ZERO-ENERGY QUANTUM STATES FOR A CLASS OF NONCENTRAL POTENTIALS AND AN EXACT CLASSICAL LIMIT

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We extend results of the recent paper by Kobayashi and Shimbori [*Phys. Rev.* **A65**, 042108 (2002)] to a large class of noncentral potentials. Namely, we have shown that zero-energy states of the central potentials considered by these Authors [$V_a(\rho) = -a^2 g_a \rho^{2(a-1)}$ with $\rho = \sqrt{x^2+y^2}$ and $a \neq 0$] and noncentral potentials discussed here, have both common set of solutions given by wave functions of the parabolic potential barrier (PPB). Moreover, it is observed that first few members of the infinite set of functions cancel the quantum correction to the classical Hamilton–Jacobi equation. The exact classical limit of quantum mechanics is thus precisely reached for them with no approximation involved.

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1. Introduction

Studying of the Schrödinger equation with the parabolic potential barrier (PPB), that is, with the inverted harmonic oscillator potential, has a very long history [1]. The one-dimensional (1D) barrier of this shape has been considered as a convenient model of an unstable system in quantum mechanics [2–6]. Some other important applications of the model include masers [7], reactive scattering [8], paradoxical aspects of the tunneling [9], anomalous symmetry breaking in quantum mechanics [10], semiconductor physics [11] and chemical problems [12, 13].

Solutions of the Schrödinger equation for the PPB are well-known both for the 1D [1, 2, 5, 11] and 2D [14, 15] variants of the barrier. The wave functions for the latter case have two very interesting features. Namely, it is a quite surprising and remarkable observation made recently by Kobayashi

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and Shimbori [15], that all the zero-energy ($E = 0$) solutions for the class of the 2D central potentials

$$V_a(\rho) = -a^2 g_a \rho^{2(a-1)} \tag{1}$$

with $\rho = \sqrt{x^2 + y^2}$, $(a \neq 0) \in \mathbb{R}$, $g_a = \text{const.} > 0$, can be represented by the same functions as for the 2D PPB. To prove that, one needs to use a suitable conformal mapping. The other interesting property of the states is that the infinite degeneracy in the PPB case also appears for all potentials $V_a(\rho)$.

The states corresponding to the value of $E = 0$ were recently used for creation of vortex patterns [15] and vortex lattices [16] in quantum mechanics. It was also shown [17–21] that some of the states lead to the exact classical limit of quantum mechanics, *i.e.* to the case when the quantum correction to the classical Hamilton–Jacobi equation is exactly zero.

In this paper, we shall show that the recent results of Kobayashi and Shimbori [15] can all be almost trivially extended to a large class of *noncentral* potentials. In consequence, their wave functions have again the same two important features as those mentioned above for the 2D *central* parabolic potential barrier.

The plan of our paper is as follows. In Sec. 2 we shall show how to reduce the Schrödinger equation to the problem of the inverted 2D oscillator. Thus, the known solutions of the model can be used at once to a large class of potentials, including noncentral ones. In Sec. 3, vanishing of the Bohm potential, that is, the quantum correction to the classical equation of the motion, will be proved for some of the solutions. The paper concludes with Sec. 4.

2. Conformal mapping

Let us consider the stationary Schrödinger equation in 2D space of variables (x, y)

$$\left[-\frac{\hbar^2}{2m}\Delta_{x,y} + V(x,y)\right]\psi(x,y) = E\psi(x,y)\,. \tag{2}$$

Now, we can use a conformal transformation, which is any analytic transformation between two complex variables, say $w = f(z)$, with dw/dz different from zero. If the function $f(z) = u(x,y) + iv(x,y)$ is analytic at the point $z = x + iy$, then with the help of the Cauchy–Riemann equations $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$, we can write for the Laplacian operator $\Delta_{x,y}$

$$\Delta_{x,y} = \left|f'(z)\right|^2 \Delta_{u,v}\,, \tag{3}$$

where $f'(z) = df/dz$ and $\varDelta_{u,v} = \partial^2/\partial u^2 + \partial^2/\partial v^2$. This well-known relation can be used for an extension of the wave functions discussed in Ref. [15] to the case of noncentral potentials.

To this end, let us specify the potentials as

$$V(u,v) = -\frac{c^2}{2m} \left|f'(z)\right|^2 \left(u^2 + v^2\right) . \tag{4}$$

Then, for $E = 0$, we have

$$\left[\varDelta_{u,v} + \frac{c^2}{\hbar^2} \left(u^2 + v^2\right)\right] \psi(u,v) = 0 . \tag{5}$$

This is the well-known equation of the 2D isotropic parabolic potential barrier in the (u,v) plane. The exact solutions of the eigenvalue problem for arbitrary E [14] correspond to the imaginary eigenvalues given by $\pm i(n_u + n_v + 1)\hbar c/m$ and $\pm i(n_u - n_v)\hbar c/m$. Zero-energy states are in the latter case obtained for zero and positive integers satisfying $n_u = n_v$.

In both cases, the solutions of Eq. (5) can be given in an explicit form. Since the way of finding them and their properties has already been discussed in detail [14, 15], we shall restrict ourselves to the list of solutions for zero-energy states only, and propose a concise comment on them. This choice is motivated by the content of the next section of our paper.

Let us begin with the simplest zero-energy solutions of Eq. (5) which are represented by two-dimensional plane waves

$$\psi_0(u,v) = N \exp\left\{\pm\frac{ic}{2\hbar}\left[\left(u^2 - v^2\right)\cos\alpha + 2uv\sin\alpha\right]\right\} \tag{6}$$

with N being, in general, a complex constant, and where α is an arbitrary angle. Because of the freedom in choosing the value of α there are infinitely many functions of the form of Eq. (6) corresponding to the energy $E = 0$. The signs $+/-$ distinguish between outward/inward moving particles.

Another set of solutions can be generated by an analytic continuation method from the solutions of the ordinary 2D harmonic oscillator. Thus, we have

$$\psi_n(u,v) = M H_n^{\pm}\left(u\sqrt{\frac{c}{\hbar}}\right) H_n^{\mp}\left(v\sqrt{\frac{c}{\hbar}}\right) \exp\left[\pm\frac{ic}{2\hbar}\left(u^2 - v^2\right)\right] , \tag{7}$$

where the polynomials $H_n(\xi)$ are derivable from the relation [5] $H_n^{\pm}(\xi) = (\mp i)^n \exp(\mp i\xi^2)(d^n/d\xi^n)\exp(\pm i\xi^2)$ with $[H_n^{\pm}(\xi)]^* = H_n^{\mp}(\xi)$, and $n = 1, 2, \ldots$, whereas M is an arbitrary complex constant. The functions (7) also correspond to the energy $E = 0$ and we meet here one more type of infinite degeneracy arising from the freedom in using various values of n.

For completeness, we add a solution of Eq. (5) which is not derivable by the method outlined in Refs. [14] and [15]. It reads

$$\psi_s(u,v) = \left(u^2 - v^2\right) \exp\left[\pm \frac{ic}{\hbar} uv\right] \tag{8}$$

and the method of obtaining the function will be given in the next section.

Once we have described explicitly the solutions of Eq. (5) we can give few examples of potentials to which they apply. If the mapping $f(z) = z^{a/2}$ is used, then, with $c^2/8m = g_a$, we can derive at once from Eq. (4) all the central potentials in Eq. (1). For $a = 2$, the inverted 2D oscillator potential is mapped to itself. Other choices for the function $f(z)$ lead, in principle, to any number of noncentral potentials. If, for example $f(z) = \ln z$, then from Eq. (4), we have $V(x,y) = (-c^2/2m)[(\ln\rho)^2 + (\arctan(y/x))^2]/\rho^2$ [21].

Whenever the potential $V(x,y)$ can be transformed to the form of Eq. (4), we are able to write exact solutions of the corresponding Schrödinger equation. Obviously, finding such transformations may be a difficult task in particular cases. What is, however, the most important fact from the point of view of the next section, is that there exist a huge number not only central but also noncentral potentials with the well-known zero-energy states. It is additionally important and surprising that among them there exist in each case states cancelling the quantum correction to the classical Hamilton–Jacobi equation.

According to Ref. [15] the zero-energy states are interpreted as stationary flows around the parabolic potential barrier. They represent incoming and outgoing flows, corresponding, respectively, to the formation and decay processes of an infinite number of resonances with equal probability. The states do not belong to the ordinary Hilbert space and are generally not normalizable. Except for the solutions $\psi_0(u,v)$, which can be normalized in terms of Dirac δ functions, those given in Eqs. (7) and (8) have to be treated as the eigenfunctions of the conjugate Schwartz space $\mathcal{S}(\mathbb{R}^2)^\times$ of the Gel'fand triplets $\mathcal{S}(\mathbb{R}^2) \subset \mathcal{L}^2(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)^\times$, also called the rigged Hilbert space [4,22,23]. The symbol $\mathcal{L}^2(\mathbb{R}^2)$ stands for the Lebesgue space and both $\mathcal{S}$ and $\mathcal{L}$ are determined in two dimensions.

Even with the above properties, the states from the rigged Hilbert space are important for physics. One may construct, for example, the probability currents, which are observable in scattering processes with possible applications in cold-atom collisions [24] and in hydrodynamics [15, 16]. The states play also very important role in finding the exact classical limit of quantum mechanics [17–21].

3. Exact classical limit

In the literature, the classical limit of quantum mechanics is approached in a number of ways (see paper [20] and references therein). However, its exact form is understood as vanishing of the Bohm or quantum potential [25]

$$Q = \frac{-\hbar^2}{2m}\,\frac{\Delta_{u,v}R}{R}\,, \tag{9}$$

where R represents a real valued amplitude of the wave function under consideration.

Introducing a real phase S, we can write the wave function in the polar form $\psi = R\exp\left[(i/\hbar)S\right]$. Then, with the help of Eq. (5), we can derive two equations satisfied by the functions R and S, which are the classical Hamilton–Jacobi equation, supplemented with the quantum correction Q, and a continuity equation for the stationary case. The classical limit of quantum mechanics is reached for the potentials in Eq. (4), only for the wave functions with amplitudes obeying the relation

$$\Delta_{u,v}R = 0\,, \tag{10}$$

and besides

$$\left(\nabla_{u,v}S\right)^2 = c^2\left(u^2+v^2\right)\,, \tag{11}$$

$$\nabla_{u,v}\cdot\left(R^2\nabla_{u,v}S\right) = 0\,. \tag{12}$$

Thus, for all potentials reducible to the form of Eq. (4), we can easily check which of their states cancel the quantum correction to Hamilton–Jacobi equation. In this way, the motions in classical and quantum mechanics are identical [17–21, 25].

Simple calculations show that $\psi_0(u,v)$ given in Eq. (6) obeys Eqs. (10), (11) and (12) for arbitrary real values of α. Thus, this particular state can be considered for a large number of central and noncentral potentials as a classical wave function and the classical limit of quantum mechanics is thus reached exactly without no approximation involved. Among the states in Eq. (7) there is only one state canceling the Bohm's correction. This is the case just for $n=1$, *i.e.*

$$\psi_1(u,v) = M_1\frac{c}{\hbar}uv\exp\left[\pm\frac{ic}{2\hbar}\left(u^2-v^2\right)\right]. \tag{13}$$

Obviously, Eq. (10) is fulfilled since $R \sim uv$ and Eqs. (11) and (12) too, since $S = \pm(c/2)(u^2-v^2)$. No further states with this property can be traced back from Eq. (7).

We have to point out that there can be noncentral potentials with at least one classical zero-energy state, *i.e.*, obeying Eqs. (10), (11) and (12), for which the Schrödinger equation (2) is not reducible to the 2D PPB problem represented by Eq. (5). An example is given by the potential $V(x,y) = -[\rho^2 + \rho^{-6} - 2\rho^{-2}\sin(4\varphi)]$, where $\rho^2 = x^2 + y^2$ and $\varphi = \arctan(y/x)$. This case and other similar ones were discussed in [21].

As the final point of this section we shall comment on the solution (8). It can be obtained observing that Eq. (12) has *de facto* the form of $2\nabla_{u,v}R \cdot \nabla_{u,v}S + R\Delta_{u,v}S = 0$. Then, whenever $\Delta_{u,v}S = 0$, it follows that the roles played by R and S may be interchanged. In our paper, this is the case for uv and $u^2 - v^2$. The solution, found in this way, is given in Eq. (8) and one can easily check that it obeys Eqs. (10), (11) and (12) as well.

4. Conclusions

In our study of the class of noncentral potentials we have shown that their eigenvalue equation can be reduced to the inverted oscillator problem and then exact solutions were given explicitly. We have thus extended the recent paper by Kobayashi and Shimbori [15] to a large class of noncentral potentials. The close connection of their solutions with the ordinary parabolic potential barrier (PPB), and the fact that some of them realize an exact classical limit of quantum mechanics, are quite surprising.

All the states we have considered here are zero-energy states in Gel'fand triplets. Properties of the states from such rigged Hilbert space are not so well understood as those from the ordinary Hilbert space. Among other reasons it is due to the restricted number of potentials for which exact solutions in terms of Gel'fand triplets are known. Nevertheless, the states are used in a growing number of physical applications. Numerous examples, especially in an analysis of vortices, can be found in [16] and [26]. The infinite degeneracy of the states is the origin of new entropy [27, 28] different from the Boltzmann entropy.

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