DISTRIBUTION OF THE LARGEST EIGENVALUES OF THE LÉVI–SMIRNOV ENSEMBLE

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(Received July 21, 2003)

We calculate the distribution of the k-th largest eigenvalue in the random matrix Lévi–Smirnov Ensemble (LSE), using the spectral dualism between LSE and chiral Gaussian Unitary Ensemble (GUE). Then we reconstruct universal spectral oscillations and we investigate an asymptotic behavior of the spectral distribution.

PACS numbers: 02.50.–r, 05.40.Fb, 05.90.+m

1. Introduction

Random matrix ensembles provide a powerful and generic formalism allowing addressing several statistical problems of the energy spectra in various complex quantum systems. In most applications, the ensembles used belong to the basin of attraction of the Gaussian ensembles which are relatively easy to calculate. Their another important feature is stability. It means that convolution of Gaussian matrices is still the Gaussian matrix. Because of this they are widely used in many applications, however they are useless when we describe processes with large contributions coming from fluctuations. To tackle with this we need to introduce new class of matrices, which can be used in those cases. Such ensembles can be viewed as the matrix generalization of the stable probability distributions of the probability theory, usually known as Lévy distributions.

Historically, such ensembles of matrices were proposed by Bouchaud and Cizeau [1]. Matrix elements of the corresponding matrices were drawn from stable, one-dimensional probability distributions. This construction, however, was breaking the rotational invariance of the ensemble, imposed in most Gaussian-like and polynomial matrix models. Recently, borrowing from the mathematical concepts of free random variables analysis [2], a new class of stable, rotationally invariant matrix models was introduced in [3]. Exploiting the Coulomb gas analogy, the authors [3] propose a general method of constructing explicit matrix ensembles characterized by stable power spectra with asymptotic behavior $\lambda^{-1-\alpha}$. Relatively little is known about the mathematical properties of the matrix ensembles with so strongly fluctuating elements that the average and the variance of the ensemble diverge.

The stability condition means that the matrix convolution of the identical independent ensembles with power-like spectra exhibits, modulo the rescaling and shift, the same power-like spectral distribution. In this sense, the ensembles provide a generalization of central limit theorem of Gaussian ensembles, where *e.g.* the convolution of two independent identical Gaussian Unitary Ensembles (GUE) with unit variance is also the Gaussian Unitary Ensemble, however with the semi-circle law rescaled by $\sqrt{2}$ in the longitudinal direction. In fact they are much more interested as an effective models. The ensembles introduced in [3] allows us to take a look on different models from the Gaussian ones.

Due to the growing interest in stochastic processes with long (sometimes called also heavy or fat) tails, intermittency and anomalous diffusion, the theory of stable random matrices may play an important role in describing generic features of complex systems using new concepts of universality. Very recently, new multi-critical random ensembles labeled by *continuous* scaling exponents were proposed [4], with eigenvalue density near zero behaving like $|x|^a$. Similar continuous series of universality class for eigenvalues appear naturally in stable random matrix ensembles with a non-compact support [3]. It is plausible to conjecture strongly [4] that by simple mapping the scaling behavior of small eigenvalues of the continuous multi-fractal regime can be related to the large eigenvalue behavior of the Lévy random matrix models.

This generalization leads us to mathematically complicated objects and right now is hard to obtain analytical results, so there are not many works on this subject. In this paper, we study one of the simplest ensembles of Lévy matrices, so-called Lévy–Smirnov matrix model. The name origins from the fact that the model constitutes a matrix analog of the Lévy–Smirnov probability distribution [5], *i.e.* the spectrum for large eigenvalues is power-like with $\alpha = 1/2$. It was possible because there is a duality between LS and chiral GUE, which allow us to employ methods used in solving GUE. In the next chapter, we remind the definition and some properties of the LS ensemble. Then, we exploit the duality of this model [3] to the massless chiral Gaussian Unitary Ensemble [6]. Since the chiral GUE plays a crucial role in describing the universal properties of low-lying eigenvalues of QCD Dirac operator, an impressive number of analytical tools and methods was developed in the last few years. Using the aforementioned duality and some of these methods [7], we calculate the distribution of k-th largest eigenvalue in the Lévy–Smirnov ensemble (hereafter LSE), and show explicit results for $k = 1, 2, 3, 4$. Finally, we analyze an asymptotic behavior of the spectral function, demonstrating how this behavior can be obtained from the microscopic spectral function. We close the paper with the discussion.

2. Lévi–Smirnov ensemble

The Random Matrix Theory is usually defined by the partition function

$$Z = \int dM \mathrm{e}^{-N\mathrm{Tr}V(M)}\,, \tag{1}$$

where M is N by N positive defined matrix and $V(M)$ is a potential, which in general, can be even a non-analytical function of M. Spectral distribution of the ensemble is defined as

$$\rho(\lambda) = \frac{1}{N}\langle \mathrm{Tr}\,\delta(\lambda - M)\rangle\,, \tag{2}$$

where the averaging is done with the measure defined by the partition function. The Gaussian Unitary Ensemble is characterized by quadratic potential $V(M) = 1/2M^2$ (we put variance to 1), leading to Wigner semi-circle law for the eigenvalues, $\rho(\lambda) = 1/\pi\sqrt{4-\lambda^2}$, *i.e.* the eigenvalues are localized on the compact interval $[-2, 2]$.

Lévy–Smirnov ensemble is characterized by a rather non-trivial potential [3]

$$V(M) = \exp[-N\mathrm{Tr}\,(M^{-1} + 2\ln M)]\,, \tag{3}$$

leading to the spectral function

$$\rho_{\mathrm{LS}}(\lambda) = \frac{1}{2\pi}\frac{\sqrt{4\lambda - 1}}{\lambda^2} \tag{4}$$

localized on the non-compact support $[1/4, \infty]$, with asymptotic behavior for large spectra $1/\lambda^{1+\alpha}$, with $\alpha = 1/2$. This behavior explains the name of the ensemble, since the probability distribution function for stable density with similar α and similar maximal asymmetry bears this name

$$p_{\mathrm{LS}}(x) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-1}{2x}\right)x^{-3/2} \qquad x > 0\,. \tag{5}$$

Note that LS matrix ensemble is *not* obtained simply by populating the matrix elements from the probability distribution (5).

3. A duality between the chiral and LS ensembles

The LS ensemble is one of the simplest example of Lévi matrices because there exist an exact, analytic expression for $V(M)$. This model has an unitary symmetry, so we can diagonalize $M \to U\Lambda U^\dagger$ and, after integration over the angular variables, rephrase the partition in terms of the eigenvalues only, the last term being the Vandermonde determinant coming from the Jacobian (see *e.g.* [9] for standard discussion)

$$Z \equiv \int_0^\infty \prod_i d\lambda_i \left(\frac{\mathrm{e}^{-N/\lambda_i}}{\lambda_i^{2N}} \right) \prod_{i>j} (\lambda_i - \lambda_j)^2 \, . \tag{6}$$

In this section we show that the distribution of eigenvalues in LSE behaves, up to a trivial transformation, in the same way as a distribution of the eigenvalues for the Chiral Gaussian Unitary Ensemble (chGUE). We will show that the equations describing both LS and Gaussian ensembles can be proved by the same method, using almost identical expressions.

We assume that N eigenvalues are ordered ($\lambda_1 > \lambda_2 > \ldots > \lambda_N$). A joint probability density $\mathit{\Omega}_{N,k}$ defined as in [7] in our case takes a form

$$\begin{aligned} \mathit{\Omega}_{N,k}(\lambda_1, \lambda_2, \ldots, \lambda_k) \; &= \; \mathcal{N} \frac{1}{(N-k)!} \int_0^{\lambda_k} d\lambda_{k+1} \ldots d\lambda_N \\ &\times \prod_{i=1}^{N} \left(\frac{\mathrm{e}^{-N/\lambda_i}}{\lambda_i^{2N}} \right) \prod_{i>j} (\lambda_i - \lambda_j)^2 \, , \end{aligned} \tag{7}$$

where $\mathcal{N}$ stands for the normalization

$$\mathcal{N} = \frac{1}{N!} \int_0^\infty \prod_{i=1}^{N} d\lambda_i \left(\frac{\mathrm{e}^{-N/\lambda_i}}{\lambda_i^{2N}} \right) \prod_{i>j} (\lambda_i - \lambda_j)^2 \, . \tag{8}$$

The main difference between Eq. (7) and its analog in chGUE lies in the range of integration. In LS case we are interested in the large eigenvalues so we integrate over an area with eigenvalues *smaller* than λ_k.

Using the methods originally developed for chGUE in [7], we can calculate the joint probability taking a microscopic limit. To do this, we change variables in the integral: $\lambda_i = 1/x_i$, for $i = k+1, \ldots, N$ and we shift variables $x_i \to x_i + 1/\lambda_k$. This leads us to the expression

$$\begin{aligned} \mathit{\Omega}_{N,k}(\{\lambda_k\}) = \frac{\mathcal{N}}{(N-k)!} \, \mathrm{e}^{-\frac{N(N-k)}{\lambda_k}} \prod_{i=1}^{k} \frac{\mathrm{e}^{-N/\lambda_i}}{\lambda_i^{2N}} \prod_{i<j}^{k} (\lambda_i - \lambda_j)^2 \\ \times \int_0^\infty \prod_{i=k+1}^{N} dx_i \mathrm{e}^{-N x_i} \lambda_k^2 x_i^2 \prod_{j=1}^{k-1} \left(1 - \lambda_j x_i - \frac{\lambda_j}{\lambda_k} \right)^2 \prod_{i>j\geq k+1} (x_i - x_j)^2 \, . \end{aligned} \tag{9}$$

Now we take a microscopic limit. Motivated by [3,4,7] we go with both λ_i and N to infinity keeping $\xi_i^2 = \lambda_i/4N^2$ fixed. It is useful to change notation to $\mathit{\Lambda}_i = 1/\xi_i$, which helps us to find the analogy mentioned before.

We calculate rescaled function Ω_k called (according to [7]) $\omega_k(\Lambda_1,\dots,\Lambda_k)$

$$\begin{aligned}\omega_k(\Lambda_1,\dots,\Lambda_k) &= \lim_{N\to\infty}\left(\prod_{i=1}^{k}\frac{4N^2}{\Lambda_i^3}\right)\Omega_{N,k}\left(\frac{4N^2}{\Lambda_1^2},\frac{4N^2}{\Lambda_2^2},\dots,\frac{4N^2}{\Lambda_k^2}\right)\\ &= \lim_{N\to\infty}\left(\prod_{i=1}^{k}\frac{4N^2}{\Lambda_i^3}\right)\mathcal{N}\frac{\mathrm{e}^{-\Lambda_k^2/4}}{(N-k)!}\prod_{i=1}^{k}(\Lambda_i^2)^2\\ &\quad\times\prod_{i<j}^{k-1}\left(\Lambda_i^2-\Lambda_j^2\right)^2\,(4N^2)^{-2k^2+k(k-1)}\\ &\quad\times\int_0^\infty\prod_{i=k+1}^{N}dx_i\,\mathrm{e}^{-Nx_i}x_i^2\prod_{j=1}^{k-1}\left(x_i-\frac{\mu_j^2}{4N^2}\right)^2\prod_{i>j\geq k+1}^{N}(x_i-x_j)^2\,,\end{aligned} \tag{10}$$

where we have introduced $\mu_j^2=\Lambda_j^2-\Lambda_k^2$.

Now, we are able to write this expression using the partition functions for chGUE $\mathcal{Z}$ in the microscopic limit

$$\begin{aligned}\omega_k(\Lambda_1,\dots,\Lambda_k) &= \mathrm{const}\,\mathrm{e}^{-\Lambda_k^2/4}\prod_{i=1}^{k}\Lambda_i\prod_{i<j}^{k-1}\left(\Lambda_i^2-\Lambda_j^2\right)^2\\ &\quad\times\frac{\mathcal{Z}_2(\mu_1,\mu_1,\mu_2,\mu_2,\dots,\mu_{k-1},\mu_{k-1})}{\mathcal{Z}_0(0)}\,.\end{aligned} \tag{11}$$

This expression shows the analogy between chGUE and LS ensembles. With the help of the above expression, in the next section, we will derive the rescaled joint probability distribution of the k largest eigenvalues in the LSE.

4. Distribution of the k-th eigenvalue

In this section we derive an analytical expression for the distribution of k-th eigenvalue. This distribution is defined as

$$p_k(\Lambda)=\int_0^\Lambda d\Lambda_1\int_{\Lambda_1}^{\Lambda}d\Lambda_2\dots\int_{\Lambda_{k-2}}^{\Lambda}d\Lambda_{k-1}\,\omega_k(\Lambda_1,\dots,\Lambda_{k-1},\Lambda)\,. \tag{12}$$

Using (11) and changing variables $x_i=\sqrt{\Lambda_i^2-\Lambda_k^2}$ for $i=1,\dots k-1$ we can write the distribution in the form

$$p_k(\Lambda) = \text{const} \int_0^{\Lambda} dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{k-2}} dx_{k-1}\, \mathrm{e}^{-\Lambda^2/4} \times \prod_{i=1}^{k} x_i \prod_{i<j}^{k-1} \left(x_i^2 - x_j^2\right)^2 \mathcal{Z}_2(x_1, x_1, \ldots, x_{k-1}, x_{k-1})\,. \tag{13}$$

An expression showed above contains the partition function for massive chiral GUE model, with twofold degeneracy for masses. With the help of the formula for this partition function, proven in [8], we can rewrite our final result. In our variables, equation (2.1) from [8] reads

$$\prod_{i<j}^{k-1} \left(x_i^2 - x_j^2\right)^2 \mathcal{Z}_2^{(2k)}(x_1, x_1, \ldots, x_{k-1}, x_{k-1}) = \text{const} \det_{1\le a,b\le k} \left[\mathcal{Z}_2^{(2)}(x_a, x_b)\right]. \tag{14}$$

Using (14) we write down our final result

$$p_k(\Lambda) = \text{const} \int_0^{\Lambda} dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{k-2}} dx_{k-1}\, \mathrm{e}^{-\Lambda^2/4} \prod_{i=1}^{k} x_i \det_{1\le a,b\le k} \left[\mathcal{Z}_2^{(2)}(x_a, x_b)\right], \tag{15}$$

where the partition function for two masses was already calculated in [8]

$$\mathcal{Z}_2^{(2)}(x_a, x_b) = \frac{x_b I_2(x_a) I_3(x_b) - x_a I_2(x_b) I_3(x_a)}{(-x_a^2 + x_b^2)}\,. \tag{16}$$

Now, we can go back to the original rescaled variable ξ (proportional to unscaled λ). Using substitution $\Lambda = 2/\sqrt{\xi}$ we can write the expression for the probability density of the k-th eigenvalue

$$p_k(\xi) = \frac{\text{const}}{\xi^2} \int_0^{2/\sqrt{\xi}} dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{k-2}} dx_{k-1}\, \mathrm{e}^{-1/\xi} \prod_{i=1}^{k-1} x_i \det_{1\le a,b\le k} \left[\mathcal{Z}_2^{(2)}(x_a, x_b)\right]. \tag{17}$$

This probability needs to be normalized. We calculate this expression for several cases $k = 1, 2, 3, 4$. The case $k = 1$ is very simple

$$p_1(\xi) = \frac{1}{\xi^2}\, \mathrm{e}^{-1/\xi}\,. \tag{18}$$

For higher k's (greater than 2) calculations are more complicated and cannot be performed by analytical methods, so we use the numerical method instead.

Fig. 1 shows the result for probability distribution for first four largest eigenvalues of the LS ensemble.

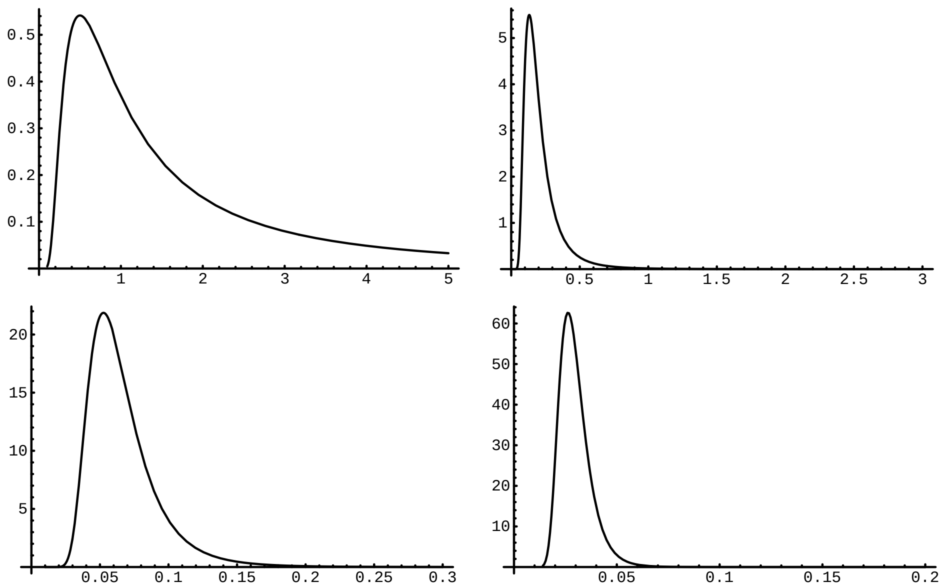

Fig. 1. The normalized distribution of the first four eigenvalues $k = 1, 2, 3, 4$ (from left to right). We can see that the peaks for higher k are closer to 0 (note that an x axis has different range on each figure).

By definition, the sum of p_k distributions reproduces the microscopic spectral density for LSE

$$\rho_{\mathrm{S}}(\xi) = \sum_{k=1}^{\infty} p_k(\xi) = (\xi)^{-2} \left(J_0^2 \left(\frac{2}{\sqrt{\xi}} \right) + J_1^2 \left(\frac{2}{\sqrt{\xi}} \right) \right) \tag{19}$$

calculated in [3]. Note also that the change of variables rewrites this density in terms of microscopic spectral density of chiral GUE, calculated in [10].

As an independent check of our formulae, we can approximately reconstruct $\rho(\xi)$ by adding first four distributions. Because the $\rho(\xi)$ asymptotic behavior is like $1/\xi^2$, it is more convenient to plot $\rho\,\xi^2$ instead ρ. Fig. 2 shows how oscillatory universal pattern for large eigenvalues is reproduced by the sum of the four largest eigenvalues.

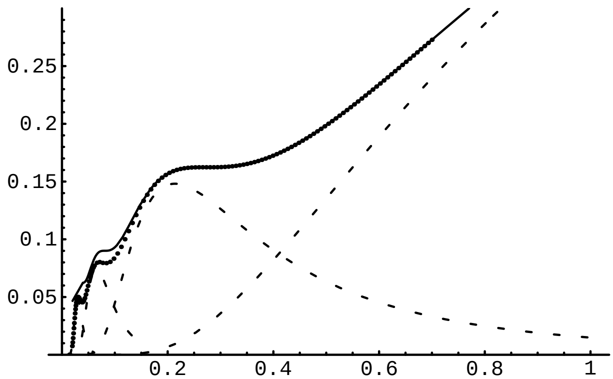

Fig. 2. The distribution of the first four eigenvalues $p_k(\xi)$ for $k = 1, 2, 3, 4$ (dashed lines) their sum (dotted line) and $\rho_{\rm S}(\xi)$ (solid line) — all functions are multiplied by ξ^2.

5. Asymptotic behavior

In this section, for completeness, we reconstruct the asymptotic behavior of the distribution function $\rho(\lambda)$ for LS ensemble, given by (4). The asymptotic behavior in infinity is $\lambda^{-3/2}$

$$\rho_{\rm LS}(\lambda) = \frac{1}{2\pi}\frac{\sqrt{4\lambda - 1}}{\lambda^2} \approx \frac{1}{\pi}\frac{1}{\lambda^{3/2}}\,. \tag{20}$$

We can reconstruct this result from equation (19) which we got after taking a microscopic limit. This means that we blow up the region close to infinity, because we took the limit $\lambda \to \infty$ keeping λ/N^2 fixed. This corresponds to asymptotic behavior of the scaling functions for ξ close to 0. We use well-known property of the Bessel functions (for x real)

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\left(\cos\left(x - \frac{1}{2}\nu\pi - \frac{1}{4}\right) + \mathcal{O}(x^{-1})\right), \text{ for } x \approx 0\,. \tag{21}$$

Applying this result to the (19) we obtain the expected result

$$\rho_{\rm S}(\xi) \approx \frac{1}{\xi^2}\frac{\sqrt{\xi}}{\pi}\left(\cos^2\left(\frac{2}{\sqrt{\xi}} - \frac{\pi}{4}\right) + \sin^2\left(\frac{2}{\sqrt{\xi}} - \frac{\pi}{4}\right)\right) = \frac{1}{\pi}\frac{1}{\xi^{3/2}}\,. \tag{22}$$

6. Discussion

The main result of this paper is the equation (17) which allows the calculations of $p_k(\xi)$ in an explicit way. The analytical form of these probability distributions can be used for comparison to similar empirical probability distributions of the *largest* eigenvalues of stable matrix ensembles, alike it is done in QCD lattice analysis for the *smallest* eigenvalues [11]. To obtain this result we used well-known methods in chiral random matrix models [7,8] and we exploited a non-trivial duality between the chiral Gaussian ensemble and LS ensemble. Naively, both ensembles are very different in nature. Chiral Gaussian ensemble is governed by a simple quadratic potential and exhibits a non-trivial symmetry due to the off-diagonal block structure. Its spectrum is localized on the compact support with novel (comparing to non-chiral Gaussian ensemble) microscopic oscillations in the vicinity of zero. LS ensemble belongs to non-Gaussian stability region, potential is a non-analytic function of M, and spectrum has an infinite support. On the basis of the duality known between α and $1/\alpha$ distributions for stable one-dimensional probability distributions [12] one may expect the similar duality between the bulk spectral function for GUE and LSE. Indeed, the semi-circle law transmutes into (4), by changing λ^2 into $1/\lambda$ and multiplying by Jacobian. It is important to stress that it is not only a trivial change of variables. It leads us to the new *stable* class of ensembles which can be used in theories with large influences coming from fluctuations. This duality allows us to obtain exact results but their interpretation is wider than ordinary exchange of variables in formulae. This transformation also misses the chiral character of the dual ensemble. It is the equivalence of the partition functions at the microscopic level that allows to transmute all non-trivial spectral properties of the chiral ensemble in the vicinity of zero to the spectral, universal properties of LS ensemble for large eigenvalues. In some sense, LS model is unique, allowing, despite having divergent mean and variance, to use the methods reserved for finite variance and compact support models. In general, the study of microscopic properties of Lévy ensembles requires mapping described in [4] and probably gives only approximate universal scalings. It is interesting to study other spectral statistics of the Lévy ensembles, especially in connection with the critical level statistics in disordered media [13].

This work was supported by the Polish State Committee for Scientific Research (KBN) projects no. 2P03B 09622 (2002–2004).

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