EQUALITY OF THE INERTIAL AND THE GRAVITATIONAL MASSES FOR A QUANTUM PARTICLE

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We investigate the interaction of the gravitational field with a quantum particle. We derive the wave equation in the curved Galilean space-time from the very broad Quantum Mechanical assumptions and from covariance under the Milne group. The inertial and gravitational masses are equal in that equation. So, we give the proof of the equality for the non-relativistic quantum particle, without applying the equivalence principle to the Schrödinger equation and without imposing any relation to the classical equations of motion. This result constitutes a substantial strengthening of the previous result obtained by Herdegen and the author.

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1. Introduction

We assume that the coefficients of the Schrödinger equation describing a particle in the fundamental fields are local functions of the space-time coordinates; compare the scalar or the electromagnetic potential for example. At the same time we expect that the coefficients of the wave equation of a freely falling quantum particle in the gravitational field are local functions of space-time coordinates which can be built up in a local way from the fields which describe the space-time. In this paper we consider only the non-relativistic case and represent the Newtonian gravity in the geometric way compatible with the equivalence principle [1–3]. Now the rather unexpected fact comes. Namely, if we assume the wave equation to be generally covariant, then the inertial mass of the quantum particle has to be equal to its gravitational mass. Strictly speaking the covariance under the Milne group is sufficient. Note that the equality is proved independently of the equivalence principle applied to the wave equation. Moreover, as we will see, the form of the wave equation is almost uniquely determined by the covariance condition. The covariance condition should be distinguished from the symmetry condition. The covariance with respect to a group means that the transform of a solution to the wave equation is a solution to the transformed wave equation. The symmetry under the group means in addition that the group is the symmetry group of the absolute elements of the theory in question, compare [4] for the standard terminology. The geometrical objects describing the space-time structure are the absolute objects in our case on account of the fact that we neglect the influence of the particle on the space-time. As we know, this is justified up to the second order effects even for the electromagnetic interactions, compare the semi-classical theory of radiation. The gravitational interaction is extremely weak. By this the neglecting the influence on the space-time is justified.

It should be stressed here that the equality of the inertial and the gravitational mass would not hold if the standard covariance condition with respect to the Galilean group were considered. In general it is impossible to restrict the covariance group to the Galilean one if the gravity is present. But in the wide class of space-times, those which correspond to isolated systems, one can restrict the covariance group in a consistent manner by distinguishing the class of asymptotically inertial frames.

In our previous paper [5] we get the same result, but with the help of the equivalence principle. That is we have assumed that the gravitational effects may be "transformed away" by an appropriate reference frame. By this the ordinary "flat" form can be given to the wave equation by an appropriate coordinate transformation. Here we do not make any use of the equivalence principle.

2. Derivation of the wave equation

Our assumptions are, more precisely, as follows:

(i) The quantum particle, when its kinetic energy is small in comparison to its rest energy mc^2, does not exert any influence on the space-time structure.

(ii) The Born interpretation for the wave function is valid, and the transition probabilities in the Newton–Cartan space-time which describes geometrically Newtonian gravity, are equal to the ordinary integral over a simultaneity hyperplane and are preserved under the coordinate transformations.

(iii) The wave equation is linear, of second order at most, generally covariant, and can be built in a local way with the help of the geometrical objects describing the space-time structure.

(iv) The probability density $\rho(X) = \psi^*\psi(X)$ is a scalar field, that is it has the following transformation law $\rho'(X') = \rho(X)$ (where all space-time coordinates are denoted by X).

In fact the conditions *(i)*, *(ii)*, *(iii)* and *(iv)* are somewhat interrelated. For example the linearity of the wave equation is deeply connected with the Born interpretation. It will be shown below, that the equality of inertial and gravitational masses for a spin-less non-relativistic particle is a consequence of *(i)*, *(ii)*, *(iii)* and *(iv)*.

We derive the most general form of the wave equation fulfilling *(i)*, *(ii)*, *(iii)* and *(iv)*. A great simplification follows from the fact that in the Newton-Cartan theory the absolute elements exist[1]. The absolute elements fix the privileged, *i.e.* non-rotating Cartesian, coordinates. In those coordinates the absolute elements take on a particularly simple form. The transformations connecting any two privileged coordinate frames form a group called the Milne group [7]. The existence of such privileged frames largely simplifies the investigation of the consequences of general covariance. The simplification has its source in the fact that the absolute elements are invariant under the Milne group and have the same *canonical* form in all privileged frames. This implies that the Newtonian potential ϕ is the only object, which describes the geometry and does not trivially simplify to a constant equal to 0 or 1, in these coordinates. The wave equation written in the privileged coordinates is covariant under the Milne group in consequence of the general covariance. The Milne group is sufficiently rich to determine the wave equation as the covariant equation under the group if we use the assumptions *(i)*, *(ii)*, *(iii)*. We confine ourselves then, to the privileged frames and the Milne group of transformations r:

$$(t, x_j) \rightarrow (t + b, R^i_j x_i + A_j(t))\,, \tag{1}$$

where R^i_j is a rotation matrix, and $A_j(t)$ are "arbitrary" functions of time. Strictly speaking it is sufficient to consider a finite-dimensional subgroup of the Milne group of polynomial $A_j(t)$ of appropriately high degree. It follows from assumption *(iv)* that the wave function $\psi(X)$ of a spin-less particle has the following transformation form

$$\psi'(X) = T_r\psi(X) = e^{-i\theta(r,X)}\psi(r^{-1}X), \tag{2}$$

where we denote the Milne transformation (1) by r. $\theta(r, X)$ is a real function.

[1] The simplification does not exist in the General Relativity. But we expect the same result for the bare masses in the relativistic theory. But the argumentation should be within the path-integral formalism for the Feynman propagator of a structureless particle, see [6].

The exponent $\xi(r,s,t) = \theta(rs,X) - \theta(r,X) - \theta(s,r^{-1}X)$ in the relation

$$T_r T_s = e^{i\xi(r,s,t)} T_{rs}, \tag{3}$$

depends on r, s and also on the time t. The nontrivial time dependence of ξ originates from the gauge freedom of the wave equation which cannot *a priori* be excluded if the gravitational field is present. A well-known Bargmann's theory [8] provides a classification of exponents ξ which are time independent. In Ref. [9] a general classification of ξ's has been presented for a time dependent exponents as is necessary for the present article. In fact this explicit time dependence of the exponent ξ is necessary to account for the experiments [10–12]. That is, the gauge freedom is needed, compare [13,14] or [15] for the simplest wave equation in the gravitational field which does possess the gauge freedom.

The classification of all possible ξ-s gives us the classification of all possible θ-s in (2). The most general θ has the form [9]

$$\theta(r,X) = -\gamma_1 \frac{d}{dt} A_j x^j - \ldots - \gamma_n \frac{d^n}{dt^n} A_j x^j + \tilde{\theta}(r,t), \tag{4}$$

where $\tilde{\theta}$ is any function of r and time t and γ_i in (4) are some arbitrary constants. The coefficients $a, b^i, \ldots$ in the wave equation

$$\Big[a\partial_t^2 + b^i \partial_i \partial_t + c^{ij} \partial_i \partial_j + f^i \partial_i + d\partial_t + g \Big] \psi = 0$$

are local functions of the potential and cannot depend on the arbitrary high order derivatives of the potential. From *(iii)* it follows then, that the coefficients are functions of the potential and its derivatives up to a (say) k-th order. In the mathematical terminology this means that $a, \ldots, g$ are *differential concomitants* of the potential, see [16]. We assume in addition that $k = 2$. We do not lose any generality by this assumption, beside this the whole reasoning could be applied for any finite k. But the case with $k > 2$ would not be physically interesting. Namely, it is *a priori* possible that the derivatives of second order are discontinuous, such that the derivatives of order $k > 2$ do not exist, at least the classical geometry does allow such a situation. On the other hand there does not exist any mathematical obstruction for a discontinuity of the wave equation coefficients, take for example the wave equation with the "step-like" potential. Then the assumptions about the existence of higher oder derivatives which are not necessary for the space-time geometry, confines our reasoning rather then generalizes it.

To simplify the reading we write the explicit form of the transformation laws for ϕ and its derivatives.

$$\begin{aligned}
\phi'(X') &= \phi(X) - \ddot{A}^i x_i\,, \\
\partial'_j\phi'(X') &= {R^{-1}}^i_{\ j}\partial_j\phi(X) - {R^{-1}}^i_{\ j}\ddot{A}_i\,, \\
\partial'_i\partial'_j\phi'(X') &= {R^{-1}}^k_{\ i}{R^{-1}}^s_{\ j}\partial_k\partial_s\phi(X)\,, \\
\partial'_t\phi'(X') &= \partial_t\phi(X) - \dddot{A}^i x_i - {R^{-1}}^i_{\ k}\dot{A}^k\partial_i\phi(X) + {R^{-1}}^i_{\ k}\dot{A}^k\ddot{A}_i\,, \\
\partial'_j\partial'_t\phi'(X') &= {R^{-1}}^i_{\ j}\partial_i\partial_t\phi(X) \\
&\quad -{R^{-1}}^r_{\ k}{R^{-1}}^i_{\ k}\dot{A}^k\partial_r\partial_i\phi(X) - {R^{-1}}^r_{\ j}\dddot{A}_r\,, \\
{\partial'_t}^2\phi'(X') &= {\partial_t}^2\phi(X) - 2{R^{-1}}^i_{\ k}\dot{A}^k\partial_i\partial_t\phi(X) \\
&\quad +{R^{-1}}^j_{\ k}{R^{-1}}^i_{\ s}\dot{A}^k\dot{A}^s\partial_j\partial_i\phi(X) \\
&\quad -\ddddot{A}_i x^i - {R^{-1}}^i_{\ k}\ddot{a}^k\partial_k\phi(X) + {R^{-1}}^i_{\ k}\ddot{A}^k\ddot{A}_i + 2{R^{-1}}^i_{\ k}\dot{A}^k\dddot{A}_i\,.
\end{aligned}$$

Now, we insert the formulas (2) and (4) to the covariance condition of the wave equation. The covariance condition gives us the transformation formulas for the coefficients in the wave equation under the Milne group. For the coefficients a, b^i, c^{ij} the transformations reads

$$b'^i(X') = R^i_j b^j(X) + 2a(X)\dot{A}^i, \tag{5}$$

$$a'(X') = a(X), \tag{6}$$

$$c'^{ij}(X') = R^i_s R^j_k c^{sk}(X) + a(X)\dot{A}^i\dot{A}^j + b^k R^i_k \dot{A}^j, \tag{7}$$

where the dot stands for the time derivative. The formula (5) is valid in each privileged system and for any potential, and implicitly at any space-time point. Let us take then, such a system and let X_0 be any (but fixed) space-time point. We consider the formula (5) for the special transformations with $R = \mathbf{1}$ and $b = 0$, $\vec{A}(t) = A(t)\vec{n}$, where $\vec{n}$ is a constant in space and time space-like unit vector. The analysis of (5) for $A(t) = \lambda(t - t_0)^4$, then for $A(t) = \lambda(t - t_0)^3$ and at last for $A(t) = \lambda(t - t_0)^2$ with any value of the parameter λ gives the following general form for the coefficient b^k

$$b^k(X) = b^k(\phi, \partial_i\phi, \partial_i\partial_j\phi, \partial_t\phi, \partial_j\partial_t\phi, \partial_t^2\phi) = b^k(\partial_i\partial_j\phi).$$

It means, that b^k is a vector concomitant, at least under rotations, spatial inversion and spatial reflections, of a tensor $\partial_i\partial_j\phi$ of valence 2. As is well known from the theory of geometric objects, such a vector concomitant has to be zero. The argumentation is as follows. Take any privileged system and any point X_0. Apply now the space inversion with the origin in X_0, *i.e.* $R = -\mathbf{1}$ and $\vec{A} = 2\vec{x}_0$, $b = 0$. Then, (5) at X_0 with this inversion gives the equation: $\vec{b}(X_0) = -\vec{b}(X_0)$ because the valence of $\partial_i\partial_j\phi$ is even and $\partial_i\partial_j\phi$ does not change the sign under the inversion. Because the point X_0 and the privileged reference frame can be chosen in an arbitrary way the concomitant $\vec{b} = 0$. From (5) immediately follows, that also $a = 0$. We have reduced our equation to the following form

$$[c^{ij}\partial_i\partial_j + d\partial_t + f^i\partial_i + g]\psi = 0.$$

Covariance condition of the equation under the Milne group gives the following transformation law for f^j

$$f'^j(X') = R^j_i f^i(X) - d\dot{A}^j - 2ic^{ij}\partial_i\theta. \tag{8}$$

First of all let us take notice of the fact that $\gamma_j = 0$ for $j > 4$. Indeed, let $X_0 = (\vec{x_0}, t_0)$ be any point. We apply now a Milne transformation for which all derivatives of $\vec{A}(t)$ disappear at t_0 with the exception of the j-th order derivative. For example, we can choose such a transformation as in the preceding considerations $A(t) = (t-t_0)^j$. Then, we insert the transformation to the law (8). Because the derivatives of the order higher then the 4-th do not appear in the transformation laws for $\phi, \partial_i\phi, \ldots, \partial_t^2\phi$, then (8) at X_0 implies that $\gamma_j = 0$. Note, that f^i is an algebraic function of the potential and its finite order derivatives with the order less or equal then $k = 2$. The natural number n in (4) is then finite and it is equal $k + 2 = 4$ at most. We define the following object

$$\widetilde{f^j} \equiv f^j + 2i\gamma_2 c^{ij}\partial_i\phi + 2i\gamma_3 c^{ij}\partial_t\partial_i\phi,$$

with the following transformation law

$$\begin{aligned} \widetilde{f}'^i(X') &= R^i_s\widetilde{f}^s(X) - (d - 2i\gamma_1 c^{sj}R^i_s \\ &\quad -2i\gamma_3 {R^{-1}}^i_s {R^{-1}}^q_p c^{sk}\partial_q\partial_k\phi\delta^{pj})\dot{A}_j - 2i\gamma_4 c^{sj}R^i_s\dddot{A}_j\,. \end{aligned} \tag{9}$$

A similar analysis as this applied to b^k shows that $\widetilde{f}^k = 0$, or equivalently

$$f^k = -2i\gamma_2 c^{ij}\partial_j\phi - 2i\gamma_3 c^{ij}\partial_j\partial_t\phi.$$

But this is possible only if $\gamma_2 = \gamma_3 = 0$ or equivalently, only if $f^k = 0$. Indeed, applying the transformation laws for $\partial_i\phi$ and $\partial_i\partial_t\phi$ to the above formula one gets the transformation law for f^k

$$\begin{aligned} f'^i(X') &= R^i_s f^s(X) - 2i\gamma_3 {R^{-1}}^i_s {R^{-1}}^q_p c^{sk}\partial_k\partial_q\phi\dot{A}^p \\ &\quad -2i\gamma_2 R^i_s c^{sk}\ddot{A}_k - 2i\gamma_3 R^i_s c^{sk}\dddot{A}_k\,. \end{aligned} \tag{10}$$

Consider the Milne transformation with $R \neq \mathbf{1}$ and $A^i(t) = (t - t_0)^2 n^i$ such that $v^j \equiv c_0^{ij} n_i \neq 0$, where $c_0^{ij} = c_{ij}(X_0)$. This is possible because $c_0^{ij} \neq 0$. Note, that if $c_0^{ij} = 0$, the analysis for f^k reduces to the case such as with b^k and $f^k = 0$. Comparing (10) with (8) at X_0 for this Milne transformation one gets

$$\gamma_2 R_i^j v^i = \gamma_2 v^j \,,$$

for all orthogonal R and $\vec{v} \neq 0$, which means that $\gamma_2 = 0$. In the similar way, but with $A^i = (t - t_0)^3 n^i$, one shows that $\gamma_3 = 0$. Summing up $f^k = 0$. Now, looking back to the transformation law (8) we realize that

$$\partial_j \theta = -\gamma_1 \dot{A}_j, c^{ij} = c\delta^{ij}, c \equiv \frac{d}{2i\gamma_1}, \tag{11}$$

where c is a scalar field: $c'(X') = c(X)$ which follows from the fact that c^{ij} is a tensor field, compare (7) and recall that $b^k = 0$ as well as $a = 0$. Note that γ_1 is the inertial mass of the particle in question and by this $\gamma_1 \neq 0$. The wave equation must be of the form

$$\left[\frac{k}{2\gamma_1}\delta^{ij}\partial_i\partial_j + ik\partial_t + g\right]\psi = 0,$$

where we introduce $ik \equiv d$. The covariance condition of the equation gives the following transformation law of g

$$\begin{aligned} g'(X') \;&=\; g(X) - \frac{k\gamma_1}{2}\dot{A}_i\dot{A}^i + k\gamma_1\dot{A}_i\dot{A}^i \\ &\quad -k\partial_t\widetilde{\theta}(A^k, t) - k\gamma_1\ddot{A}_i x^i \,. \end{aligned}$$

Let us define a new object

$$\Lambda(X) = g(X) + \gamma_1 k(X)\phi(X).$$

It is clear that the transformation law of Λ is as follows

$$\Lambda'(X') = \Lambda(X) + \frac{k\gamma_1}{2}\dot{A}_i\dot{A}^i - k\partial_t\widetilde{\theta}(A^k, t).$$

Both $\widetilde{\theta}$ and Λ taken separately are not uniquely defined. This is because the potential ϕ is determined up to a time dependent additive term, namely, the gauge freedom term. So, one can assume any form for $\widetilde{\theta}$ by an appropriate gauge redefinition of ϕ changing the first one by $G(t)$ and the second one by $(1/\gamma_1)\dot{G}(t)$. Assume then, that $\widetilde{\theta}$ is chosen in such a way that $\partial_t\widetilde{\theta} = \gamma_1/2\dot{A}_i\dot{A}^i$. After this the above transformation law for Λ takes on the following form $\Lambda'(X') = \Lambda(X)$ and Λ is a scalar field. In the identical way as for b^k we show that $\Lambda = \Lambda(\partial_i \partial_j \phi)$. So, Λ is one of the Kronecker's invariants of the matrix $(\partial_a \partial_b \phi)$. Now, we come back to the equation and easily show that it can be covariant if and only if k is a constant. We get, then, the Schrödinger equation which after the standard notation of constants has the form

$$\left[\frac{\hbar^2}{2m}\delta^{ij}\partial_i\partial_j + i\hbar\partial_t - m\phi + \Lambda\right]\psi = 0, \tag{12}$$

with the θ in T_r given by

$$\theta = \frac{m}{2\hbar}\int\limits_0^t \dot{\vec{A}}^2(\tau)\,\mathrm{d}\tau + \frac{m}{\hbar}\dot{A}_i x^i .$$

Note that the inertial mass m in the equation is equal to the parameter at the gravitational potential. That is, the gravitational mass must be equal to the inertial mass.

It is remarkable that the wave equation would be covariant with respect to the Galilean group even when $m_i \neq m_g$ because the potential ϕ transforms as a scalar under this group. This is a consequence of the Galilean covariance of the ordinary Schrödinger equation with a scalar potential.

In the paper [5] the same wave equation has been derived with the additional result $\Lambda = 0$ as a consequence of the equivalence principle applied to the wave equation.

Equality of inertial and gravitational masses in the quantum regime was verified experimentally, see [10, 11]. In the beautiful experiment of Kasevich and Chu [11] the Eötvös parameter for the sodium atom was estimated to be $\leq 10^{-6}$.

3. Comparison with a classical point particle

The same result can be obtained for the classical *test* particle, but under slightly stronger assumptions. The second theorem of Nöther connected with the Milne covariance and with the gauge freedom of the potential gives us the identities equivalent to the equation of motion of the particle, that is, the geodetic equation.

Now we present this in details. Suppose we have a matter fields $b^A = b^A(X)$ with the Lagrange density function $\mathcal{L} = \mathcal{L}(X, b^A, \phi)$. We consider the Nöther identities. In our case the covariance transformations depend on arbitrary functions of the time only, but not of all space-time coordinates. Compare the Milne transformations or the time dependent gauge transformation of the potential $\phi \to \phi + \dot{\epsilon}(t)$. This arbitrariness does not allows us to obtain the differential identities, but it is sufficient to obtain some integral identities, if we suppose that the matter density goes to zero sufficiently fast when the space coordinates goes to infinity. Namely, let us denote the arbitrary functions of time defining the covariance transformations by $\epsilon^i = \epsilon^i(t)$. The variations of the fields $\equiv \{b^A, \phi\}$ under the action of the transformation are as follows

$$\overline{\delta} y^{\mathcal{A}} = c_i^{\mathcal{A}} \epsilon^i + d_i^{\mathcal{A}\mu} \partial_\mu \epsilon^i + g_i^{\mathcal{A}\mu\nu} \partial_\mu \partial_\nu \epsilon^i.$$

Let us write $\mathcal{L}_{\mathcal{A}} \equiv \{\mathcal{L}_A, \mathcal{L}_0\}$ for the Euler–Lagrange derivatives of $\mathcal{L}$ with respect to $y^{\mathcal{A}}$, $\mathcal{L}_0$ is the Euler–Lagrange derivative with respect to ϕ. Then we have the following Nöther integral identities:

$$\int\limits_{R^3} \left[\mathcal{L}_{\mathcal{A}} c_i^{\mathcal{A}} - \partial_\mu (\mathcal{L}_{\mathcal{A}} d_i^{\mathcal{A}\mu}) + \partial_\mu \partial_\nu (\mathcal{L}_{\mathcal{A}} g_i^{\mathcal{A}\mu\nu}) \right] \mathrm{d}^3 x \equiv 0.$$

The identities for the gauge transformation $\phi \to \phi + \dot{\epsilon}(t)$ and the Milne transformations $x^i \to x^i + \epsilon^i(t)$ read

$$\frac{d}{dt} \left[\int\limits_{R^3} \mathcal{L}_0 \, \mathrm{d}^3 x \right] \equiv 0, \tag{13}$$

$$\frac{d^2}{dt^2} \left[\int\limits_{R^3} (-\mathcal{L}_0 x_i) \, \mathrm{d}^3 x \right] \equiv \int\limits_{R^3} \mathcal{L}_0 \partial_i \phi \, \mathrm{d}^3 x, \tag{14}$$

by virtue of $\mathcal{L}_A = 0$. At this place we have to make additional assumption as compared to quantum level, that in the limit for the point particle moving along a trajectory $z_i = z_i(t)$ we have

$$\mathcal{L} = \mathfrak{L}(x_i, t) \delta(x_i - z_i), \tag{15}$$

where δ is the three-dimensional Dirac delta function. Suppose first that the matter is minimally coupled: $\mathcal{L}$ does not depend on derivatives of the potential. From the identity (13) it follows that $\mathcal{L}_0 = \mathrm{const}\, \delta(x - z) \equiv -m\delta(x - z)$. Inserting this to the identity (14) one obtains

$$m \ddot{z}_i = -m \partial_i \phi. \tag{16}$$

Now, suppose that the matter is not minimally coupled and the Lagrange density does depend on the first degree derivatives of the potential. We assume, however, that the Lagrange density does not depend on the second and the higher derivatives of the potential. As a consequence of this assumption and from the identity (13) we get $\mathcal{L}_0 = -\mathrm{const}\,\delta(x-z) - \partial_i(Q^i\delta(x-z))$, where $Q^i = \partial\mathfrak{L}/\partial(\partial_i\phi)$. The identity (1) gives

$$m\ddot{z}_i = -m\partial_i\phi + Q^j\partial_i\partial_j\phi + \ddot{Q}_i\,. \tag{17}$$

As is seen from this equation the quantity $W_i = Q^j\partial_i\partial_j\phi + \ddot{Q}_i$ has to be a vector. In accord to our general assumption W_i is an algebraic function of ϕ, its derivatives up to the third order, $\dot{z}_i$ and $\ddot{z}_i$. An analysis similar to that presented above shows that $W_i = W_i(\partial_k\phi + \ddot{z}_i, \partial_t\partial_j\partial_k\phi + \dot{z}^a\partial_a\partial_j\partial_k\phi, \partial_a\partial_b\phi, \partial_a\partial_b\partial_c\phi)$. Taking this into account and the specific form of the W_i in (17) we prove that $W_i = 0$ and the equation (17) takes on the form of the geodetic equation (16). We have not analyzed the situation in which the Lagrange density $\mathcal{L}$ of matter can *a priori* depend on second order derivatives of ϕ. Because it is natural to assume that the field equations of matter and gravity are of second order at most it is natural to assume that $\mathcal{L}$ does not depend on second and higher order derivatives of the potential.

So, if one assume that the Lagrange density does not depend on second and higher order derivatives of ϕ, and that the equations of matter are generally covariant and can be constructed in a local way from the geometry of space-time, then the equations of motion for the particle are determined by the space-time geometry. Moreover, the equations are in accord with the equivalence principle. Summing up, we have shown exactly the same for the quantum particle[2]. Note, that the equation (12) cannot be derived from the Lagrange density which does not contain the second derivatives of the field ϕ if $\Lambda \neq 0$. Our result is nontrivial if one takes into account (1) the observation of Trautman [17] that the equivalence principle can be violated by a field with the Lagrange function containing the first degree derivatives of ϕ and (2) the equation of motion for any matter field cannot be derived in this way.

[2] The equality of both masses can be obtained of course by considering the WKB approximation of the corresponding classical particle. But this is the whole point: in the classical equations of motion mass cancels out on both sides if the gravitational mass is equal to the inertial one. WKB approximation is just equivalent to classical mechanics. In full Schrödinger equation the mass does not cancel. The identities (13) and (14) are also true for the Lagrange density function $\mathcal{L}$ of the Schrödinger equation. The assumption (15) can be fulfilled approximately for a finite time interval. So, we get the equivalence principle at the classical level without imposing any condition on the inertial mass in the Schrödinger equation, because the identities (13) and (14) do not involve the kinetic term which contains the inertial mass. So, there is no immediate relation between the equality of the inertial and the gravitational masses for a classical particle and the equality for a quantum particle. Therefore the ability to prove the equality is nontrivial.

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