frac{d\tilde{\tau}}{d\zeta}\right] \\ &\times\exp\left[-\beta m_\perp\cosh(\alpha_\perp)\cosh(\alpha_\parallel - y) + \beta p_\perp\sinh(\alpha_\perp)\cos(\phi-\varphi)\right].\end{aligned} \tag{28}$$

The form of Eq. (29) shows explicitly that the distribution $dN/(dyd^2p_\perp)$ is independent of y and φ, in accordance with our assumptions of the boost-invariance and cylindrical symmetry. The integrals over $\alpha_\parallel$ and ϕ in Eq. (29) are analytic and lead to the Bessel functions K and I,

$$\frac{dN}{dyd^2p_\perp} = \frac{e^{\beta\mu}}{2\pi^2} m_\perp K_1\left[\beta m_\perp \cosh(\alpha_\perp)\right] I_0\left[\beta p_\perp \sinh(\alpha_\perp)\right] \int_0^1 d\zeta\ \rho(\zeta)\tilde{\tau}(\zeta)\frac{d\rho}{d\zeta}$$
$$- \frac{e^{\beta\mu}}{2\pi^2} p_\perp K_0\left[\beta m_\perp \cosh(\alpha_\perp)\right] I_1\left[\beta p_\perp \sinh(\alpha_\perp)\right] \int_0^1 d\zeta\ \rho(\zeta)\tilde{\tau}(\zeta)\frac{d\tilde{\tau}}{d\zeta}\,. \tag{29}$$

In the spirit of the blast-wave model of Siemens and Rasmussen we have assumed here that the radial velocity is constant, $v_r = \tanh\alpha_\perp(\zeta) = \text{const}$, otherwise the Bessel functions should be kept under the integral over ζ.

In order to achieve the simplest possible form of the model, the common practice is to neglect the second line of Eq. (29). This procedure means that one assumes implicitly the freeze-out condition $d\tilde{\tau}/d\zeta = 0\,(\tilde{\tau} = \tau = \text{const})$. In this case the boost-invariant blast-wave model is reduced to the formula

$$\frac{dN}{dyd^2p_\perp} = \text{const}\, m_\perp K_1\left[\beta m_\perp \cosh(\alpha_\perp)\right] I_0\left[\beta p_\perp \sinh(\alpha_\perp)\right]\,, \tag{30}$$

where the constant has absorbed the factor $e^{\beta\mu}\tau\rho_{\max}^2/(4\pi^2)$. Eq. (30) forms the basis of numerous phenomenological analyses of the transverse-momentum spectra measured at the SPS and RHIC [13] energies.

5. Resonances

The main drawback of the formalism outlined above is that it neglects the effect of the decays of hadronic resonances. Such an approach may be justified at lower energies but should be improved at the relativistic energies where most of the light particles are produced in the decays of heavier resonance states. The expressions giving the rapidity and transverse momentum spectra of particles originating from two- and three-body decays of the resonances with a specified momentum distribution were worked out by Sollfrank, Koch and Heinz [14, 15]. Their formulae may also be used to account for the feeding from the resonances in the blast-wave model, as already proposed in Ref. [12]. In other words, we wish to stress that the choice of the freeze-out hypersurface and of the flow profile are elements *completely independent* of the treatment of the resonances. Both are important with

the latter being the basic ingredient in the calculation of particle abundances and the key to success of thermal models.

At the SPS and RHIC energies it is important to include not only the decays of the most common resonances such as η, ρ, ω, K^* or Δ, but also of much heavier states. Although their contributions are suppressed by the Boltzmann factor, their number increases strongly with the mass [16, 17], hence their role can be easily underestimated. The effects of sequential decays of heavy resonances were first realized in statistical analyses of the ratios of hadronic abundances/multiplicities (for recent results see [18–21]) which showed that the statistical models give a very good description of the data, provided most of the hadrons appearing in the Particle Data Tables are included in the calculations.

In order to discuss the role of the sequential decays of the resonances it is convenient to start with a general formalism giving the Lorentz-invariant phase–space density of the measured particles [22]

$$\begin{aligned} n_1(x_1,p_1) &= E_1 \frac{dN_1}{d^3p_1 d^4x_1} \\ &= \int \frac{d^3p_2}{E_{p_2}} B(p_2,p_1) \int d\tau_2 \Gamma_2 e^{-\Gamma_2\tau_2} \int d^4x_2 \delta^{(4)}\left(x_2 + \frac{p_2\tau_2}{m_2} - x_1\right) \dots \\ &\times \int \frac{d^3p_N}{E_{p_N}} B(p_N,p_{N-1}) \int d\tau_N \Gamma_N e^{-\Gamma_N\tau_N} \\ &\times \int d\Sigma_\mu(x_N)\; p_N^\mu\; \delta^{(4)}\left(x_N + \frac{p_N\,\tau_N}{m_N} - x_{N-1}\right) f_N\left[p_N \cdot u(x_N)\right] . \end{aligned} \tag{31}$$

Here the indices $1, 2, \dots, N$ label hadrons in one chain of the sequential decays. The first resonance is produced on the freeze-out hypersurface and has the label N. The final hadron has the label 1, for more details see [6, 7]. The function $B(k,q)$ is the probability distribution for a resonance with momentum k to produce a particle with momentum q in a two-body decay

$$B(k,q) = \frac{b}{4\pi p^*}\,\delta\left(\frac{k\cdot q}{m_R} - E^*\right). \tag{32}$$

The function $B(k,q)$ satisfies the normalization condition

$$\int \frac{d^3q}{E_q} B(k,q) = b\,, \tag{33}$$

where b is the branching ratio for a given decay channel and $p^*(E^*)$ is the momentum (energy) of the emitted particle in the resonance's rest frame

(a generalization to three-body decays is straightforward and explained in Ref. [18]).

Integration of Eq. (31) over all space–time positions gives the formula for the momentum distribution

$$\begin{aligned} E_{p_1}\frac{dN_1}{d^3p_1} &= \int d^4x_1\, n_1\left(x_1,p_1\right) \\ &= \int \frac{d^3p_2}{E_{p_2}} B\left(p_2,p_1\right)\ldots \int \frac{d^3p_N}{E_{p_N}} B\left(p_N,p_{N-1}\right) \\ &\quad \times \int d\Sigma_\mu\left(x_N\right)\, p_N^\mu\; f_N\left[p_N\cdot u\left(x_N\right)\right]. \end{aligned} \tag{34}$$

Eq. (34) serves as the starting point to prove that for constant values of the thermodynamic parameters on the freeze-out hypersurface the ratios in the full phase–space (4π) are the same as in the local fluid elements. In this way, a connection between the measured ratios and the local thermodynamic parameters is obtained [23]. One may also check that for the boost-invariant systems it is enough to consider the ratios at any value of the rapidity to infer the values of the thermodynamic parameters [5, 7].

The experimental RHIC data show, however, that the rapidity distributions are of Gaussian shape[3] and the thermodynamic parameters vary with rapidity (the measured $\bar{p}/p$ ratio depends on y), hence, the system created at RHIC is, strictly speaking, not boost-invariant. In this situation the relation between the measured ratios and thermodynamic parameters is not obvious. Fortunately, the RHIC data show also a rather flat rapidity distribution and constant ratios in the rapidity range $-1 < y < 1$ [27, 28]. In this region (the central part of the broad Gaussian) the system to a good approximation may be treated as boost-invariant and the standard analysis of the ratios may be performed to obtain the thermodynamic parameters at $y = 0$.

6. Single-freeze-out model

The analysis of the ratios of hadron multiplicities measured at RHIC gives a typical temperature of 170 MeV. On the other hand, the analysis of the spectra based on Eq. (30) gives a lower temperature of about 100–140 MeV. Such a situation was observed already at the SPS energies, which motivated the introduction of the concept of two different freeze-outs.

Certainly, if the spectra contain important contributions from high lying states, the value of T obtained from the blast-wave formula fitted to the spectra cannot be interpreted as the temperature of the system in the precise

[3] This feature has revived the interest in the Landau hydrodynamic model [24, 25], see, for example, Ref. [26].

thermodynamic sense. First, the contributions from the resonances (feeding mostly the low-momentum region) should be subtracted from the spectra of light hadrons, giving the insight to the properties of the primordial particles. Using other words, we may argue that the calculation of the ratios should include the same number of the resonances as the corresponding calculation of the spectra.

An example of such a calculation is the single-freeze-out model formulated in Refs. [5,6]. In this model the decays of the resonances as well as the transverse flow change the spectra of the primordial particles in such a way that it is possible to describe well the spectra and the ratios with a single value of the temperature. The basic effect here is that the hadronic decays lead to effective cooling of the spectra.

Similarly to the original blast-wave models discussed above, the single freeze-out model assumes a certain form of the freeze-out hypersurface in the Minkowski space. In this case it is defined by the constant value of the proper time

$$\tau = \sqrt{t^2 - r_x^2 - r_y^2 - r_z^2} = \text{const}\,. \tag{35}$$

The transverse size of the system is defined by the parameter $\rho_{\max}$,

$$\rho = \sqrt{r_x^2 + r_y^2}\,, \qquad \rho < \rho_{\max} \tag{36}$$

and the velocity field at freeze-out is taken in the Hubble-like form[4]

$$u^\mu = \frac{x^\mu}{\tau} = \frac{t}{\tau}\left(1, \frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right)\,. \tag{37}$$

The natural parameterization of the freeze-out hypersurface has the form

$$\begin{aligned} t &= \tau \cosh\alpha_\parallel \cosh\alpha_\perp, \quad z = \tau \sinh\alpha_\parallel \cosh\alpha_\perp, \\ x &= \tau \sinh\alpha_\perp \cos\phi\,, \qquad y = \tau \sinh\alpha_\perp \sin\phi\,, \end{aligned} \tag{38}$$

which may be considered as the special case of the formula (22). Eq. (5) leads to the following expression defining the volume element

$$d\Sigma^\mu(x) = u^\mu(x)\,\tau^3 \sinh(\alpha_\perp)\cosh(\alpha_\perp)\,d\alpha_\perp\,d\alpha_\parallel\,d\phi\,. \tag{39}$$

A very important feature of the choice (35)–(37) is that the volume element is proportional to the four-velocity field. This feature holds also in the model of Siemens and Rasmussen. In this case the treatment of the resonance is very much facilitated. In particular, Eq. (34) may be rewritten in the form:

[4] For a recent attempt to connect the parameterization (35)–(37) with hydrodynamic calculation see Ref. [29].

$$E_{p_1}\frac{dN_1}{d^3p_1} = \int d\Sigma\,(x_N)\int\frac{d^3p_2}{E_{p_2}}B\,(p_2,p_1)\ldots$$
$$\times\int\frac{d^3p_N}{E_{p_N}}B\,(p_N,p_{N-1})\,p_N\cdot u\,(x_N)\;f_N\,[p_N\cdot u\,(x_N)]$$
$$= \int d\Sigma\,(x_N)\,p_1\cdot u\,(x_N)\;f_1\,[p_1\cdot u\,(x_N)]\;, \qquad (40)$$

where we have introduced the notation

$$p_{i-1}\cdot u\,(x_N)\;f_{i-1}\,[p_{i-1}\cdot u\,(x_N)] = \int\frac{d^3p_i}{E_{p_i}}B\,(p_i,p_{i-1})$$
$$\times\,p_i\cdot u\,(x_N)\;f_i\,[p_i\cdot u\,(x_N)]\;. \qquad (41)$$

In the local rest frame, the iterative procedure defined by Eq. (41) becomes a simple one-dimensional integral transform

$$f_{i-1}\,(q) = \frac{b\,m_R}{2E_q\,p^*q}\int\limits_{k_-(q)}^{k_+(q)} dk\,k\,f_i\,(k)\;, \qquad (42)$$

where $k_\pm(q) = m_R|E^*q \pm p^*E_q|/m_1^2$. Eqs. (41) and (42) allow us to deal with a very large number of decays in the very efficient way, very similar to that used in the calculation of the hadron abundances.

7. Non boost-invariant single-freeze-out model

The model described above may be generalized to the non boost-invariant version in the minimal way by the modification of the system boundaries. Introducing a dependence of the transverse size on the longitudinal coordinate z (or $\alpha_\parallel$), we break explicitly the assumption of the boost-invariance. At the same time, however, the local properties of the hypersurface and flow remain unchanged allowing us to treat the resonances in the same simple way as described in the previous section.

Since the measured rapidity distributions are approximately Gaussian, it is natural to start with the Gaussian ansatz for the dependence of the transverse size on the parameter $\alpha_\parallel$ and restrict the region of the integration over $\alpha_\perp$ to the interval

$$0 \le \alpha_\perp \le \alpha_\perp^{\max}\exp\left[-\alpha_\parallel^2/(2\Delta^2)\right]\;. \qquad (43)$$

The original boost-invariant version is recovered in the limit $\Delta \longrightarrow \infty$. Using the values of the thermodynamic parameters obtained from the boost-invariant version of the model applied to the hadronic ratios measured at midrapidity ($T = 165.6$ MeV and $\mu_B = 28.5$ MeV), we are left with three extra parameters (τ, $\rho_{\max} = \tau \sinh \alpha_{\perp}^{\max}$, and Δ), which should be fitted to the $p_{\perp}$-spectra collected at different values of the rapidity.

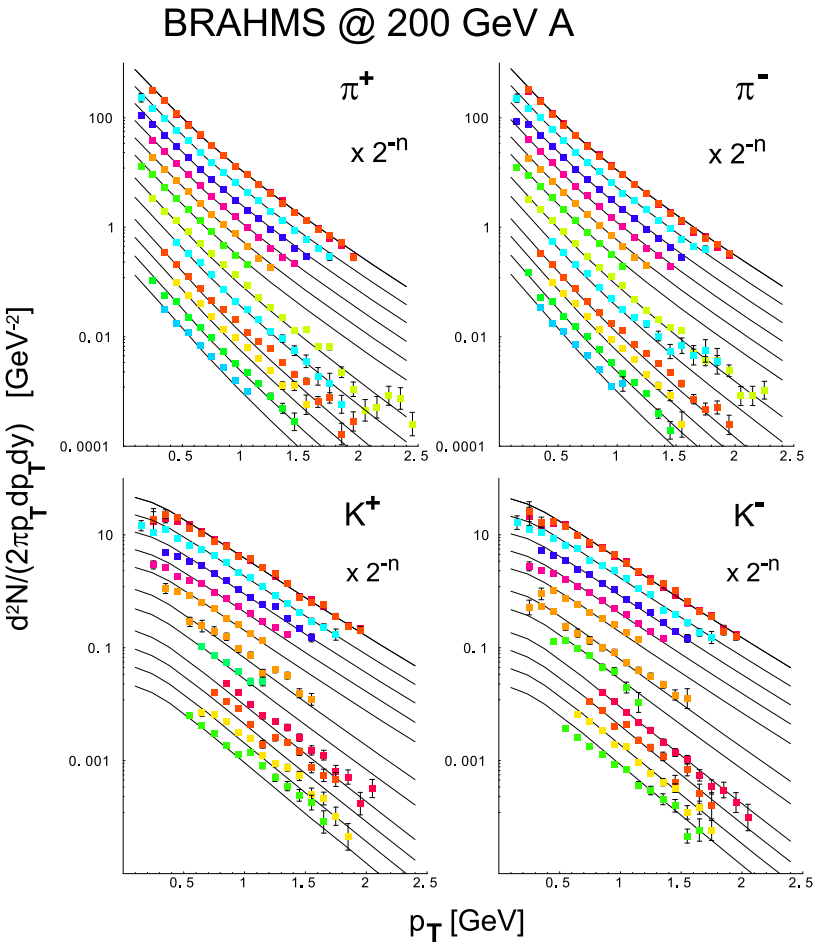


Fig. 2. The single-freeze-out model fit to the transverse-momentum spectra measured by the BRAHMS Collaboration for different values of the rapidity [28]. The successive curves correspond, from top to bottom, to the rapidity bins centered at: $y = -0.05$, 0.05, 0.5, 0.7, 0.9, 1.1, 1.3, 2.2, 2.5, 3.05, 3.15, 3.25, 3.35, 3.53 [28]. The data in the third, fourth *etc.* bin are subsequently divided by factors of 2, which is indicated by the 2^{-n} label.

The result of such a fit to the available BRAHMS data on π^+, π^-, K^+, and K^- production are shown in Fig. 2. The optimal values of the parameters found in the fit are: $\tau = 8.33$ fm, $\alpha_\perp^{\max} = 0.825$, and $\Delta = 3.33$. One can see that the model reproduces the data very well in a wide range of the transverse-momentum and rapidity. In Fig. 3 we show the model rapidity distributions compared to the data. Small discrepancies (of about 10%) between the model and the data may be seen for the pions at $y = 0$. Note that the comparison of the rapidity distributions in Fig. 3 is done with a linear scale; definitely, small discrepancies may be expected for a such simplified description of the freeze-out.

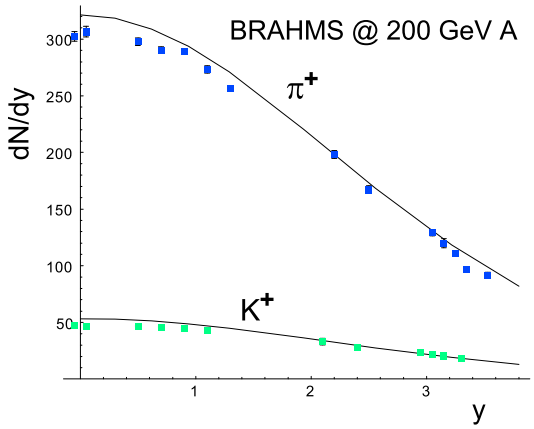


Fig. 3. Comparison of the measured rapidity distributions [28] with the results of the single-freeze-out model. The theoretical curves were obtained by the integration of the spectra shown in Fig. 2.

It should be emphasized that the non-boost invariant version of the model presented above is not capable of describing correctly the $\bar{p}/p$ ratio. In the present framework this requires an introduction of the rapidity dependence of the baryon chemical potential.

8. Conclusions

We have discussed several parameterizations of the freeze-out conditions in the relativistic heavy-ion collisions. We have argued that the single freeze-out model used to describe the RHIC data is a natural development of the blast-wave models worked out, among others, by Siemens, Rasmussen, Heinz, Schnedermann and Sollfrank. The main advantage of the single-freeze-out model is that it includes all well established resonance decays, allowing us to treat the chemical and thermal freeze-out as essentially one phenomenon. In this respect, the single-freeze-out model is very similar to

the original blast-wave model. Further similarities concern the shape of the freeze-out hypersurface (only the time-like parts are considered) and the strict use of the Cooper–Frye formula. Due to the limited space, we have not discussed here the variety of models where, instead of the Cooper–Fry formula, the so-called emission functions are introduced and modeled. An example of such an approach is the Buda–Lund model [30].

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