ADIABATIC LIMIT INTERFERENCE EFFECTS FOR TWO ENERGY LEVEL TRANSITION AMPLITUDES AND NIKITIN–UMANSKII FORMULA STUDIED BY FUNDAMENTAL SOLUTION METHOD

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A method of fundamental solutions has been used to study adiabatic transition amplitudes in two energy level systems for a class of Hamiltonians allowing some simplifications of Stokes graphs corresponding to such transitions. It has been shown that for simplest such cases the amplitudes take the Nikitin–Umanskii form but for more complicated ones they are formed by a sum of terms strictly related to a structure of Stokes graph corresponding to such cases. These results are in a full agreement with the ones of Joye, Mileti and Pfister [*Phys. Rev.* **A44**, 4280 (1991)] found by other method.

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1. Introduction

In our previous paper [1] we have applied a formalism of fundamental solutions to obtain formulae for adiabatic transition amplitudes in two level energy systems. The corresponding formalism has been developed under quite general assumptions about a nature of Hamiltonians perturbing a system adiabatically providing in this way still one more example that the fundamental solution method applies universally to most cases of standard problems of the one dimensional quantum mechanics (bound states, scattering, barrier penetration,) with a large variety of potentials.

Unfortunately, in its applications to particular examples considered in the Section 5 and the furthers of the paper mentioned we have made an error in detailed calculations of the corresponding transition amplitudes. As a consequence of this we have also drawn in this paper erroneous conclusions which followed from the obtained erroneous formulae. In the present paper we would like to correct the corresponding calculations as well as to draw correct conclusions.

However in order to avoid a permanent referring to the material presented and discussed in the first four sections of the paper [1] and to make our present paper selfsufficient and selfconsistent we shall repeat below to large extent the contents of these sections. Therefore we shall start with reminding shortly main reasons for studying transitions in two energy level systems.

First of all such systems provide us with the simplest models to investigate transition amplitudes between different energy levels by different approaches [2]. On the other side these systems play an important role in experimental investigations of basic principles of quantum mechanics [3]. Recently a lot of efforts has been devoted by Joye *et al.* to obtain more rigorous results on the adiabatic limit of transition amplitudes for these and similar systems with distinguished two energy levels well separated from the others.

Systems with two energy levels only are formally equivalent to a one-half spin system put into time dependent magnetic field. However good approximate results and the more so the exact ones are difficult to obtain for such systems even for simple time evolutions of the effective 'magnetic' field. Therefore each opportunity of improving this situation is worth trying. A treatment of the problem by the method of fundamental solutions (so fruitful in its application to stationary problems of 1-dim Schrödinger equation [9–11]) is of first importance, the more so that to our knowledge, the method was not used so far to this goal. A possibility of application of the method is related to the fact that transition amplitudes which enter a linear system of first order differential equations describing their time evolution can be represented by Fröman–Fröman amplitudes [9] appropriate for studying the adiabatic limit of interest. Equivalently, a linear system mentioned can be first transformed into a system of decoupled second order equations having a form of the stationary Schrödinger equation, one for each amplitude, which next allows us to apply all advantages of the fundamental solution method [10, 11]. The only obstacle related with this approach is a complexity of effective 'potentials' which appear in the final system of the Schrödinger-type equations.

The paper is organized as follows. In the next section the problem of transitions in two energy level systems is stated and corresponding assumptions about the effective 'magnetic field' are formulated. A linear system of two differential equations for the transition amplitudes is rewritten in a form of two decoupled equations of the Schrödinger type. In Sec. 3 properties of the fundamental solution method are recalled. In Sec. 4 some subtleties of the application of the fundamental solution method to the problems considered in the paper are discussed. In Sec. 5 a class of Hamiltonians with so called NED property is distinguished for further considerations. In Sec. 6 an exact form of a transition amplitude for the NED systems is obtained and its adiabatic limit is found. In Sec. 7 two examples of the NED systems are considered and the Nikitin–Umanskii formula is reconstructed. Finally in Sec. 8 we discuss our results stressing their coincidence with the corresponding ones of Joye *et al.* [5] despite the fact that the method of construction of solutions used by the latter authors as well as their respective analytical continuations by the plane of the complex time are completely different than ours.

2. Adiabatic transitions in two energy level systems

First let us remind that, in general, any two energy level system is formally equivalent to a one-half spin system put into an external magnetic field $\boldsymbol{B}(t)$. Its Hamiltonian $H(t)$ is given then by $H(t) = \frac{1}{2}\mu\boldsymbol{B}(t)\cdot\boldsymbol{\sigma}$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli's matrices so that two energy levels $E_\pm(t)$ of $H(t)$ are given by $E_\pm(t) = \pm\frac{\mu}{2}B(t)$ where $B(t) = \sqrt{\boldsymbol{B}^2(t)}$.

Typically, considering adiabatic transitions between the two energy levels $E_\pm(t)$ the following properties of the field $\boldsymbol{B}(t)$ are assumed:

1° The field $\boldsymbol{B}(t)$ is real being defined for the real t, $-\infty < t < +\infty$;

2° The field $\boldsymbol{B}(t)$ can be continued analytically off the real values of t as a meromorphic function which is defined on some t-Riemann surface $\boldsymbol{R}_B$. A sheet of $\boldsymbol{R}_B$ from which $\boldsymbol{B}(t)$ is originally continued is called physical;

3° There is an infinite strip $\Sigma = \{t : |\Im t|\langle\delta, \delta\rangle 0\}$ on the physical sheet in which the field $\boldsymbol{B}(t)$ is holomorphic without roots in the strip and achieves there finite limits for $\Re t = \pm\infty$, *i.e.* $\boldsymbol{B}(\Re t = \pm\infty) = \boldsymbol{B}_\pm \neq \boldsymbol{0}$ in the strip;

4° The 'absolute' value $B(t) = \sqrt{\boldsymbol{B}^2(t)}$ of the field $\boldsymbol{B}(t)$ is a ramified function of t on $\boldsymbol{R}_B$ with square root branch points coinciding with crossing points of the two energy levels $E_\pm(t)$;

It is also assumed that the field $\boldsymbol{B}(t)$ depends additionally on a parameter $T(>0)$ *i.e.* $\boldsymbol{B}(t) \equiv \boldsymbol{B}(t, T)$ which introduces a "natural" scale of time to the system. Therefore a time evolution of the system can be expressed most naturally in units of T. For T small in comparison with the actual period of the process considered the latter is "fast" or "sudden". In the opposite case, however, when T is large in this comparison the process is "slow" or "adiabatic".

In the adiabatic evolution of the system the following is assumed about the field $\boldsymbol{B}(t, T)$:

5° A dependence of $\boldsymbol{B}(t,T)$ on T is such that a rescaled field $\boldsymbol{B}(sT,T)$ has the following asymptotic behavior for $T \to +\infty$

$$\boldsymbol{B}(sT,T) \sim \boldsymbol{B}_0(s) + \frac{1}{T}\boldsymbol{B}_1(s) + \frac{1}{T^2}\boldsymbol{B}_2(s) + \cdots \tag{1}$$

and accordingly its s-Riemann surface $\boldsymbol{R}_B(T)$ approaches 'smoothly' the topological structure of the Riemann surface corresponding to the first term $\boldsymbol{B}_0(s)$ of the expansion (1).

6°As a function of s the field $\boldsymbol{B}_0(s)$ satisfies the properties **1°** $-$ **4°** above with substitutions $t \to s$ and $\boldsymbol{B}(s) \to \boldsymbol{B}_0(s)$.

7° For purposes of this paper we shall assume also an algebraic dependence of $\boldsymbol{B}(sT,T)$ on s so that its asymptotic behaviour in the strip Σ as $s \to \pm\infty$ is the following:

$$\begin{aligned} \boldsymbol{B}(sT,T) \;&\sim\; \boldsymbol{B}_\pm + \frac{\boldsymbol{B}_1^\pm}{s^{\alpha_1}} + \frac{\boldsymbol{B}_2^\pm}{s^{\alpha_2}} + \ldots + \frac{\boldsymbol{B}_k^\pm}{s^{\alpha_k}} + \ldots\;, \\ \frac{1}{2} \;&<\; \alpha_1 < \alpha_2 < \ldots < \alpha_k < \ldots\;, \end{aligned} \tag{2}$$

where $\alpha_1, \ldots, \alpha_k$, are assumed to be rational.

The next assumption which validity will become clear in Sec. 6 needs to formulate a notion of Stokes lines for the function $B_0(s)$. These are lines which starts at roots of $B_0(s)$ and are governed by conditions $\Re\Big(i\int_{s_k}^{s} B_0(\sigma)d\sigma\Big)$ $= 0$ where $s_k,\ k = 1, \ldots,$ are roots of $B_0(s)$. We shall assume the following about roots of $B_0(s)$, its Stokes lines and about components of the limiting field $\boldsymbol{B}_0(s)$.

8°

- Roots of $B^2(sT,T)$ and $B_0^2(s)$ are simple.
- A boundary of the central strip $\Sigma = \{t : |\Im t|\langle\delta,\delta\rangle 0\}$ in which $B(sT,T)$ is holomorphic and with no roots inside it consists of two infinite Stokes lines which become complex conjugated with each other in the adiabatic limit.
- Each component of $\boldsymbol{B}(sT,T)$ and $\boldsymbol{B}_0(s)$ is holomorphic and nonvanishing at roots of $B^2(sT,T)$ and $B_0^2(s)$ respectively.

The following family of the fields $\boldsymbol{B}(t,T)$ can easily satisfy the assumptions **1°**–**8°** above

$$B_i(t,T) = \frac{(P_{1i}(t,T))^{1/m}}{(P_{2i}(t,T))^{1/n}}\,, \qquad i = 1,2,3\,, \tag{3}$$

where $P_{1i}(t,T)$ and $P_{2i}(t,T),\ i=1,2,3$ are polynomials in t of degrees k and l respectively satisfying the condition $m/k=n/l$, with integers $m,n\geq 1$ and with the product $P_{21}P_{22}P_{23}$ vanishing nowhere on the real axis of the t-plane and with the real polynomial coefficients $a_{ijr}(T),\ i=1,2,\ j=1,2,3,\ r=1,\ldots,k,l$ at t^r having the following forms

$$a_{ijr}(T)=\frac{1}{T^r}\sum_{s=0}^{u}\frac{\alpha_{ijrs}}{T^s}\,. \tag{4}$$

Consider now the time-dependent Schrödinger equation corresponding to $H(t)$. It takes the form

$$\frac{i}{T}\frac{d\Psi(s,T)}{ds}=\frac{1}{2}\mu\boldsymbol{B}(sT,T)\cdot\boldsymbol{\sigma}\,\Psi(s,T)\,. \tag{5}$$

An adiabatic evolution of the wave function $\Psi(s,T)$ corresponds now to taking a limit $T\to+\infty$ in (5). Being more precise we are going to find in this limit the transition amplitude between the two energy levels of the system for $s\to+\infty$ under the assumptions that $\Psi(-\infty,T)$ coincides with one of the two possible eigenstates $\Psi_\pm(-\infty,T)$ of $H(-\infty)$ $(=H(+\infty))$ (corresponding to $E_\pm(-\infty)$ $(=E_\pm(+\infty))$ and that there is no level crossing for real t *i.e.* $\liminf\limits_{-\infty<t<+\infty} B(t)\geq\epsilon>0$. Well known approximate solutions of this problem are that of Landau [12] and Zener [13] in a form of the so called Landau–Zener formula and that of Dykhne [14] who have shown that such an amplitude should be exponentially small in the limit $T\to+\infty$. We are going to show in the next sections how to get an exact (*i.e.* not approximate) result for this amplitude as well as its adiabatic limit with the help of the fundamental solutions.

There is a standard way of proceeding when the adiabatic limit is investigated. It is defined by using eigenvectors $\Psi_\pm(s,T)$ of $H(sT,T)$ satisfying $(\Psi_\pm,\dot{\Psi}_\pm)=0$. As such eigenvectors $\Psi_\pm(s,T)$ can be chosen as the following ones

$$\begin{aligned}\Psi_+(s,T)&=\mathrm{e}^{-i\int_0^s\dot{\phi}\sin^2\frac{\Theta}{2}d\sigma}\begin{bmatrix}\cos\frac{\Theta}{2}\\ \sin\frac{\Theta}{2}\mathrm{e}^{i\phi}\end{bmatrix},\\ \Psi_-(s,T)&=\mathrm{e}^{-i\int_0^s\dot{\phi}\cos^2\frac{\Theta}{2}d\sigma}\begin{bmatrix}\sin\frac{\Theta}{2}\\ -\cos\frac{\Theta}{2}\mathrm{e}^{i\phi}\end{bmatrix},\end{aligned} \tag{6}$$

where Θ and ϕ are respective polar and azimuthal angles of the vector $\boldsymbol{B}(t,T)$ and dots over different quantities mean derivatives with respect to s-variable.

The wave function $\Psi(s,T)$ can now be represented as

$$\begin{aligned}\Psi(s,T) \;=\; & a_+(s,T)\mathrm{e}^{-iT\int_0^s E_+(\xi,T)d\xi}\,\Psi_+(s,T)\\ & +a_-(s,T)\mathrm{e}^{-iT\int_0^s E_-(\xi,T)d\xi}\,\Psi_-(s,T)\,. \end{aligned} \tag{7}$$

Rewriting the equation (5) in terms of the coefficients $a_\pm(s,T)$ we arrive at the following linear system of two equations

$$\begin{aligned}\dot a_+(s,T) \;&=\; -c^*(s,T)\mathrm{e}^{i\int_0^s \omega(\xi,T)d\xi}a_-(s,T)\,,\\ \dot a_-(s,T) \;&=\; c(s,T)\mathrm{e}^{-i\int_0^s \omega(\xi,T)d\xi}a_+(s,T)\,,\end{aligned} \tag{8}$$

where

$$\begin{aligned} c(s,T) \;&=\; \frac{\dot\Theta}{2}+\frac{i\dot\phi}{2}\sin\Theta = -\frac{1}{2}\frac{\left[\boldsymbol{B}\times\left(\boldsymbol{B}\times\dot{\boldsymbol{B}}\right)\right]_z}{B^2\sqrt{B_x^2+B_y^2}}+\frac{i}{2}\frac{\left(\boldsymbol{B}\times\dot{\boldsymbol{B}}\right)_z}{B\sqrt{B_x^2+B_y^2}}\,,\\ \omega(s,T) \;&=\; T\left(E_+-E_-\right)-\dot\phi\cos\Theta = \mu T B-\frac{B_z}{B}\frac{\left(\boldsymbol{B}\times\dot{\boldsymbol{B}}\right)_z}{B_x^2+B_y^2}\,. \end{aligned} \tag{9}$$

According to our assumptions we are looking for a solution to the system (8) satisfying the following initial conditions $a_+(-\infty,T) = 1$ and $a_-(-\infty,T) = 0$ and under this condition we are interested in the limits $\lim_{s\to+\infty} a_-(s,T)$ and $\lim_{T\to+\infty} a_-(+\infty,T)$.

The system (8) of equations can be easily solved directly by iterations [17]. However the obtained form of solutions is not appropriate for taking the adiabatic limit $T\to+\infty$. A well known form of solutions appropriate for such a goal is provided by their Fröman–Fröman representation [9] which can be further standardized as corresponding solutions to second order equations satisfied by each of the amplitudes $a_\pm(s,T)$. Coefficients of the latter equations can depend of course on the coefficients c and ω only but in a quite complicated way (see (21) below). Therefore a success of the method we are going to apply depends strongly on a dependence of the coefficients c and ω on the $\boldsymbol{B}$-field components. The above one given by (9) is however quite complicated. Fortunately it can be simplified by a suitable unitary transformation leaving the corresponding equations (8) invariant. The following unitary operator does the job

$$\begin{aligned} U \;&=\; \mathrm{e}^{\frac{1}{2}i\mu T\int\limits_0^s B_z(sT,T)ds\sigma_z}\\ &=\; \left[\begin{array}{cc} \mathrm{e}^{\frac{1}{2}i\mu T\int\limits_0^s B_z(sT,T)ds} & 0\\ 0 & \mathrm{e}^{-\frac{1}{2}i\mu T\int\limits_0^s B_z(sT,T)ds}\end{array}\right]. \end{aligned} \tag{10}$$

For the new amplitudes $\left[\begin{array}{c} a_1(s,T) \\ a_2(s,T) \end{array}\right]$ we get

$$\begin{aligned} a_1(s,T) &= \mathrm{e}^{-i\int_0^s(\dot\phi+\mu TB)\sin^2\frac{\Theta}{2}d\sigma}\cos\frac{\Theta}{2}a_+(s,T) \\ &\quad +\mathrm{e}^{-i\int_0^s(\dot\phi-\mu TB)\cos^2\frac{\Theta}{2}d\sigma}\sin\frac{\Theta}{2}a_-(s,T)\,, \\ a_2(s,T) &= \mathrm{e}^{i\int_0^s(\dot\phi-\mu TB)\cos^2\frac{\Theta}{2}d\sigma+i\phi(0)}\sin\frac{\Theta}{2}a_+(s,T) \\ &\quad -\mathrm{e}^{i\int_0^s(\dot\phi+\mu TB)\sin^2\frac{\Theta}{2}d\sigma+i\phi(0)}\cos\frac{\Theta}{2}a_-(s,T)\,. \end{aligned} \tag{11}$$

The transformation (10) does not change the form of Eq. (8) changing only the corresponding functions c and ω. Namely we have

$$\begin{aligned} \dot a_1(s,T) &= -Tc_1^*(s,T)\mathrm{e}^{iT\int_0^s\omega_1(\xi,T)d\xi}a_2(s,T)\,, \\ \dot a_2(s,T) &= Tc_1(s,T)\mathrm{e}^{-i\int_0^s\omega_1(\xi,T)d\xi}a_1(s,T)\,, \end{aligned} \tag{12}$$

where

$$c_1 = -\frac{i}{2}\mu B\sin\Theta\mathrm{e}^{i\phi}\,, \qquad \omega_1 = \mu B\cos\Theta\,. \tag{13}$$

It is worth to note that the form (11) of the considered transformation provides us immediately with the asymptotic forms of the amplitudes $a_1(s,T)$ and $a_2(s,T)$ for $s\to-\infty$ since the amplitudes $a_\pm(s,T)$ can take arbitrary values $a_\pm(-\infty,T)$ in this limit satisfying only the condition $|a_+(-\infty,T)|^2+|a_-(-\infty,T)|^2=1$. Namely we have simply in this limit

$$\begin{aligned} a_1(s,T) &\sim \mathrm{e}^{-i\int_0^s(\dot\phi+\mu TB)\sin^2\frac{\Theta}{2}d\sigma}\cos\frac{\Theta}{2}a_+(-\infty,T) \\ &\quad +\mathrm{e}^{-i\int_0^s(\dot\phi-\mu TB)\cos^2\frac{\Theta}{2}d\sigma}\sin\frac{\Theta}{2}a_-(-\infty,T)\,, \\ a_2(s,T) &\sim \mathrm{e}^{i\int_0^s(\dot\phi-\mu TB)\cos^2\frac{\Theta}{2}d\sigma+i\phi(0)}\sin\frac{\Theta}{2}a_+(-\infty,T) \\ &\quad -\mathrm{e}^{i\int_0^s(\dot\phi+\mu TB)\sin^2\frac{\Theta}{2}d\sigma+i\phi(0)}\cos\frac{\Theta}{2}a_-(-\infty,T)\,. \end{aligned} \tag{14}$$

However, since we are going to consider the case $a_+(-\infty,T)=1$ and $a_-(-\infty,T)=0$ then for this case we get for $s\to-\infty$

$$\begin{aligned} a_1(s,T) &\sim \mathrm{e}^{-i\int_0^s(\dot\phi+\mu TB)\sin^2\frac{\Theta}{2}d\sigma}\cos\frac{\Theta}{2}\,, \\ a_2(s,T) &\sim \mathrm{e}^{i\int_0^s(\dot\phi-\mu TB)\cos^2\frac{\Theta}{2}d\sigma+i\phi(0)}\sin\frac{\Theta}{2}\,. \end{aligned} \tag{15}$$

We can express further the amplitude $a_-(s,T)$, in which we are interested, by the $a_{1,2}$ ones inverting the transformation (11) to get

$$\begin{aligned} a_-(s,T) \; &= \; \mathrm{e}^{i\int_0^s(\dot\phi-\mu TB)\cos^2\frac{\Theta}{2}d\sigma}\sin\frac{\Theta}{2}a_1(s,T) \\ &\quad + \mathrm{e}^{-i\int_0^s(\dot\phi+\mu TB)\sin^2\frac{\Theta}{2}d\sigma-i\phi(0)}\cos\frac{\Theta}{2}a_2(s,T)\,. \end{aligned} \tag{16}$$

Moreover we can always assume that a limiting value of the field $\boldsymbol{B}(sT,T)$ for $s\to+\infty$ coincides with its z-component to be $\boldsymbol{B}_+=\boldsymbol{B}(+\infty,T)=(0,0,B_+)$, so that $\Theta(+\infty,T)=0$. Therefore in the limit considered we get from (16)

$$a_-(+\infty,T)=\lim_{s\to+\infty}\mathrm{e}^{-i\int_0^s(\dot\phi+\mu TB)\sin^2\frac{\Theta}{2}d\sigma-i\phi(0)}a_2(s,T)\,. \tag{17}$$

Consequently it is the amplitude $a_2(s,T)$ for which the above limit we have to consider.

The system (12) can be rewritten further as the following linear system of second order equations

$$\begin{aligned} \ddot a_1-\left(\frac{\dot c_1^*}{c_1^*}+iT\omega_1\right)\dot a_1+|c_1|^2a_1 \; &= \; 0\,, \\ \ddot a_2-\left(\frac{\dot c_1}{c_1}-iT\omega_1\right)\dot a_2+|c_1|^2a_2 \; &= \; 0\,, \end{aligned} \tag{18}$$

where the amplitudes $a_{1,2}$ decouple from each other being however still related by (8).

By the following transformations

$$\begin{aligned} a_1(s,T) \; &= \; \mathrm{e}^{\frac12\int_0^s\left(\frac{\dot c_1^*}{c_1^*}+iT\omega_1\right)d\xi}b_1(s,T)\,, \\ a_2(s,T) \; &= \; \mathrm{e}^{\frac12\int_0^s\left(\frac{\dot c_1}{c_1}-iT\omega_1\right)d\xi}b_2(s,T) \end{aligned} \tag{19}$$

we bring the equations (18) to Schrödinger types

$$\ddot b_{1,2}(s,T)+T^2q_{1,2}(s,T)b_{1,2}(s,T)=0\,, \tag{20}$$

where

$$q_2(s,T)=-\frac{1}{4T^2}\left(\frac{\dot c_1}{c_1}-iT\omega_1\right)^2+|c_1|^2+\frac{1}{2T^2}\left(\frac{\dot c_1}{c_1}-iT\omega_1\right)^{\cdot} \tag{21}$$

while (for real s and T) we have

$$q_1(s,T)=q_2^*(s,T)\,. \tag{22}$$

A dependence of the function $q_2(s,T)$ on T is given by

$$q_2(s,T) = \frac{1}{4}\mu^2 B^2 - \frac{i\mu B_z}{2T}\left(\frac{\dot{B}_z}{B_z} - \frac{\dot{c}_1}{c_1}\right) + \frac{1}{2T^2}\left[\left(\frac{\dot{c}_1}{c_1}\right)^{\cdot} - \frac{1}{2}\left(\frac{\dot{c}_1}{c_1}\right)^2\right], \quad (23)$$

where a dependence of B, B_z, c_1 on T in (23) is also anticipated. By (22) we get a corresponding dependence of $q_1(s,T)$ on T.

The equations (20) are now basic for our further analysis since their form is just of the stationary 1-D Schrödinger equation.

Taking into account (1) and (9) it is easy to check that the last formula provides us with the following type of asymptotic behavior of $q_{1.2}(s,T)$ for large T:

$$q_{1,2}(s,T) = q_{1,2}^{(0)}(s) + \frac{1}{T}q_{1,2}^{(1)}(s) + \frac{1}{T^2}q_{1,2}^{(2)}(s) + \dots \quad (24)$$

Therefore the above form of dependence of $q_{1,2}(s,T)$ on T permits us to apply to the considered case the method of fundamental solutions. For this reason we shall start the next section with a review of basic principles of the method suitably adapted to the considered case.

3. Fundamental solutions and their properties

A construction of fundamental solutions related to a given $q_{1,2}(s,T)$ has been described in many of our earlier papers (see for example [10, 11, 18]). This includes a notion of a Stokes graph and according to Fröman and Fröman [9] and Fedoriuk [10], can be performed in the following way [11].

First let us note that both $q_{1,2}(s,T)$ as functions of s are defined completely by an s-dependence of field $\boldsymbol{B}(Ts,T)$. Since the latter by our assumptions is meromorphic on some Riemann surface $\boldsymbol{R}_B(T)$ then, by (23), $q_{1,2}(s,T)$ are algebraic functions of $\boldsymbol{B}$, $\dot{\boldsymbol{B}}$ and $\ddot{\boldsymbol{B}}$. Therefore, they are also meromorphic functions of s defined again on some other Riemann surfaces $\boldsymbol{R}_{1,2}$ determined by these algebraic dependencies. In spite of a possible complexity of their topological structures limiting forms of the latter when $T \to +\infty$ can be quite simple being determined basically by the first terms $q_{1,2}^{(0)}(s)$ of the expansions (24). In consequence, by (23), it should be determined by $\mu\boldsymbol{B}^{(0)}(s)$ *i.e.* by the first term of the expansion (1).

We shall introduce and discuss the fundamental solutions to the equations (20) focusing on the function $q_2(s,T)$. An extension of the discussion to the $q_1(s,T)$ case will be obvious.

Let Z denote a set of all the points of $\boldsymbol{R}_2$ at which $q_2(s,T)$ has its single or double poles. Let $\delta(x)$ be a meromorphic function on $\boldsymbol{R}_2$, the unique singularities of which are double poles at the points collected by Z with coefficients at all the poles equal to $1/4$ each. (In a case when $\boldsymbol{R}_2$ is simply a complex plain the latter function can be constructed in general with the help of the Mittag–Leffler theorem [15]. But for a case of branched $\boldsymbol{R}_2$ the general procedure is unknown to us). Consider now a function

$$\tilde{q}_2(s,T) = q_2(s,T) + \frac{1}{T^2}\delta(s)\,. \tag{25}$$

The δ-term in (25) (called the Langer term [11, 16]) plays an important role in a construction of the fundamental solutions explained below. This term contributes to (25) if and only when the corresponding 'potential' function $q_2(s,T)$ contains simple or second order poles. (Otherwise the corresponding δ-term is put to zero.)

To define a Stokes graph corresponding to the function $\tilde{q}_2(s,T)$ we have to define first Stokes lines emerging from roots (turning points) of $\tilde{q}_2(s,T)$. They satisfy one of the following equations:

$$\Im\int_{s_i}^{s}\sqrt{\tilde{q}_2(\xi,T)}d\xi = 0 \tag{26}$$

with s_i being a root of $\tilde{q}_2(s,T)$. We shall assume further a generic situation when all roots s_i are simple.

Stokes lines which are not closed end at these points of $\boldsymbol{R}_2$ (*i.e.* have the latter points as their boundaries) for which the action integral in (26) becomes infinite. Of course, such points are singular for $\tilde{q}_2(s,T)$ and they can be its finite poles or its poles lying at an infinity.

Each such a singularity z_k of $\tilde{q}_2(s,T)$ defines a domain called a sector. This is the connected domain of $\boldsymbol{R}_2$ bounded by Stokes lines and z_k itself. The latter is also a boundary for the Stokes lines being an isolated boundary point of the sector (as it is in the case of the second order pole).

In each sector the LHS in (26) is only positive or only negative.

Considering now the equation (20) for $b_2(s,T)$ and following Fröman and Fröman [9] and Fedoriuk [10] we can define in each sector S_k having a singular point z_k at its boundary the following solution:

$$b_{2,k}(s,T) = \tilde{q}_2^{-\frac{1}{4}}(s,T)\cdot \mathrm{e}^{\sigma i T W(s,T)}\chi_{2,k}(s,T) \quad k = 1,2,\ldots\,, \tag{27}$$

where

$$\chi_{2,k}(s,T) = 1+\sum_{n\geq 1}\left(-\frac{\sigma}{2iT}\right)^n\int_{z_k}^{s}d\xi_1\int_{z_k}^{\xi_1}d\xi_2\ldots\int_{z_k}^{\xi_{n-1}}d\xi_n$$
$$\times\Omega(\xi_1)\Omega(\xi_2)\ldots\Omega(\xi_n)\left(1-\mathrm{e}^{-2\sigma iT(W(s)-W(\xi_1))}\right)$$
$$\times\left(1-\mathrm{e}^{-2\sigma iT(W(\xi_1)-W(\xi_2))}\right)\cdots\left(1-\mathrm{e}^{-2\sigma iT(W(\xi_{n-1})-W(\xi_n))}\right) \tag{28}$$

with

$$\Omega(s,T)=\frac{\delta(s)}{\tilde{q}_2^{\frac{1}{2}}(s,T)}-\frac{1}{4}\frac{\tilde{q}_2''(s,T)}{\tilde{q}_2^{\frac{3}{2}}(s,T)}+\frac{5}{16}\frac{\tilde{q}_2'^2(s,T)}{\tilde{q}_2^{\frac{5}{2}}(s,T)} \tag{29}$$

and

$$W(s,T)=\int_{s_i}^{s}\sqrt{\tilde{q}(\xi,T)}d\xi\,, \tag{30}$$

where s_i is a root of $\tilde{q}(s,T)$ lying at the boundary of S_k.

A sign of σ ($=\pm1$) and an integration path are chosen in (27) and (28) in such a way to have:

$$\sigma\Im\left(W(\xi_j)-W(\xi_{j+1})\right)\leq 0 \tag{31}$$

for any ordered pair of integration variables (with $\xi_0=s$). A path with this property is called canonical. Obviously, the condition (31) means that $b_{2,k}(s,T)$ vanishes in its sector when $s\to z_k$ along the canonical path.

Now to ensure all the integrals in (28) to converge at the limit z_k when the latter is a first or a second order pole of $\tilde{q}_2(s,T)$ or when the solutions (27) are to be continued to such poles the Langer δ-term has to appear in (25) and (29) what compensates possible divergencies. It follows from (29) that each such pole z_k demands a contribution to $\delta(s)$ of the form $(2(s-z_k))^{-2}$, what has been already assumed in the corresponding construction of $\delta(s)$.

It is now necessary to mention the main property of the fundamental solution method which is that analytic continuations of fundamental solutions along canonical paths ensures an immediate pass to adiabatic limit on every stage of calculations performed with their use. This property can be always utilized if all zeros of $\tilde{q}$-functions are simple and distributions of their zeros and poles are discrete *i.e.* there are no accomodation points for these singularities. We shall assume in the remainder that the two level energy systems we are going to consider will satisfy the last conditions.

4. The adiabatic limit in the fundamental solution approach

Let us consider a Stokes graph $\boldsymbol{G}_2$ corresponding to a given $\tilde{q}_2(s,T)$ which is drawn on the Riemann surface $\sqrt{\boldsymbol{R}_2}$ corresponding to $\sqrt{\tilde{q}_2(s,T)}$.

First let us notice that, in general, positions of singular points of $\tilde{q}_2(s,T)$ such as its branch points and poles on the Riemann surface $\boldsymbol{R}_2$ depend on T. This also means a dependence on T of jumps of $\tilde{q}_2(s,T)$ on its cuts as well as a T-dependence of coefficients of its poles.

Further, we can expect that, according to the property **5°** of the magnetic field $\boldsymbol{B}$ (see Sec. 2), a singular structure of $\tilde{q}_2(s,T)$, *i.e.* positions of its roots and poles, as well as cut jumps and pole coefficients, should change smoothly if the adiabatic limit is taken achieving their final positions and values respectively. This limiting structure is defined by a singularity structure of $\tilde{q}_2^{(0)}(s,T)$ (see expansion (24)). Therefore, both a topology of $\sqrt{\boldsymbol{R}_2}$ and the associated Stokes graph $\boldsymbol{G}_2$ changes accordingly coinciding eventually with the Riemann surface $\sqrt{\boldsymbol{R}_2^{(0)}}$ and with the Stokes graph $\boldsymbol{G}_2^{(0)}$ corresponding to $\sqrt{\tilde{q}_2^{(0)}(s,T)}$. There are the following ways by which the limiting structure can be achieved:

a. some of branch points and poles of $\tilde{q}_2(s,T)$ escape to infinities of $\boldsymbol{R}_2$;

b. some of branch points and poles of $\tilde{q}_+(s,T)$ approach the respective singularities of $\tilde{q}_2^{(0)}(s,T)$;

c. some of branch points and poles of $\tilde{q}_2(s,T)$ disappear because their respective jumps and coefficients vanish in the limit $T \to +\infty$.

To be more specific we expect that a set $\boldsymbol{S}_2$ of all singular points of $\tilde{q}_2(s,T)$ (*i.e.* containing all its branch points and poles) will consist of three well separated subsets $\boldsymbol{S}_2^{\rm inf}$, $\boldsymbol{S}_2^{\rm van}$ and $\boldsymbol{S}_2^{\rm fin}$ when T becomes large enough. The set $\boldsymbol{S}_2^{\rm inf}$ contains points which run to infinities of $\boldsymbol{R}_2$ when $T \to +\infty$. The points contained in $\boldsymbol{S}_2^{\rm van}$ disappear in the latter limit while those of the set $\boldsymbol{S}_2^{\rm fin}$ will coincide in this limit with the set $\boldsymbol{S}_2^{(0)}$ of singular points of $\tilde{q}_2^{(0)}(s,T)$.

Now let us remove the points contained in $\boldsymbol{S}_2^{\rm inf} \cup \boldsymbol{S}_2^{\rm van}$ from the Riemann surface $\boldsymbol{R}_2$, *i.e.* let us consider these points as regular for $\tilde{q}_2(s,T)$. Then $\boldsymbol{R}_2$ will transform into $\boldsymbol{R}_2^{\rm fin}$ — a Riemann surface which singular points coincide with those of the set $\boldsymbol{S}_2^{\rm fin}$.

Parallelly with the previous operation let us remove from $\sqrt{\boldsymbol{R}_2}$ also the Stokes lines generated by the points of $\boldsymbol{S}_2^{\rm inf} \cup \boldsymbol{S}_2^{\rm van}$ so that the remaining Stokes lines can be uniquely continued to form the Stokes graph $\boldsymbol{G}_2^{\rm fin}$ generated by the set $\boldsymbol{S}_2^{\rm fin}$. It is then clear that the graph $\boldsymbol{G}_2^{\rm fin}$ will coincide with $\boldsymbol{G}_2^{(0)}$ in the limit $T \to +\infty$.

We shall call both the above two operations — the adiabatic limit reduction or simply the reduction operation.

According to what we have mentioned earlier a set of sectors associated with the graph $\boldsymbol{G}_2$ and a corresponding set of fundamental solutions defined in them are transformed by the reduction operation to reduced forms *i.e.* under this operation some sectors of $\boldsymbol{G}_2$ transform into corresponding sectors of $\boldsymbol{G}_2^{\rm fin}$ whereas the others disappear. Obviously, the latter sectors are those which disappear when the limit $T \to +\infty$ is taken.

A good illustration for the above discussion can be an example considered in Sec. 7, namely the Nikitin model of the atom–atom scattering, for which the corresponding rescaled $\boldsymbol{B}$-field is the following $\boldsymbol{B}(sT,T) = \left(\left(b^2+s^2\right)^{-3/2},0,1\right)\frac{\Delta\epsilon}{\mu}$. We can write for this case the corresponding Schrödinger equation (20) using the amplitudes $a_\pm(s,T)$ for which the respective $q_\pm(s,T)$ are following

$$q_\pm(s,T) = \left[\frac{\Delta\epsilon}{2}\left(1+\frac{1}{(b^2+s^2)^3}\right)^{\frac{1}{2}} \pm \frac{i}{2T}\left(\frac{6s(b^2+s^2)^2}{1+(b^2+s^2)^3} - \frac{s}{b^2+s^2} - \frac{1}{s}\right)\right]^2$$
$$-\frac{3}{2}\frac{i\Delta\epsilon}{T}\frac{s}{(1+(b^2+s^2)^3)^{\frac{1}{2}}(b^2+s^2)^{\frac{5}{2}}} - \frac{1}{2T^2}\left[\frac{2s^2+b^2(b^2+s^2)}{s^2(b^2+s^2)}\right.$$
$$\left. -\frac{3}{2}\;\frac{4(b^2+s^2)^4(s^2-b^2)-4(b^2+s^2)(b^2+5s^2)+3s^2(b^2+s^2)}{(1+(b^2+s^2)^3)^2}\right]. \tag{32}$$

Equations (32) show that in the limit $T \to +\infty$ the Stokes graph for the considered problem is determined by the function

$$q^{(0)}(s,T) = \frac{(\Delta\epsilon)^2}{4}\left(1+\frac{1}{(b^2+s^2)^3}\right). \tag{33}$$

The graph is shown in Fig. 1.

While each of $q_\pm(s,T)$ has 40 roots, five branch points at $s=\pm ib$ and at $s = s_k = \pm\left(\mathrm{e}^{\frac{(2k+1)\pi i}{3}} - b^2\right)^{\frac{1}{2}}$, $k=1,2,3$, as well as two poles at $s=0$, there are only six roots at $s=s_k$, $k=1,2,3$ and only two poles at $s=\pm ib$ for $q^{(0)}(s,T)$.

The functions $q_\pm(s,T)$ are determined on two sheeted Riemann surfaces $\boldsymbol{R}_\pm$ respectively with the branch points at $s=\pm ib$ and at $s=s_k$, $k=1,2,3$ and with 40 roots distributed into halves on each sheet of the surfaces. Therefore the Riemann surfaces $\sqrt{\boldsymbol{R}_\pm}$ corresponding to $\sqrt{q_\pm(s,T)}$ are four-sheeted with these 40 roots being square root branch points on them. When $T\to+\infty$ only six of these branch points survive coinciding with the six roots of $q^{(0)}(s,T)$ at $s=\pm s_k$, k=1,2,3 whereas $\boldsymbol{R}_\pm$ transform into the complex

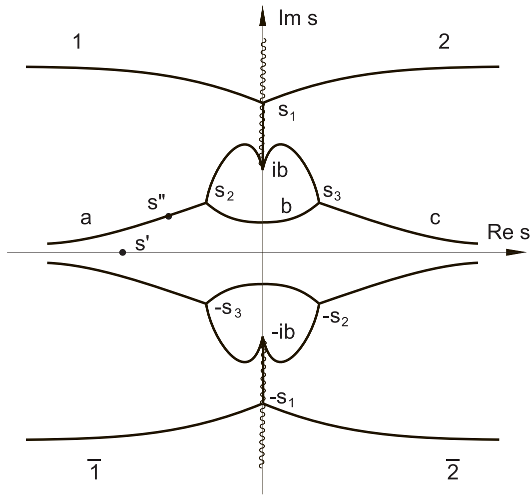

Fig. 1. The Stokes graph corresponding to $q_-^{(0)}(s)$.

s-plane since the branch points of $q_\pm(s,T)$ at $s = \pm ib$ disappear, being transformed into the second order poles of $q^{(0)}(s,T)$. It is easy to check however that for a finite but large T these six roots of $q^{(0)}(s,T)$ are each split initially into two. The split is the result of the square root branch points at $s = \pm ib$ to which the recovering of the finite T transforms the poles of $q^{(0)}(s,T)$ at the same points. The two copies of each of these six roots lie of course on different sheets of $\boldsymbol{R}_\pm$. Next, each of these 12 roots is still split into three by the same reason of finiteness of T. In this way, on each of the two sheets of $\boldsymbol{R}_\pm$ there are 36 roots grouped by three around their limit $s = \pm s_k$, $k = 1,2,3$ achieved for $T \to +\infty$.

The remaining four roots of $q_\pm(s,T)$ are displaced in two pairs, one pair on each sheet of $\boldsymbol{R}_\pm$, close to the points $s = 0$ at which the second order poles of $q_\pm(s,T)$ are localized. When $T \to +\infty$ the roots in each pair collapse into $s = 0$ multiplying the corresponding second order poles and thus causing mutual cancellations of the latter and themselves in this limit.

Now let us focus our attention on the Stokes graph $\boldsymbol{G}_-$ generated by $q_-(s,T)$ on the first sheet of $\boldsymbol{R}_-$ as well as on the remaining ones. It looks as in Fig. 2.
(The Stokes graph $\boldsymbol{G}_+$ corresponding to $q_+(s,T)$ can be obtained from $\boldsymbol{G}_-$ by complex conjugation of the latter.) On the figure the wavy lines denote the cuts corresponding to the branch points of the fundamental solutions defined on $\boldsymbol{R}_-$. The sheet in Fig. 2 cut along the wavy lines defines a domain where all the fundamental solutions $b_{-,1}(s,T), \ldots, b_{-,\bar{2}}(s,T)$ defined in the corresponding sectors $S_1, \ldots, S_{\bar{2}}$ (shown in the figure) are holomorphic.

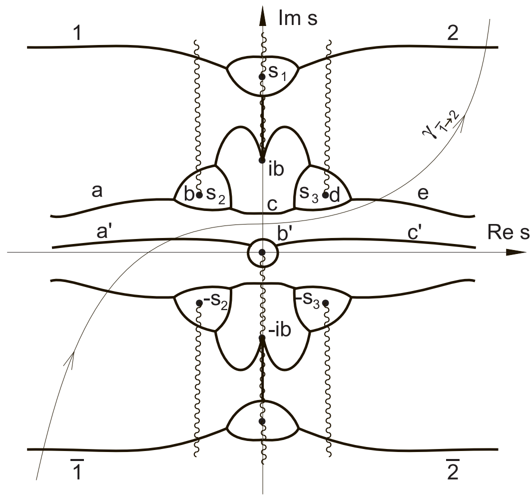

Fig. 2. The Stokes graph corresponding to $q_-(s,T)$.

According to our earlier description of the behavior of the Riemann surface $\sqrt{R_+}$ when $T \to +\infty$ the set $\boldsymbol{S}_-^{\rm inf}$ corresponding to the considered case is empty, $\boldsymbol{S}_-^{\rm van}$ contains four points at $s=0$ on each of the four sheets of $\sqrt{\boldsymbol{R}_-}$ (these four points correspond to the second order poles of $q_-(s,T)$) and the four branch points close to $s=0$, while $\boldsymbol{S}_-^{\rm fin}$ contains all the remaining singular points of $\sqrt{q_-(s,T)}$.

5. Systems which are not essentially different from their adiabatic limits (NED systems)

The last example considered above shows us also that by changing the amplitude representation to the $a_{1,2}$ ones we get much simpler s-dependence for the corresponding functions $q_{1,2}$ defining Eqs. (20) and for their adiabatic limit $q_{1,2}^{(0)}$. Namely for the interesting us amplitude a_2 we have

$$\begin{aligned} q_2(s,T) &= \frac{1}{4}(\Delta\epsilon)^2\left(1+\frac{1}{(b^2+s^2)^3}\right) - \frac{3\imath\Delta\epsilon}{2T}\frac{s}{b^2+s^2} - \frac{3}{4T^2}\frac{2b^2+s^2}{(b^2+s^2)^2}\,, \\ q_2^{(0)}(s) &= \frac{1}{4}(\Delta\epsilon)^2\left(1+\frac{1}{(b^2+s^2)^3}\right)\,. \end{aligned} \tag{34}$$

It is seen from (34) that both the functions $q_2(s,T)$ and $q_2^{(0)}(s)$ have the same Riemann surfaces, namely the simple complex plain on which they have poles in exactly the same points. They differ only by positions of their zeros the latter being in a mutual one-to-one correspondence so that each zero of

$q_2^{(0)}(s)$ is an adiabatic limit of the corresponding zero of $q_2(s,T)$. Therefore the Stokes graphs corresponding to both these functions are topologically equivalent having the forms of Fig. 1

As a consequence of this an application of the fundamental solution method to the cases of Eqs. (20) with respective $q_2(s,T)$ and $q_2^{(0)}(s)$ functions gives the same adiabatic limit for both these cases. We shall describe such a situation as corresponding to a system which do not differ essentially from its adiabatic limit and we shall call such a system the not-essentially-different one (the NED-system).

It follows from the above discussion that the NED property is not an immanent one for a system but can be achieved by choosing a suitable amplitude representation for a system.

6. Transition amplitudes for NED systems

As it follows from the discussion of the previous section systems with the NED properties allow us for as easy canonical continuations of fundamental solutions of interests as they are for their corresponding adiabatically reduced forms. Therefore for such systems we can consider them applying an exact procedure or using the adiabatical limit for the latter to get correct results for adiabatical limit transition amplitudes.

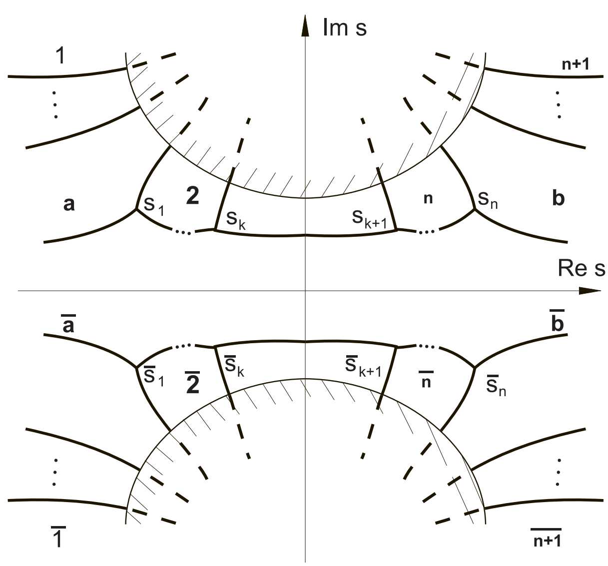

Fig. 3. The Stokes graph corresponding to $q_2(s,T)$ of a NED system.

We shall apply the procedure of canonical continuation of fundamental solutions to the amplitude $a_2(s,T)$. First we have to express this amplitude by the fundamental solutions and to satisfy the second of the conditions (15). Canonically continued to $-\infty$ with simple results of such continuations are the solutions $b_{2,1}(s,T)$and $b_{2,\bar{1}}(s,T)$ corresponding to the sectors S_1 and $S_{\bar{1}}$ respectively shown in Fig. 3 representing a Stokes graph corresponding to a general NED system. We have

$$a_2(s,T) = \mathrm{e}^{\int_0^s \frac{1}{2}\left(\frac{\dot{c}_1}{c_1} - i\omega_1\right)(\sigma,T)d\sigma} \left(Ab_{2,\bar{1}}(s,T) + Db_{2,1}(s,T)\right), \tag{35}$$

where

$$\begin{aligned} b_{2,\bar{1}}(s,T) &= q_2^{-1/4}(s,T)\mathrm{e}^{-iT\int_{s_{\bar{1}}}^s \sqrt{q_2(\xi,T)}d\xi}\chi_{2,\bar{1}}(s,T)\,, \\ b_{2,1}(s,T) &= q_2^{-1/4}(s,T)\mathrm{e}^{iT\int_{s_1}^s \sqrt{q_2(\xi,T)}d\xi}\chi_{2,1}(s,T)\,, \end{aligned} \tag{36}$$

and where we have assumed the positive real value of $\sqrt{q_2(s,T)}$ on the physical sheet. The lower integration limits in (36) are respective zeros of $q_2(s,T)$ shown in Fig. 3.

Taking into account that

$$\begin{aligned} \frac{\dot{c}_1}{c_1} &= \frac{\dot{B}}{B} + \dot{\Theta}\cot\Theta + i\dot{\phi}\,, \\ \frac{\dot{B}_z}{B} &= \frac{\dot{B}}{B}\cos\Theta - \dot{\Theta}\sin\Theta\,, \end{aligned} \tag{37}$$

we get in the limits $s \to \pm\infty$ along the real axis

$$\begin{aligned} iT\sqrt{q_2(s,T)} &\sim \frac{1}{2}i\mu TB + \frac{1}{2}\frac{B_z}{B}\left(\frac{\dot{B}_z}{B_z} - \frac{\dot{c}_1}{c_1}\right) \\ &= \frac{1}{2}i\mu TB - \frac{\dot{\Theta}}{2}\frac{1}{\sin\Theta} - \frac{1}{2}i\dot{\phi}\cos\Theta\,, \end{aligned} \tag{38}$$

so that

$$\begin{aligned} \frac{1}{2}\left(\frac{\dot{c}_1}{c_1} - i\omega_1\right) + iT\sqrt{q_2} &\sim (i\dot{\phi} + i\mu TB)\sin^2\frac{\Theta}{2} + \frac{1}{2}\frac{\dot{B}}{B} - \frac{\dot{\Theta}}{2}\tan\frac{\Theta}{2}\,, \\ \frac{1}{2}\left(\frac{\dot{c}_1}{c_1} - i\omega_1\right) - iT\sqrt{q_2} &\sim (i\dot{\phi} - i\mu TB)\cos^2\frac{\Theta}{2} + \frac{1}{2}\frac{\dot{B}}{B} + \frac{\dot{\Theta}}{2}\cot\frac{\Theta}{2} \end{aligned} \tag{39}$$

in both the limits.

It is now easy to show that Eqs. (35)–(39) provide us with the following asymptotic form of the amplitude $a_2(s,T)$ for $s \to -\infty$

$$
\begin{aligned}
&a_2(s,T) \sim A\left(\frac{\mu B_-}{2}\right)^{-1/2} \\
&\times \frac{1}{\sin\frac{\Theta_0}{2}} \mathrm{e}^{\int_0^{-\infty}\left[\frac{1}{2}\left(\frac{\dot c_1}{c_1}-i\omega_1\right)-iT\sqrt{q_2}-i(\dot\phi-\mu TB)\cos^2\frac{\Theta}{2}-\frac{\dot\Theta}{2}\cot\frac{\Theta}{2}\right]d\sigma-iT\int_{s_{\bar 1}}^0\sqrt{q_2}d\sigma-i\phi_0} \\
&\times \mathrm{e}^{\int_0^s i(\dot\phi-\mu TB)\cos^2\frac{\Theta}{2}d\sigma+i\phi_0}\sin\frac{\Theta}{2} + D\left(\frac{\mu B_-}{2}\right)^{-\frac{1}{2}}\frac{1}{\cos\frac{\Theta_0}{2}} \\
&\times \mathrm{e}^{\int_0^{-\infty}\left[\frac{1}{2}\left(\frac{\dot c_1}{c_1}-i\omega_1\right)+iT\sqrt{q_2}-i(\dot\phi+\mu TB)\sin^2\frac{\Theta}{2}-\frac{\dot\Theta}{2}\tan\frac{\Theta}{2}\right]d\sigma+iT\int_{s_1}^0\sqrt{q_2}d\sigma-i\phi_0} \\
&\times \mathrm{e}^{\int_0^s i(\dot\phi+\mu TB)\cos^2\frac{\Theta}{2}d\sigma+i\phi_0}\cos\frac{\Theta}{2}\,, \qquad (40)
\end{aligned}
$$

where $\Theta_0 = \Theta(0)$, $\phi_0 = \phi(0)$ and the infinite integrals in the above formulae are finite.

Comparing now the formula (40) with (14) and (15) we see that we have to put $D = 0$ in the formula (35) while for the coefficient A we get

$$
\begin{aligned}
A &= \left(\frac{\mu B_-}{2}\right)^{\frac{1}{2}}\sin\frac{\Theta_0}{2} \\
&\times \mathrm{e}^{\int_{-\infty}^0\left[\frac{1}{2}\left(\frac{\dot c_1}{c_1}-i\omega_1\right)-iT\sqrt{q_2}-i(\dot\phi-\mu TB)\cos^2\frac{\Theta}{2}-\frac{\dot\Theta}{2}\cot\frac{\Theta}{2}\right]d\sigma+iT\int_{s_{\bar 1}}^0\sqrt{q_2}d\sigma+i\phi_0}. \qquad (41)
\end{aligned}
$$

Consequently it is the solution $b_{2,\bar 1}(s,T)$ which will be continued canonically to the sectors $n+1$ and $\overline{n+1}$ from which it is subsequently continued to $+\infty$ of the real s-axis. According to the figure this canonical continuation can be done by representing $b_{2,\bar 1}(s,T)$ first as a linear combination of the next two fundamental solutions $b_{2,2}(s,T)$ and $b_{2,\bar 2}(s,T)$ defined in the respective sectors 2 and $\bar 2$. Next the latter two solutions have to be expressed in the same way by a pair of fundamental solutions of the sectors 3 and $\bar 3$ and so on up to the moment when the fundamental solutions of the sectors $n+1$ and $\overline{n+1}$ enter this procedure. Representing the corresponding fundamental solutions in the form

$$
\begin{aligned}
b_{2,k}(s,T) &= q_2^{-\frac{1}{4}}(s,T)\mathrm{e}^{iT\int_{s_{k-1}}^s\sqrt{q_2(\xi,T)}d\xi}\chi_{2,k}(s,T)\,, \\
b_{2,\bar k}(s,T) &= q_2^{-\frac{1}{4}}(s,T)\mathrm{e}^{-iT\int_{s_{k-1}}^s\sqrt{q_2(\xi,T)}d\xi}\chi_{2,\bar k}(s,T),\ \ k = 2,3,\ldots,n+1
\end{aligned}
\qquad (42)
$$

this chain of operations can be handled by the following multiplications of matrices

$$
\begin{aligned}
M &= M_1 M_2 \ldots M_n\,,\\
M_1 &= \frac{1}{\chi_{2,2\to\bar{2}}}\left[\begin{array}{cc} 0 & 0\\ -i\alpha_{\bar{1},1}\chi_{2,\bar{1}\to\bar{2}} & \chi_{2,\bar{1}\to 2}\end{array}\right],\\
M_k &= \frac{1}{\chi_{2,k+1\to\overline{k+1}}}\left[\begin{array}{cc} e^{\beta_k}\chi_{2,k\to\overline{k+1}} & i\alpha_{\bar{k},k}e^{\beta_k}\chi_{2,k\to k+1}\\ -i\alpha_{\bar{k},k}e^{\beta_{\bar{k}}}\chi_{2,\bar{k}\to\overline{k+1}} & e^{\beta_{\bar{k}}}\chi_{2,\bar{k}\to k+1}\end{array}\right],\\
\alpha_{\bar{k},k} &= e^{iT\int_{s_{\bar{k}}}^{s_k}\sqrt{q_2(s,T)}ds},\\
\beta_{k+1} &= iT\int\limits_{s_k}^{s_{k+1}}\sqrt{q_2(s,T)}ds\,,\\
\beta_{\overline{k+1}} &= -iT\int\limits_{s_{\bar{k}}}^{s_{\overline{k+1}}}\sqrt{q_2(s,T)}ds\,,\quad k=1,\ldots,n
\end{aligned}
\tag{43}
$$

so that

$$
b_{2,\bar{1}}(s,T) = M_{21}b_{2,n+1}(s,T) + M_{22}b_{2,\overline{n+1}}(s,T)\,. \tag{44}
$$

Let us note that the phase integrals defining the coefficients β_k and $\beta_{\bar{k}}$, $k=2,\ \ldots\ n$, are purely imaginary. Moreover the coefficients $\alpha_{\bar{k},k}$, $k=1,\ldots,n$, become pure real and equal to each other while the coefficients β_k become equal to $-\beta_{\bar{k}}$, $k=2,\ldots,\ n$ in the adiabatic limit $T\to+\infty$. To be more precise in these latter statements let $s'_{\bar{k}}$, $k=1,\ldots,n$, denote points where the antiStokes line emanating from $s_{\bar{k}}$, $k=1,\ldots,n$, crosses the Stokes line passing by the points $s_1,s_2,\ldots,s_n$. Then by the assumption **8**° of Sec. 2 we have

$$
\begin{aligned}
\int\limits_{s_{\bar{k}}}^{s_k}\sqrt{q_2(s,T)}ds &= \int\limits_{s_{\bar{k}}}^{s'_{\bar{k}}}\sqrt{q_2(s,T)}ds + \int\limits_{s'_{\bar{k}}}^{s_k}\sqrt{q_2(s,T)}ds\\
&= \int\limits_{s_{\bar{k}}}^{s'_k}\sqrt{q_2(s,T)}ds + O\left(\frac{1}{T}\right),\\
\int\limits_{s_{\bar{k}}}^{s'_k}\sqrt{q_2(s,T)}ds &= \int\limits_{s_{\bar{1}}}^{s'_1}\sqrt{q_2(s,T)}ds\,,
\end{aligned}
$$

$$
\begin{aligned}
\int\limits_{s_{\bar{k}}}^{s_{\overline{k+1}}} \sqrt{q_2(s,T)}ds &= \int\limits_{s'_{\bar{k}}}^{s'_{\overline{k+1}}} \sqrt{q_2(s,T)}ds \\
&= \left(\int\limits_{s'_{\bar{k}}}^{s_k} + \int\limits_{s_k}^{s_{k+1}} + \int\limits_{s_{k+1}}^{s'_{\overline{k+1}}} \right) \sqrt{q_2(s,T)}ds \\
&= \int\limits_{s_k}^{s_{k+1}} \sqrt{q_2(s,T)}ds + O\left(\frac{1}{T}\right), \quad k = 1,\ldots,n \qquad (45)
\end{aligned}
$$

i.e. each point $s'_{\bar{k}}$, $k = 2,\ldots,$ n tends to its corresponding limit s_k, $k = 2,\ldots,n$ when $T \to +\infty$ with the rates shown in (45).

Rewriting Eqs. (35) as

$$
a_2(s,T) = A\mathrm{e}^{\frac{1}{2}\int_0^s \left(\frac{\dot{c}_1}{c_1} - iT\omega_1\right)d\xi} b_{2,\bar{1}}(s,T) \tag{46}
$$

and taking into account (17) we get

$$
\begin{aligned}
&a_-(+\infty,T) = M_{21}\left(\frac{B_-}{B_+}\right)^{\frac{1}{2}} \sin\frac{\Theta_0}{2} \\
&\times \exp\left\{ \int\limits_{-\infty}^{0} \left[\frac{1}{2}\left(\frac{\dot{c}_1}{c_1} - i\omega_1\right) - iT\sqrt{q_2} - i(\dot{\phi} - \mu TB)\cos^2\frac{\Theta}{2} - \frac{\dot{\Theta}}{2}\cot\frac{\Theta}{2} \right] ds \right. \\
&+ \int\limits_{0}^{+\infty} \left[\frac{1}{2}\left(\frac{\dot{c}_1}{c_1} - iT\omega_1\right) + iT\sqrt{q_2} - i(\dot{\phi} + \mu TB)\sin^2\frac{\Theta}{2} \right] ds \\
&\left. + iT\int_{s_{\bar{1}}}^{0} \sqrt{q_2}d\sigma + iT\int_{s_n}^{0} \sqrt{q_2}ds \right\}
\end{aligned} \tag{47}
$$

since the second term in (44) vanishes in the limit $s \to +\infty$ along the real axis (because $\Theta(+\infty,T) = 0$ by assumption).

The formula (47) is just the one which corrects the erroneous formula (29) of the paper [1] (as well as the other formulae corresponding to other cases considered in the cited paper).

It should be stressed that the formula (47) is *exact*. In this form it looks however very complicate because of the complicated structure of the matrix element M_{21}. The latter is polynomial with respect to the coefficients

$\alpha_{\bar{k},k}$, $k = 1, \ldots, n$, and rational with respect to the χ-coefficients. Exposing its linear terms in α's we get for it

$$\begin{aligned} M_{21} &= -i \prod_{k=2}^{n} \chi_{\bar{k}\to k}^{-1} \sum_{k=1}^{n} \mathrm{e}^{iT\int_{s_{\bar{k}}}^{s_k} \sqrt{q_2(s,T)}ds - iT\int_{s_{\bar{1}}}^{s_{\bar{k}}} \sqrt{q_2(s,T)}ds + iT\int_{s_k}^{s_n} \sqrt{q_2(s,T)}ds} \\ &\quad \times \chi_{\bar{k}\to\overline{k+1}} \prod_{l=1}^{k-1} \chi_{\bar{l}\to l+1} \prod_{l=k+1}^{n} \chi_{l\to\overline{l+1}} + \cdots . \end{aligned} \tag{48}$$

Nevertheless in the adiabatic limit $T \to +\infty$ the formulae (47) and (48) simplify greatly since then all χ's coefficients of M_{21} become equal to 1 and in the multiplication of the limiting matrices M_k's, $k = 1, \ldots, n$, all terms containing powers of the factors $\alpha_{\bar{k},k}$, $k = 1, \ldots, n$ higher than the first ones have to be neglected. Further everywhere where $B(s,T) \neq 0$ the adiabatic limits $T \to +\infty$ of $\frac{1}{2}\left(\frac{\dot{c}}{c} - iT\omega\right) \pm iT\sqrt{q_2(s,T)}$ are exactly the same in their forms as those for $s \to \pm\infty$ given by (39). The condition $B(s,T) \neq 0$ is satisfied obviously for the integrals in the formula (47) taken along the real axis. However the phase integrals defining the element M_{12} are taken between zeros of $\sqrt{q_2(s,T)}$ which in the adiabatic limit coincide with the ones of $B_0(s)$. Nevertheless this trouble can be easily avoided by representing the corresponding integrations linking pairs of zeros $(s_{\bar{k}}, s_k)$, $(s_{\bar{1}}, s_{\bar{k}})$ and (s_k, s_n), $k = 1, \ldots, n$, by the ones along closed contours $C_{s_{\bar{k}}s_k}$, $C_{s_{\bar{1}}s_{\bar{k}}}$ and $C_{s_k s_n}$, surrounding respective pairs of zeros. The same idea applies to the two integrations between the pair of points $(s_{\bar{1}}, 0)$ and $(s_n, 0)$ except that the corresponding contours $C_{s_{\bar{1}}0}$ and $C_{s_n 0}$ are not closed but starts and ends at $s = 0$ points lying on two different sheets of $\boldsymbol{R}_B$. Therefore making first use of Eqs. (45) we can apply the asymptotics (39) to all the phase integrals in formulae (47) and (48) so that the former takes the following form when the integration along the real axis is performed

$$\begin{aligned} a_{-}^{\mathrm{ad}}(+\infty, T) &= -i \tan\frac{\Theta_0}{2} \mathrm{e}^{\frac{1}{4}\left(\int_{C_{s_{\bar{1}}s_1}} + \int_{C_{s_{\bar{1}}0}} + \int_{C_{s_n 0}}\right)\left(i\mu TB - \frac{\dot{\Theta}}{\sin\Theta} - i\dot{\phi}\cos\Theta\right)ds} \\ &\quad \times \sum_{k=1}^{n} \mathrm{e}^{-\frac{1}{4}\left(\int_{C_{s_1 s_k}} - \int_{C_{s_k s_n}}\right)\left(i\mu TB - \frac{\dot{\Theta}}{\sin\Theta} - i\dot{\phi}\cos\Theta\right)ds}, \end{aligned} \tag{49}$$

where it is assumed that all the quantities are now determined by the asymptotic field $\boldsymbol{B}_0$.

In the present form of the formula (49) only the integrations of the middle term $-\frac{\dot{\Theta}}{\sin\Theta}$ in the exponents can be performed explicitly (since $\int \frac{\dot{\Theta}}{\sin\Theta} ds = \ln\tan\frac{\Theta}{2}$). First let us note that because $\frac{\dot{\Theta}}{\sin\Theta} = \frac{1}{2}\left(\frac{\dot{B}_0 - \dot{B}_{0,z}}{B_0 - B_{0,z}} - \frac{\dot{B}_0 + \dot{B}_{0,z}}{B_0 + B_{0,z}}\right)$, so that $\ln\tan\frac{\Theta}{2} = \frac{1}{2}\ln\frac{B_0-B_{0,z}}{B_0+B_{0,z}}$, we can always choose all the integration contours in (49) in such a way to avoid possible roots of $B_{0,z}\pm B_0=0$ so that the unique singularities which remain inside these contours are branch points of $B_0(s)$. Therefore to the corresponding integrals of $-\frac{\dot\Theta}{\sin\Theta}$ along the closed contours $C_{s_1s_k}$, $C_{s_ks_n}$, $k=1,\ldots,n$, and $C_{s_{\bar1}s_1}$ can contribute only roots of the function $F(s)\equiv\frac{B_0(s)-B_{0,z}(s)}{B_0(s)+B_{0,z}(s)}$. Net results of these contributions depends however on details of mapping of the s-Riemann surface on the F-one. If after such a mapping a closed contour C_γ rounds the zero point of the F-plane n_γ times (we take n_γ to be positive for anticlock orientation of the contour and negative for the opposite case) then a contribution of this zero point to the corresponding contour integral of $-\frac{\dot\Theta}{\sin\Theta}$ is $-i\pi n_\gamma$. The remaining two open integrals along the contours $C_{s_{\bar1}0}$ and C_{s_n0} can contribute only by their limits giving

$$-\frac{1}{4}\left(\int\limits_{C_{s_{\bar1}0}}+\int\limits_{C_{s_n0}}\right)\frac{\dot\Theta}{\sin\Theta}=-\ln\tan\frac{\Theta_0}{2}+il\frac{\pi}{2} \tag{50}$$

with some integer l since $B_{0,z}(s)$ is regular at the points $s_{\bar1},s_n$ and values of $B_0(s)$ on both the sheets differ by sign so that $F_2(0)=F_1^{-1}(0)$ where $F_{1,2}(0)$ are values (both real) of $F(s)$ at $s=0$ on the 'first' and 'second' sheets respectively.

Therefore we obtain the following final result

$$\begin{aligned}a_-^{\mathrm{ad}}(+\infty,T)&=-i^{l+1}\mathrm{e}^{-\frac{1}{4}i\pi n_{s_{\bar1}s_1}}\mathrm{e}^{\frac{1}{4}\left(\int_{C_{s_{\bar1}s_1}}+\int_{C_{s_{\bar1}0}}+\int_{C_{s_n0}}\right)\left(i\mu TB-i\dot\phi\cos\Theta\right)ds}\\&\times\sum_{k=1}^{n}\mathrm{e}^{\frac{1}{4}i\pi(n_{s_1s_k}-n_{s_ks_n})}\mathrm{e}^{-\frac{1}{4}\left(\int_{C_{s_1s_k}}-\int_{C_{s_ks_n}}\right)\left(i\mu TB-i\dot\phi\cos\Theta\right)ds}\,.\end{aligned} \tag{51}$$

Since $\dot\phi\cos\Theta=\frac{B_{0,z}}{B_0}\frac{B_{0,x}\dot B_{0,y}-B_{0,y}\dot B_{0,x}}{B_0^2-B_{0,z}^2}$ we can shrink all the integrations in (51) to paths linking respective points to get

$$\begin{aligned}a_-^{\mathrm{ad}}(+\infty,T)&=-i^{l+1}\mathrm{e}^{-\frac{1}{4}i\pi n_{s_{\bar1}s_1}}\mathrm{e}^{\frac{1}{2}\left(\int_{s_{\bar1}}^{s_1}+\int_{s_{\bar1}}^{0}+\int_{s_n}^{0}\right)\left(i\mu TB-i\dot\phi\cos\Theta\right)ds}\\&\times\sum_{k=1}^{n}\mathrm{e}^{\frac{1}{4}i\pi(n_{s_1s_k}-n_{s_ks_n})}\mathrm{e}^{-\frac{1}{2}\left(\int_{s_1}^{s_k}-\int_{s_k}^{s_n}\right)\left(i\mu TB-i\dot\phi\cos\Theta\right)ds}\,.\end{aligned} \tag{52}$$

We should remember that all the integrations in (52) run along paths avoiding roots of the equations $B_{0,z}\pm B_0=0$.

For the corresponding transition probability we obtain

$$P_-^{\rm ad}(T) = {\rm e}^{-\Im\int_{s_{\bar 1}}^{s_1}\left(\mu TB-\dot\phi\cos\Theta\right)ds} \times \left|\sum_{k=1}^{n}{\rm e}^{\frac14 i\pi(n_{s_1s_k}-n_{s_ks_n})}{\rm e}^{-\frac12\left(\int_{s_1}^{s_k}-\int_{s_k}^{s_n}\right)\left(i\mu TB-i\dot\phi\cos\Theta\right)ds}\right|^2. \tag{53}$$

Formulae similar to (52) and (53) have been found by Joye, Mileti and Pfister [5]. In fact if we apply the assumptions made in the last paper by its authors these formulae become identical, up to an overall phase in (52), with the corresponding ones found by the authors mentioned.

The last formulae take on particularly simple forms for the case of two turning points lying on the upper Stokes line drawn on Fig. 3. when the equations $B_{0,z}\pm B_0=0$ have no solutions inside the strip bounded by the two Stokes lines on Fig. 3 and on the lines themselves. We can then deform all integration paths in the formula (52) to ones along the Stokes and antyStokes lines so that the corresponding integrals will have explicitly pure real or pure imaginary values. We get for this case

$$\begin{aligned}
a_-^{\rm ad}(+\infty,T) &= -i^{l+1}{\rm e}^{-\frac14 i\pi n_{s_{\bar 1}s_1}}{\rm e}^{\frac12\left(\int_{s_{\bar 1}}^{s_1}+\int_{s_{\bar 1}}^{0}+\int_{s_2}^{0}\right)\left(i\mu TB-i\dot\phi\cos\Theta\right)ds}\\
&\times\left({\rm e}^{\frac14 i\pi n_{s_1s_2}}{\rm e}^{-\frac12\int_{s_1}^{s_2}\left(i\mu TB-i\dot\phi\cos\Theta\right)ds}+{\rm e}^{-\frac14 i\pi n_{s_1s_2}}{\rm e}^{\frac12\int_{s_1}^{s_2}\left(i\mu TB-i\dot\phi\cos\Theta\right)ds}\right)\\
&= -2i^{l+1}{\rm e}^{-\frac14 i\pi n_{s_{\bar 1}s_1}+\frac12 i\Re\left(+\int_{s_{\bar 1}}^{0}+\int_{s_2}^{0}\right)\left(\mu TB-\dot\phi\cos\Theta\right)ds}{\rm e}^{-\frac12\Im\int_{s_{\bar 1}}^{s_1}\left(\mu TB-\dot\phi\cos\Theta\right)ds}\\
&\times\cos\left(\frac12\Re\int_{s_1}^{s_2}\left(\mu TB-\dot\phi\cos\Theta\right)ds-\frac14\pi n_{s_1s_2}\right)
\end{aligned}\tag{54}$$

so that for the corresponding transition probability we get

$$P_-^{\rm ad}(+\infty,T) = {\rm e}^{-\Im\int_{s_{\bar 1}}^{s_1}\left(\mu TB-\dot\phi\cos\Theta\right)ds} \times \cos^2\left(\frac12\Re\int_{s_1}^{s_2}\left(\mu TB-\dot\phi\cos\Theta\right)ds-\frac14\pi n_{s_1s_2}\right). \tag{55}$$

7. Examples of NED systems

An example of a class of fields $\boldsymbol{B}$ with the NED property has been considered recently by Berman *et al.* [17]. The fields are defined by putting $B_z(sT,T)=B_\infty$, $B_x(sT,T)=f(s)\cos(\omega_0 sT)$, $B_y(sT,T)=f(s)\sin(\omega_0 sT)$ with $f(s)$ having the properties $\mathbf{1}^\circ$–$\mathbf{3}^\circ$ of the field $\boldsymbol{B}$ and vanishing at both infinities of the real axis. This problem is however unitary equivalent to the one with the field $\boldsymbol{B}=[f(s),0,B_\infty-\frac{\omega_0}{\mu}]$ so that for this case we have $B=\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}$ where $\Omega=\mu B_\infty-\omega_0$ and $\dot\phi\equiv 0$.

Assuming for $f(s)$ properties desired by the assumption $\mathbf{1}^\circ$–$\mathbf{8}^\circ$ of Sec. 2 we get using the formula (52)

$$a_-^{\mathrm{ad}}(+\infty,T) = -i^{l+1}\mathrm{e}^{-\frac{1}{4}i\pi n_{s_{\bar{1}}s_1}}\mathrm{e}^{\frac{1}{2}i\mu T\left(\int_{s_{\bar{1}}}^{s_1}+\int_{s_{\bar{1}}}^{0}+\int_{s_n}^{0}\right)\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}ds}$$
$$\times\sum_{k=1}^{n}\mathrm{e}^{\frac{1}{4}i\pi(n_{s_1s_k}-n_{s_ks_n})}\mathrm{e}^{-\frac{1}{2}i\mu T\left(\int_{s_1}^{s_k}-\int_{s_k}^{s_n}\right)\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}ds}, \tag{56}$$

where $s_{\bar{1}}$ and $s_k,\ k=1,\ldots,n$, are roots of the equations $f(s)=\pm i\frac{\Omega}{\mu}$.

If there are only two turning points s_1 and s_2 then according to formula (54) we get

$$a_-^{\mathrm{ad}}(+\infty,T) = -2i^{l+1}\mathrm{e}^{-\frac{1}{4}i\pi n_{s_{\bar{1}}s_1}+\frac{1}{2}i\mu T\Re\left(+\int_{s_{\bar{1}}}^{0}+\int_{s_2}^{0}\right)\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}ds}$$
$$\times\mathrm{e}^{-\frac{1}{2}\mu T\Im\int_{s_{\bar{1}}}^{s_1}\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}ds}\cos\left(\frac{1}{2}\mu T\Re\int_{s_1}^{s_2}\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}ds-\frac{1}{4}\pi n_{s_1s_2}\right). \tag{57}$$

It is now not difficult to establish that the contour $C_{s_1s_2}$ rounds the zero point on the F-plane twice (see Fig. 4 and Fig. 5). Therefore we obtain finally for this case

$$a_-^{\mathrm{ad}}(+\infty,T) = -2i^{l+1}\mathrm{e}^{-\frac{1}{4}i\pi n_{s_{\bar{1}}s_1}+\frac{1}{2}i\mu T\Re\left(+\int_{s_{\bar{1}}}^{0}+\int_{s_2}^{0}\right)\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}ds}$$
$$\times\mathrm{e}^{-\frac{1}{2}\mu T\Im\int_{s_{\bar{1}}}^{s_1}\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}ds}\sin\left(\frac{1}{2}\mu T\Re\int_{s_1}^{s_2}\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}ds\right) \tag{58}$$

and for the corresponding transition amplitude

$$P_-^{\mathrm{ad}} = 4\mathrm{e}^{-\mu T\Im\int_{s_{\bar{1}}}^{s_1}\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}ds}\sin^2\left(\frac{1}{2}\mu T\Re\int_{s_1}^{s_2}\sqrt{\left(\frac{\Omega}{\mu}\right)^2+f^2(s)}ds\right). \tag{59}$$

The last two formulae have been obtained earlier by Nikitin and Umanskii [18] as well as by Crothers [20] and by Davies and Pechukas [21] using the steepest-descent methods.

As a second example we shall consider again the Nikitin Hamiltonian for the atom–atom scattering. The model of Nikitin [19] describes the scattering $A*+B\to A+B+\Delta\epsilon$ of the exited atom $A*$ moving with a small

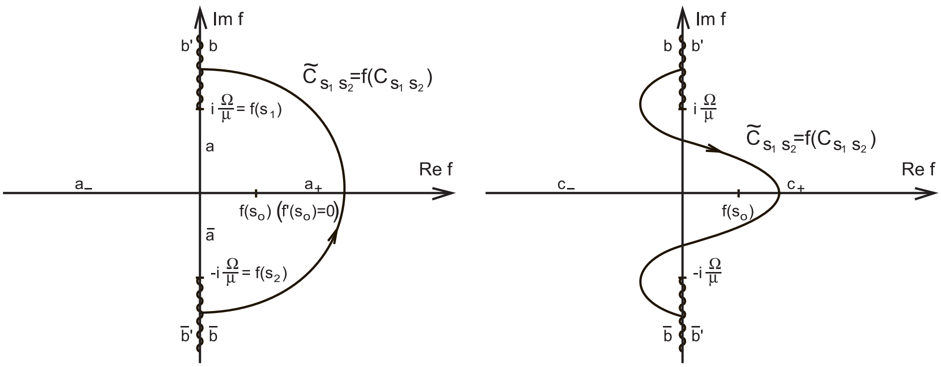

Fig. 4. The integration contour $C_{s_1 s_2}$ mapped into the f-plane.

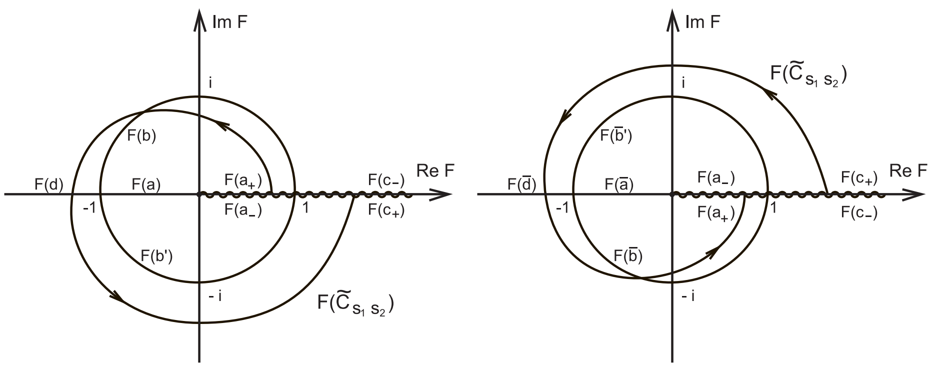

Fig. 5. The integration contour $C_{s_1 s_2}$ mapped into the F-plane.

velocity v with the impact parameter b' and scattered by the atom B. The interaction between the atoms is of the dipol–dipol type. The latter example was analyzed in the context of the adiabatic limit $v \to 0$ also by Joye *et al.* [5].

The Hamiltonian for this system reads ([18], paragraph 9.3.2 and [19]):

$$H(R) = \begin{bmatrix} \frac{\Delta\epsilon}{2} & \frac{C}{R^3} \\ \frac{C}{R^3} & -\frac{\Delta\epsilon}{2} \end{bmatrix}, \tag{60}$$

where $\Delta\epsilon$ and C are constants and $R = \sqrt{b'^2 + v^2 t^2}$ is the distance between the atoms. Introducing $d = (2C/\Delta\epsilon)^{\frac{1}{3}}$ as a natural distant unit for this case and $T = d/v$ as the corresponding adiabatic parameter and rescaling:

$t \to sT$ and $b' \to bd$ we get from (60):

$$H(s) = \frac{\Delta\epsilon}{2}\begin{bmatrix} 1 & \frac{1}{(b^2+s^2)^{\frac{3}{2}}} \\ \frac{1}{(b^2+s^2)^{\frac{3}{2}}} & -1 \end{bmatrix}. \tag{61}$$

In the 'magnetic field' language we have of course $\boldsymbol{B}(sT,T) = \left(\left(b^2+s^2\right)^{-\frac{3}{2}},0,1\right)\frac{\Delta\epsilon}{\mu}$ so that all the assumptions $\mathbf{1}^\circ - \mathbf{8}^\circ$ above are satisfied with $\boldsymbol{B}^{\pm}(T) = \boldsymbol{B}^{\pm}(\pm\infty,T) = (0,0,1)\frac{\Delta\epsilon}{\mu}$.

Obviously the last form of the $\boldsymbol{B}$-field shows that it belongs to the class of Berman *et al.* with two turning points on the "main" Stokes line (see Fig. 1) so that the formulae (58) and (59) are applicable readily.

8. Discussion and conclusions

In our present calculations of the adiabatic limit for the transition amplitudes in the two energy level systems we have corrected erroneous formulae of our previous paper [1]. We have considered systems with the NED properties, *i.e.* for which their corresponding Stokes graphs do not differ essentially from their adiabatic limit forms. We have shown that for the two energy level systems the method of fundamental solutions provides us in an elegant way both with the exact and with approximate results the latter obtained in the adiabatic limit $T \to +\infty$.

A formula (52) which gives the corresponding transition amplitudes in the adiabatic limit shows that these amplitudes result as an interference of contributions coming from all complex conjugated pairs of turning points lying on the same complex conjugated Stokes lines of the respective limiting Stokes graph. Up to an overall phase it coincides with the one of Joye, Mileti and Pfister [5].

Let us however discuss by a moment basic differences between our approach and the authors just mentioned. In fact these approaches differ essentially by using different solutions and different methods of their analytical continuations along corresponding Stokes graphs. In our case we consider both assumed global properties of corresponding Stokes graphs and a definite set of exact solutions accompanied them — the funadamental solutions. Analytical continuations of the latter are performed along canonical paths leaving turning points of the Stokes graphs far away. Such a continuation ensures to get both an exact and immediately an approximate formulae for the transition amplitudes. No detailed estimations of the adiabatic limit at each step of such analytical continuation are necessary. This limit can be obtained almost automatically for final exact results.

Contrary to our method the one of the authors of [5] uses (approximate) solutions continued along the "basic" Stokes line and because of that passing by turning points lying on this line. Of course these are these latter points which demand detailed estimations of solutions of corresponding comparison equations used by the authors mentioned for analytic continuation, since turning points become singular for these solutions in the adiabatic limit.

A particularly simple formula for the transition amplitudes follows from a general one (54) when the latter is applied to the NED systems considered by Berman *et al.* [17] with two turning points on the "main" Stokes line. Namely, it obtains then the form (58) found earlier by Nikitin and Umanskii [18] as well as by Crothers [20] and by Davies and Pechukas [21] using the steepest-descent methods.

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REFERENCES

[1] S. Giller, C. Gonera, *Phys. Rev.* **A63**, 052102 (2001); Erratum: *Phys. Rev.* **A67**, 0699XX(E) (2003).

[2] L.D. Landau, E.M. Lifshitz, *Quantum Mechanics. Nonrelativistic Theory*, Pergamon, New York 1965.

[3] L. Davidovich, *Quantum Optics in Cavities and the Classical Limit of Quantum Mechanics*, in: *Latin-American School of Physics XXXI ELAF. New Perspectives on Quantum Mechanics 1998* Eds. Shahen Hacyan, Rocio Jauregui, AIP, Woodburg, New York 1999; S. Haroche, *Cavity Quantum Electrodynamics: A Review of Rydberg Atom — Microwave Experiments on Entanglement and Decoherence*, in: *Latin-American School of Physics XXXI ELAF. New Perspectives on Quantum Mechanics 1998* Eds. Shahen Hacyan, Rocio Jauregui, AIP, Woodburg, New York 1999.

[4] A. Joye, H. Kunz, Ch.-Ed. Pfister, *Ann. Phys.* **208**, 299 (1991).

[5] A. Joye, G. Mileti, Ch.-Ed. Pfister, *Phys. Rev.* **A44**, 4280 (1991).

[6] A. Joye, Ch.-Ed. Pfister, *J. Phys.* **24**, 753 (1991).

[7] A. Joye, Ch.-Ed. Pfister, *Phys. Lett.* **A169**, 62 (1992).

[8] A. Joye, Ch.-Ed. Pfister, *J. Math. Phys.* **34**, 454 (1993).

[9] N. Fröman, P.O. Fröman, *JWKB Approximation. Contribution to the Theory*, North-Holland, Amsterdam 1965.

[10] M.V. Fedoriuk, *Asymptotic Methods for Linear Ordinary Differential Equations*, Nauka: Moscow 1983 (in Russian).

[11] S. Giller, *Acta Phys. Pol. B* **23**, 457 (1992); *J. Phys.* **A33**, 1543 (2000); S. Giller, P. Milczarski, *J. Phys.* **32**, 955 (1999); *J. Phys.* **33**, 357 (2000); S. Giller, `quant-ph/0107021`.

[12] L.D. Landau, *Collected Papers of L.D. Landau*, Pergamon, Oxford 1965.

[13] C. Zener, *Proc. R. Soc.* **137**, 696 (1932).

[14] A.M. Dykhne, *Zh. Eksp. Teor. Fiz.* **41**, 1324 (1961).

[15] S. Lange, *Complex Analysis*, Springer, New York 1993.

[16] R.E. Langer, *Phys. Rev.* **51**, 669 (1937).

[17] P.R. Berman, Lixin Yan, Keng-Hwee Chiam, Ruwang Sung, *Phys. Rev.* **A57**, 79 (1998).

[18] E.E. Nikitin, S.Ya. Umanskii, *Theory of Slow Atomic Collisions*, Springer-Verlag, Berlin 1984.

[19] E.E. Nikitin, *Chem. Phys. Lett.* **2**, 402 (1968).

[20] D.S.F. Crothers, *J. Phys.* **A5**, 1680 (1972); *J. Phys. B* **6**, 1418 (1973).

[21] J.P. Davis, P. Pechukas, *J. Chem. Phys.* **60**, 3129 (1976).